ON THE BROWNIAN RANGE AND THE BROWNIAN REVERSAL

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Abstract

This paper studies a novel Brownian functional defined as the supremum of a weighted average of the running Brownian range and its running reversal from extrema on the unit interval. We derive the Laplace transform for the squared reciprocal of this functional, which leads to explicit moment expressions that are new to the literature. We show that the proposed Brownian functional can be used to estimate the spot volatility of financial returns based on high-frequency price observations.

Keywords: Brownian functionals; range process; Laplace transform; spot volatility estimation

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1. Introduction

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, let $W = (W_t)_{t \in [0,1]}$ denote a standard Brownian motion starting at $W_0 = 0$, and denote $M_t := \sup_{0 \leq s \leq t} W_s$ and $m_t := \inf_{0 \leq s \leq t} W_s$ as its associated supremum and infimum processes. The process $R_t := M_t - m_t$ is known as the Brownian range process, whose properties have been studied extensively in [6], [15], [18], and [30], among others. Properties of the Brownian range process are exploited to measure the variation of financial returns; see e.g. [3], [7], [16], [25], and [27]. Intuitively, when one models the log-price of an asset by a scaled Brownian motion, its range process in an interval summarizes the maximum span of the asset price during that interval, which provides more precise measurements of the price variation than using only prices at end-points.

Nevertheless, R_t only increases its value when W_t refreshes its running extrema, and is otherwise constant. Crucially, it does not reveal how much the Brownian motion has reversed back towards W_0 from the running extrema, which can contain substantial amount of information about the variability of the Brownian path. To capture the reversal part of the Brownian path, we define U_t and u_t as the Brownian reversal processes from above and below, respectively:

$$U_t := M_t - W_t^+, \quad u_t := m_t - W_t^-,$$

where $W_t^+ = (W_t \vee 0)$ and $W_t^- = (W_t \wedge 0)$ are the sections of the trajectories of W_t above and below W_0. A graphical illustration is presented in Figure 1. Intuitively, U_t and u_t are the proportions of the Brownian paths reflected at its supremum or infimum that are above or below W_0, respectively. Whenever $W_t = M_t$ (resp. $W_t = m_t$), we have $U_t = 0$ (resp. $u_t = 0$) so that the
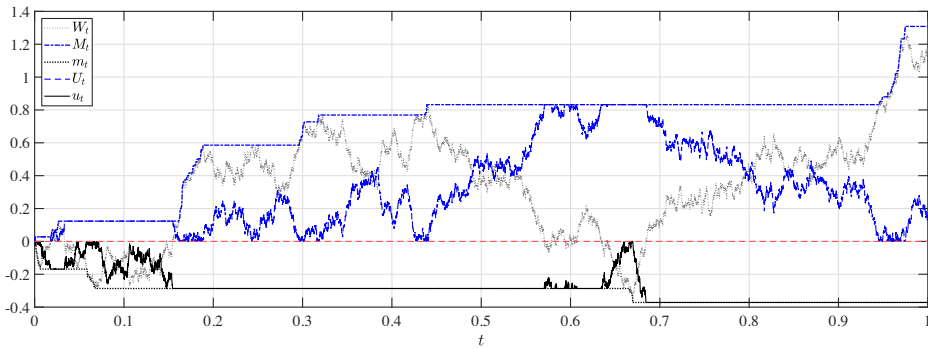

FIGURE 1. An illustration of running Brownian extrema and reversals.

Brownian motion is exactly at its supremum (resp. infimum) with zero reversal. As W_t evolves from the extrema towards W_0, the magnitudes of U_t or u_t increase, and vice versa. The maximum reversal from above (resp. below) is attained when W_t crosses W_0 from above (resp. below), in which case we have $U_t = M_t$ (resp. $u_t = m_t$), indicating that the Brownian motion at time t has returned to W_0 from above (resp. below).

Similar to the construction of the Brownian range which summarizes the magnitudes of the Brownian supremum and infimum, the total reversal of W_t from both above and below can be summarized by the following process, which we refer to as the *Brownian reversal* at time t:

$$V_t := U_t - u_t \equiv R_t - |W_t|. \tag{1.1}$$

It is easy to see that $0 \le V_t \le R_t$ for all t, as the Brownian motion cannot reverse by more than its total range. For any t, the joint law of (R_t, V_t) can be derived from the Lévy trivariate law of (W_t, M_t, m_t), which can be found in e.g. [8] and [15].

The main contribution of this paper is to document the analytical properties of a novel Brownian functional defined as the supremum of the weighted average of V_t and R_t on [0,1]:

$$S^{(\alpha)} := \sup_{t \in [0,1]} \{\alpha V_t + (1-\alpha)R_t\} \equiv \sup_{t \in [0,1]} s_t^{(\alpha)}, \quad \alpha \in [0, 1],$$

where $s_t^{(\alpha)} := R_t - \alpha|W_t|$ in view of (1.1). As two important special cases, $S^{(0)} \equiv R_1$ is the Brownian range, and $S^{(1)} \equiv \sup_{t \in [0,1]} V_t$ is termed the *maximal Brownian reversal*. According to [29], one can usually guess whether the law of a particular Brownian functional is straightforward or impossible to establish, but it is not immediately clear which category $S^{(\alpha)}$ belongs to. In this paper, we shall show that the law of $S^{(\alpha)}$ can be explicitly characterized. In particular, we derive the Laplace transform of the random variable $T^{(\alpha)} := (S^{(\alpha)})^{-2}$ for all $\alpha \in [0, 1]$, which reveals many interesting properties of $S^{(\alpha)}$. For example, we prove that all moments of $T^{(\alpha)}$ are rational functions of α, and the maximal Brownian reversal has the first moments $\mathbb{E}[T^{(1)}] = 1$ and $\mathbb{E}[S^{(1)}] = \sqrt{8/\pi} \ln 2$. As an application, we show that $S^{(\alpha)}$ can be used to construct more precise spot volatility estimators than those based on high-frequency price ranges such as [3] and [25], where the precision gain comes from the reversal part of the Brownian path not captured by the price range.

The remainder of the paper is organized as follows. Section 2 presents the main results of the paper concerning the distributional properties of $S^{(\alpha)}$. An application of $S^{(\alpha)}$ to spot volatility estimation is shown in Section 3, followed by the proofs to all theoretical results in Section 4.

2. Main results

Our key result is a semi-closed-form expression for the Laplace transform of $T^{(\alpha)}$, which allows us to obtain explicit moments of $S^{(\alpha)}$ and $T^{(\alpha)}$ as well as their density functions. To derive the Laplace transform, we introduce some notations that will be used in this section. Denote the hyperbolic sine, cosine, and tangent functions as $\sinh[x]$, $\cosh[x]$, and $\tanh[x]$ for $x \in \mathbb{R}$, respectively. The inverse function of $y = \tanh[x]$ will be denoted by $x = \operatorname{arctanh}[y]$ for $y \in (-1, 1)$. For a general process $(A_t)_{t \geq 0}$, let $T_x(A) := \inf\{t \geq 0 : A_t > x\}$ denote the first hitting time of A to the level x since time 0 with the convention $\inf\{\emptyset\} = \infty$. We shall use $n \in \mathbb{N}$ to denote a generic natural number (excluding 0).

We start from the following equal events:

$$\{S^{(\alpha)} > x\} = \left\{ \sup_{t \in [0,1]} s_t^{(\alpha)} > x \right\} = \{T_x(s^{(\alpha)}) < 1\}.$$

By the Brownian scaling law, we deduce that

$$\mathbb{P}(S^{(\alpha)} > x) = \mathbb{P}(T_x(s^{(\alpha)}) < 1) = \mathbb{P}(x^2 T_1(s^{(\alpha)}) < 1) = \mathbb{P}(T_1(s^{(\alpha)})^{-1/2} > x),$$

which implies that

$$T^{(\alpha)} := (S^{(\alpha)})^{-2} \stackrel{d}{=} T_1(s^{(\alpha)}).$$

Therefore $T^{(\alpha)}$ has the same law as the first hitting time of the non-negative process $s^{(\alpha)}$ to 1. As two special cases, $T_1(s^{(0)})$ is the hitting time of the Brownian range process to 1, which is studied in [6] and [18], and $T_1(s^{(1)})$ is the first hitting time of the Brownian reversal process V_t to 1, whose distributional properties are unknown in the literature. For all $\alpha \in [0, 1]$, we have $R_t \geq s_t^{(\alpha)}$ for all $t \geq 0$ and it follows that $T_1(s^{(1)}) \geq T_1(s^{(\alpha)})$. The Laplace transform of $T^{(\alpha)}$ is stated in Theorem 2.1 below.

Theorem 2.1. *Let $\bar{\alpha} := \alpha/(1 - \alpha)$. For all $\alpha \in [0, 1)$ and $\lambda > 0$, it holds that*

$$\mathcal{L}_\alpha(\lambda) := \mathbb{E}[e^{-\lambda T^{(\alpha)}}] = \sinh[\sqrt{2\lambda}]^{-2} \int_0^{\tanh[\sqrt{2\lambda}]^2} (1 - y)^{-1} {}_2F_1\left(\frac{1}{2}, 1; 1 + \frac{\bar{\alpha}}{2}; y\right) dy, \quad (2.1)$$

where ${}_2F_1(a, b; c; x)$ is the Gauss hypergeometric function defined by

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k,$$

in which $(a)_k := a(a + 1) \cdots (a + k - 1)$ is the rising factorial with $(a)_0 = 1$. When $\alpha = 1$, $\mathcal{L}_1(\lambda)$ takes the form

$$\mathcal{L}_1(\lambda) = \lim_{\alpha \to 1} \mathcal{L}_\alpha(\lambda) = 2 \sinh[\sqrt{2\lambda}]^{-2} \ln \cosh[\sqrt{2\lambda}]. \quad (2.2)$$

For all $\lambda \geq 0$ and $\alpha \in [0, 1]$, it holds that

$$\mathcal{L}_\alpha(\lambda) \leq \mathcal{L}_0(\lambda) = \cosh[\sqrt{\lambda/2}]^{-2} \leq 1. \quad (2.3)$$

Remark 2.1. With $\alpha = 0$, we have ${}_2F_1(\frac{1}{2}, 1; 1; y) = (1 - y)^{-1/2}$, and thus the expression of $\mathcal{L}_0(\lambda)$ can be verified by carrying out the integration in (2.1), which agrees with the Laplace

transform of the Brownian range time given in e.g. [5], [6], and [18]. When $\bar{\alpha} = n$, then $\alpha = n/(n+1)$ and the Gauss hypergeometric function can be evaluated explicitly by the following recursive relation [26, eq. (15.5.16)]:

$$F_{n+2}(y) = \frac{(n+2)(1 - (1-y)F_n(y))}{(n+1)y}, \quad F_0(y) = (1-y)^{-1/2}, \quad F_1(y) = \mathrm{arctanh}[\sqrt{y}]/\sqrt{y},$$

where

$$F_n(y) := {}_2F_1\left(\frac{1}{2}, 1; 1 + \frac{n}{2}; y\right).$$

This further leads to a series of closed-form Laplace transforms by analytical integration. Examples of the Laplace transforms for $n \in \{1, 2, 3, 4\}$ are presented as

$$\mathcal{L}_{1/2}(\lambda) = 2\lambda \sinh[\sqrt{2\lambda}]^{-2}, \quad \mathcal{L}_{2/3}(\lambda) = 8 \sinh[\sqrt{2\lambda}]^{-2} \ln \cosh[\sqrt{\lambda/2}],$$

$$\mathcal{L}_{3/4}(\lambda) = 3 \sinh[\sqrt{2\lambda}]^{-2}(\sqrt{2\lambda} \tanh[\sqrt{2\lambda}]^{-1} - 1), \tag{2.4}$$

$$\mathcal{L}_{4/5}(\lambda) = \frac{4}{3} \sinh[\sqrt{2\lambda}]^{-2}(\tanh[\sqrt{x/2}]^2 + 4 \ln \cosh[\sqrt{x/2}]),$$

and the expressions for a general n can be obtained in the same fashion. The family of Laplace transforms appears to be new in the literature.

Remark 2.2. Interestingly, the Laplace transform of $\mathcal{L}_{1/2}(\lambda)$ in (2.4) also shows up in [2] and [28]. Combining equations (1.4), (1.8), and (1.9) of [2], we deduce the following unobvious result:

$$T^{(1/2)} \overset{d}{=} \left(\max_{t,s \in [0,1]} |b_t - b_s|\right)^2,$$

where $b = (b_t)_{t \in [0,1]}$ is the standard Brownian bridge, that is, a Brownian motion W on $[0,1]$ conditioned on $W_0 = W_1 = 0$. It is intriguing to identify the probabilistic link that causes the Laplace transforms of two completely different Brownian functionals to coincide. However, this requires individual theoretical analysis which deviates from the main focus of this paper, and is thus left for future research.

The expression for $\mathcal{L}_\alpha(\lambda)$ allows us to evaluate the moments of $T^{(\alpha)}$ and $S^{(\alpha)}$. We begin by summarizing some known results in the literature corresponding to the special cases $\alpha \in \{0, \frac{1}{2}\}$. Let us denote $\kappa_k^{(\alpha)} := \mathbb{E}[(T^{(\alpha)})^k]$ and $\mu_k^{(\alpha)} := \mathbb{E}[(S^{(\alpha)})^k]$. From Table 1 of [2], we deduce the following results for all $k \in \mathbb{N}$ after some straightforward simplification:

$$\kappa_k^{(0)} = \frac{k!(-2)^{k+1}}{(2k)!(k+1)}(1 - 4^{k+1})B_{2(k+1)}, \quad \kappa_n^{(1/2)} = \frac{k!}{(2k)!}(1 - 2k)(-2)^{3k}B_{2k},$$

$$\mu_k^{(0)} = \frac{4}{\sqrt{\pi}}\Gamma\left(\frac{k+1}{2}\right)(1 - 2^{2-k})2^{k/2}\zeta(k-1), \quad \mu_k^{(1/2)} = \sqrt{\frac{2}{\pi}}2^{(1-k)/2}k\Gamma\left(\frac{3+k}{2}\right)\zeta(k+1), \tag{2.5}$$

where B_n is the nth Bernoulli number, $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt$ is the gamma function, and $\zeta(x)$ is the Riemann zeta function. The moments of the Brownian range, $\mu_k^{(0)}$, also appear in [16] and [27]. It is worth noting that $\mu_2^{(0)} = 4 \ln 2$ can be derived by taking the limit of the above

expression using properties of the zeta function. Also, as $\mu_k^{(0)} \geq \mu_k^{(\alpha)}$ for all $\alpha \in [0, 1]$ due to $R_1 \geq S^{(\alpha)}$, we conclude that all moments of $S^{(\alpha)}$ are finite.

We now turn to the general case with $\alpha \in [0, 1]$. We begin with the calculation of $\kappa_k^{(\alpha)}$, which have surprisingly simple forms. In detail, we shall show that $\kappa_k^{(\alpha)}$ are finite for all $k \in \mathbb{N}$ and are rational functions of α. To this end, we define the rational sequence $(C_{n,k})_{n,k\in\mathbb{N}}$ through the following recursive relation:

$$C_{n,1} = \frac{2^{3n+5}(1 - 4^{n+2})(2n + 3)B_{2(n+2)}}{(2(n + 2))!}, \quad C_{n,k} = \sum_{j=0}^{n} C_{j,k-1}C_{n-j,1}.$$

We also need the following sequence of rational functions for $k \in \mathbb{N} \cup \{0\}$,

$$D_k^{(\alpha)} := \frac{\bar{\alpha}}{\bar{\alpha} - 1}\left(1 - \frac{(1/2)_{k+1}}{(\bar{\alpha}/2)_{k+1}}\right), \tag{2.6}$$

which is continuous on [0,1] with well-defined limits at $\alpha \in \{0, 1/2, 1\}$,

$$D_k^{(0)} = \frac{2(1/2)_{k+1}}{k!}, \quad D_k^{(1/2)} = H_{2(k+1)} - H_{k+1}/2, \quad D_k^{(1)} \equiv 1, \tag{2.7}$$

where $H_n = \sum_{k=1}^{n} k^{-1}$ is the nth harmonic number. We have the following.

Proposition 2.1. *For all $k \in \mathbb{N}$ and $\alpha \in [0, 1]$, we have $\kappa_k^{(\alpha)} \leq \kappa_k^{(1)} < \infty$ with*

$$\kappa_k^{(\alpha)} = \sum_{n=0}^{k} \binom{k}{n}\kappa_n^{(1/2)}\gamma_{k-n}^{(\alpha)}, \tag{2.8}$$

where $\gamma_k^{(\alpha)}$ for $\alpha \in (0, 1) \setminus \{1/2\}$ is defined as the finite sum

$$\gamma_k^{(\alpha)} = (-1)^k k! \sum_{n=0}^{k} \frac{D_n^{(\alpha)}C_{k-n,n+1}}{2(n + 1)}, \tag{2.9}$$

with the limiting cases

$$\gamma_k^{(0)} = \frac{(-1)^k k! 2^{k+1}}{(2(k + 1))!}, \quad \gamma_k^{(1/2)} = \mathbb{1}_{\{k=0\}}, \quad \gamma_k^{(1)} = (-1)^k k! \frac{2^{3k+2}(4^{k+1} - 1)B_{2(k+1)}}{(k + 1)(2(k + 1))!}. \tag{2.10}$$

Remark 2.3. As $\kappa_k^{(\alpha)}$ is a finite combination of $\gamma_{k-n}^{(\alpha)}$ which is a rational function of α, we conclude that $\kappa_k^{(\alpha)}$ is a rational function of α, which implies that $\kappa_k^{(\alpha)} \in \mathbb{Q}$ whenever $\alpha \in \mathbb{Q} \cap [0, 1]$. It is also worth noting that $\gamma_k^{(1/2)} = \mathbb{1}_{\{k=0\}}$ reduces (2.8) to the trivial identity $\kappa_k^{(1/2)} = \kappa_k^{(1/2)}$.

Taking $\alpha = 1$, equation (2.5) and Proposition 2.1 lead to the following simpler form for $\kappa_k^{(1)}$.

Corollary 2.1. *It holds that*

$$\kappa_k^{(1)} = -\sum_{n=0}^{k} \binom{k}{n}\frac{(-2)^{2n+1}\kappa_n^{(0)}\kappa_{k-n}^{(1/2)}}{(2n + 1)(2n + 2)}. \tag{2.11}$$

Using the closed-form Laplace transforms in (2.4), it is possible to derive a simplified formula for $\kappa_k^{(\alpha)}$ with $\alpha \in \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}\}$ that does not involve a double sum. Regrettably, for a general $\alpha \in [0, 1]$, we are unable to simplify the polynomial expressions in Proposition 2.1. Nevertheless, Proposition 2.1 allows us to iteratively construct the exact form of $\kappa_k^{(\alpha)}$ for all $k \in \mathbb{N}$. To provide a more concrete illustration on the functional form of $\kappa_k^{(\alpha)}$ derived in Proposition 2.1, we list a few expressions for $k \leq 4$:

$$\kappa_1^{(\alpha)} = \frac{1}{2 - \alpha}, \quad \kappa_2^{(\alpha)} = \frac{4}{3(4 - 3\alpha)}, \quad \kappa_3^{(\alpha)} = \frac{4(20\alpha^2 - 62\alpha + 51)}{15(2 - \alpha)(4 - 3\alpha)(6 - 5\alpha)},$$

$$\kappa_4^{(\alpha)} = \frac{32(372 - 757\alpha + 530\alpha^2 - 126\alpha^3)}{105(2 - \alpha)(4 - 3\alpha)(6 - 5\alpha)(8 - 7\alpha)}.$$

Setting $\alpha \in \{0, \frac{1}{2}, 1\}$, we obtain the values

$$\kappa_1^{(0)} = \frac{1}{2}, \quad \kappa_2^{(0)} = \frac{1}{3}, \quad \kappa_3^{(0)} = \frac{17}{60}, \quad \kappa_4^{(0)} = \frac{31}{105},$$

$$\kappa_1^{(1/2)} = \frac{2}{3}, \quad \kappa_2^{(1/2)} = \frac{8}{15}, \quad \kappa_3^{(1/2)} = \frac{32}{63}, \quad \kappa_4^{(1/2)} = \frac{128}{225},$$

$$\kappa_1^{(1)} = 1, \quad \kappa_2^{(1)} = \frac{4}{3}, \quad \kappa_3^{(1)} = \frac{12}{5}, \quad \kappa_4^{(1)} = \frac{608}{105},$$

and it is routine to verify that these values are indeed consistent with the analytical expression in equation (2.5) and Corollary 2.1. The simple result of $\mathbb{E}[T^{(1)}] = 1$ is worth highlighting, which means that the Brownian reversal process V_t hits 1 with a unit mean hitting time. This coincides with the well-known result $\mathbb{E}[T_1(|W|)] = 1$, which is the expected time for a standard Brownian motion to exit $[-1, 1]$. However, as $\mathbb{E}[e^{-\lambda T_1(|W|)}] = \cosh[\sqrt{2\lambda}]^{-1}$, the distribution of $T_1(|W|)$ is completely different from that of $T^{(1)}$ despite the identical first moment. For example, one can verify that $2\text{Var}[T^{(1)}] = \text{Var}[T_1(|W|)] = \frac{2}{3}$, so $T^{(1)}$ is less dispersed than $T_1(|W|)$.

We proceed to examine $\mu_k^{(\alpha)}$ for $k \in \mathbb{N}$, which can be evaluated (see [9]) from the following fractional derivative of the Laplace transform of $T^{(\alpha)}$ since $\mu_k^{(\alpha)} \equiv \mathbb{E}[(T^{(\alpha)})^{-k/2}]$:

$$\mu_k^{(\alpha)} = \frac{1}{\Gamma(k/2)} \int_0^\infty \mathcal{L}_\alpha(\lambda)\lambda^{k/2-1} \, \mathrm{d}\lambda, \tag{2.12}$$

which converges for all $k \in \mathbb{N}$ as the moments of $S^{(\alpha)}$ are finite. We derive the following representation.

Proposition 2.2. *For all $k \in \mathbb{N}$ and $\alpha \in [0, 1]$, it holds that*

$$\mu_k^{(\alpha)} = \frac{2^{1-k/2}}{\Gamma(k/2)} \sum_{n=0}^\infty \frac{D_n^{(\alpha)}}{n + 1} \int_0^1 \operatorname{arctanh}[y]^{k-1} y^{2n} \, \mathrm{d}y. \tag{2.13}$$

We obtain some more tractable formulae below using (2.13).

Corollary 2.2. *For all $k \in \mathbb{N}$, it holds that*

$$\mu_k^{(1)} = -\frac{2^{1-k/2}}{\Gamma(k/2)} \int_0^1 y^{-2} \ln(1 - y^2) \operatorname{arctanh}[y]^{k-1} \, \mathrm{d}y. \tag{2.14}$$

For $\alpha \in (0, 1) \setminus \{\frac{1}{2}\}$, we have

$$\mu_1^{(\alpha)} = \sqrt{\frac{2}{\pi}} \frac{\bar{\alpha}}{\bar{\alpha} - 1} (2 \ln 2 - \Psi_0(\bar{\alpha})),$$

$$\mu_2^{(\alpha)} = \frac{\bar{\alpha}}{\bar{\alpha} - 1} \left(\frac{\pi^2}{12} + (\ln 2 - \Psi_0(\bar{\alpha})^2/2)^2 + \frac{\Psi_1(\bar{\alpha})}{4} \right),$$

(2.15)

where

$$\Psi_k(x) = \psi_k\left(\frac{1+x}{2}\right) - \psi_k\left(\frac{x}{2}\right),$$

and $\psi_k(x)$ is the kth-order polygamma function, defined as the kth derivative of the digamma function $\psi_0(x) := \mathrm{d} \ln \Gamma(x)/\mathrm{d}x$.

Remark 2.4. The definite integral in (2.14) can be calculated explicitly for $k \in \{1, 2, 3\}$ via MATHEMATICA®:

$$\mu_1^{(1)} = \sqrt{8/\pi} \ln 2, \quad \mu_2^{(1)} = \pi^2/12 + (\ln 2)^2, \quad \mu_3^{(1)} = \frac{\pi^2 \ln 2}{3\sqrt{2\pi}} + \frac{3\zeta(3)}{2\sqrt{2\pi}}.$$

(2.16)

This should be compared with the following values generated by (2.5):

$$\mu_1^{(0)} = \sqrt{\frac{8}{\pi}}, \quad \mu_2^{(0)} = 4 \ln 2, \quad \mu_3^{(0)} = \frac{\sqrt{8}}{3}\pi^{3/2},$$

$$\mu_1^{(1/2)} = \frac{\pi^{3/2}}{3\sqrt{2}}, \quad \mu_2^{(1/2)} = \frac{3\zeta(3)}{2}, \quad \mu_3^{(1/2)} = \frac{\pi^{7/2}}{30\sqrt{2}}.$$

(2.17)

Thus the moments $\mu_k^{(1)}$ appear to be more sophisticated than the special cases in (2.17). Using properties of polygamma functions and L'Hôpital's rule, one can verify that the limits of the functions in (2.15) at $\alpha \in \{0, \frac{1}{2}, 1\}$ are identical to those in (2.16) and (2.17). However, we are unable to derive a general expression for $k > 2$, in which case the integral in (2.13) does not simplify to elementary functions of n. In this case, it is more convenient to calculate the moments by numerically integrating (2.12) than from (2.14), as the latter converges very slowly when α is near zero.

Remark 2.5. The quantity $\sqrt{8/\pi} \ln 2$ also appears in [11] as $\mathbb{E}[\mathbb{D}_1] = \mu_1^{(1)}$, where $\mathbb{D}_1 = W_\sigma - \inf_{\sigma \leq t \leq 1} W_t$ and σ is the time that W attains its maximum in [0,1], that is, $\mathbb{D}_1$ is the maximum Brownian 'downfall' from its global maximum W_σ on [0,1] to the partial minimum on $[\sigma, 1]$. The distribution of $\mathbb{D}_1$ is closely related to the Brownian meander, as discussed in [12]. However, the equality $\mathbb{E}[\mathbb{D}_1] = \mu_1^{(1)}$ only holds in the first order, as we find $\mathbb{E}[\mathbb{D}_1^2] = \pi^2/6 \neq \mu_2^{(1)}$ by integrating the density of $\mathbb{D}_1$.

We conclude this section with some comments on evaluating the density function of $T^{(\alpha)}$, denoted by $f_T(t;\alpha)$, which also produces the density function of $S^{(\alpha)}$ by a standard Jacobian transform. By definition, we have $\mathcal{L}_\alpha(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda t} f_T(t;\alpha)\, \mathrm{d}t$, so that for almost all $t \geq 0$, $f_T(t;\alpha)$ can be recovered by the inverse Laplace transform of $\mathcal{L}_\alpha(\lambda)$ in principle (see e.g. [33]). However, apart from some special cases (e.g. $\alpha = 0, \frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \ldots$), the inverse Laplace transform does not seem to have analytical solutions, including the relatively simple $\mathcal{L}_1(\lambda)$. Therefore, for a general $\alpha \in [0, 1]$ we recommend calculating $f_T(t;\alpha)$ based on numerical
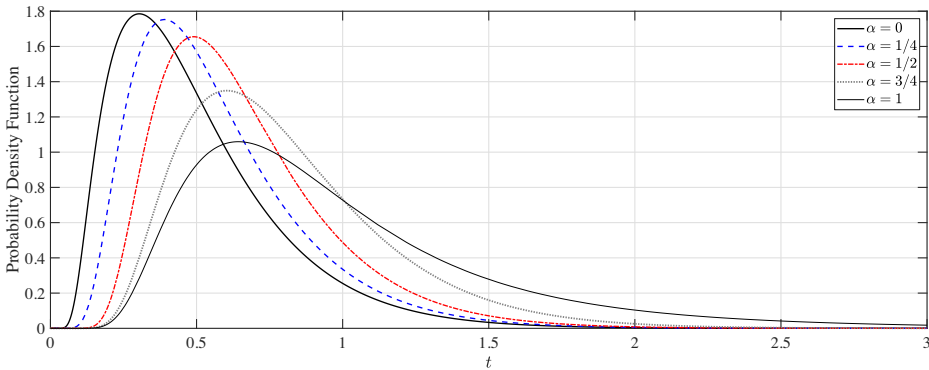
FIGURE 2. Probability density functions of $T^{(\alpha)}$ for various choices of α. Each line plots $f_T(t;\alpha)$ for the choice of α presented in the figure legend. Apart from the $\alpha = 0$ case where the expression of $f_T(t;0)$ can be found in [6, eq. (4)], all the densities are generated by numerical inverse Laplace transforms via MATHEMATICA®.

inverse Laplace transforms. For some selected values of α, we present the density functions $f_T(t;\alpha)$ in Figure 2. The figure shows that $T^{(\alpha)}$ have unimodal and right-skewed distributions across different choices of $\alpha \in [0, 1]$. The probability mass moves towards the right tail as α increases, which reflects the fact that $T^{(\alpha)}$ is increasing in α.

3. Spot volatility estimation with price ranges and reversals

In this section, we show that $S^{(\alpha)}$ can be used to construct simple yet precise spot volatility estimators of financial returns, which is a key economic variable of interest to investors and policy makers. We shall adopt the setting in [3], [4], and [25], and assume that the log-price of an asset $(P_t)_{t \geq 0}$ follows a semi-martingale of the form

$$P_t = P_0 + \int_0^t \mu_s \, \mathrm{d}s + \int_0^t \sigma_s \, \mathrm{d}W_s + J_t, \quad t \in [0, T], \tag{3.1}$$

where W is a standard Brownian driving the diffusive part of the price process, the drift process μ and the volatility process σ are both càdlàg (right continuous with left limits) adapted, and J is a pure-jump process driven by some Poisson random measure. The interval $[0, T]$ can be understood as the length of a trading session when the price is observed, and our inference target can be σ_τ or σ_τ^2 at some fixed time $\tau \in (0, T)$, which are known as the spot volatility or the spot variance at time τ.

Following [3] and [25], to facilitate our asymptotic analysis on spot volatility estimation, we impose some regularity conditions on the processes above.

Assumption 3.1. *For the price process P in (3.1), we assume that there exists a sequence of diverging stopping times $(T_m)_{m \geq 1}$ and a sequence of finite constants $(K_m)_{m \geq 0}$ such that, for all $m \geq 1$:*

(i) $|\mu_t| + |\sigma_t| + |\sigma_t|^{-1} + F_t(\mathbb{R} \setminus \{0\}) \leq K_m$ *for all $t \in [0, T_m]$, where F_t denotes the spot Lévy measure of J;*

(ii) *for all $s, t \in [0, T]$, it holds that $\mathbb{E}[|\sigma_{t \wedge T_m} - \sigma_{s \wedge T_m}|^2] \leq K_m |t - s|^{2\kappa}$ for some constant $\kappa > 0$.*

As discussed in [3] and [25], the above assumption is very mild. It only assumes local boundedness of various processes and that σ_t is κ-Hölder-continuous in the L_2-norm. The assumption encompasses almost all stylized facts of asset prices, such as stochastic volatility with a diurnal pattern and leverage effect. In particular, by choosing $\kappa < 1/2$, we allow σ_t to have a 'rough' path, which is advocated by recent studies in the financial mathematics literature [13, 14, 17].

To estimate the spot quantities at some predetermined time τ, we focus on a shrinking window starting at τ, $I_n = [\tau, \tau + \Delta_n] \subset [0, T]$, where n represents the stages of the statistical experiment, and the asymptotic limit is achieved by letting $\Delta_n \to 0$ as $n \to \infty$. This corresponds to the fixed-k asymptotics in [4] using exactly $k = 1$ block, and our theory naturally extends to the case with fixed $k > 1$ following the procedure in [25].

We construct the following functional from the observed price path on I_n, which is essentially the same functional as $S^{(\alpha)}$ constructed from P instead of W, up to a normalization factor of $\Delta_n^{1/2}$:

$$v_n^{(\alpha)} := \Delta_n^{-1/2} \sup_{t \in I_n} \{w_t - \alpha |r_t|\}, \quad \alpha \in [0, 1],$$

where $r_t := P_t - P_\tau$ is the log-return on the interval $[\tau, t]$ for some $t \in I_n$, and $w_t := h_t - l_t$ is the price range on $[\tau, t]$, in which the 'high' and 'low' returns on $[\tau, t]$ are defined as $h_t := \sup_{s \in [\tau, t]} r_s$ and $l_t := \inf_{s \in [\tau, t]} r_s$, respectively. As a comparison, a candlestick chart observed on I_n consists of the open-to-close, high, and low returns on I_n, $(r_{\tau + \Delta_n}, h_{\tau + \Delta_n}, l_{\tau + \Delta_n})$. From here, it is clear that both $v_n^{(\alpha)}$ and the candlestick data use the entire price path of P on I_n. The main difference is that $v_n^{(\alpha)}$ allows us to measure the maximum price reversal through the outer supremum operator, while the candlestick data can only measure the price reversal at the end-point of the interval, i.e. $w_{\tau + \Delta_n} - |r_{\tau + \Delta_n}|$. Note that in practice we only observe a discrete price path contaminated by measurement errors. Nevertheless, as argued in [3] and [25], we may choose Δ_n to 'not-too-finely' sample the high-frequency prices, such that the price functionals can reliably approximate those computed from the efficient price path.

In the same vein as [3], [4], [20], and [25], we show that $v_n^{(\alpha)}$ is coupled with a limit Brownian functional that has the same distribution as $S^{(\alpha)}$ scaled by σ_τ.

Theorem 3.1. *Under Assumption 3.1 as $n \to \infty$, it holds that*

$$\sup_{\alpha \in [0,1]} |v_n^{(\alpha)} - \sigma_\tau S_n^{(\alpha)}| = o_p(1), \tag{3.2}$$

where

$$S_n^{(\alpha)} := \Delta_n^{-1/2} \sup_{t \in I_n} \left\{ \sup_{s \in [\tau, t]} W_s - \inf_{s \in [\tau, t]} W_s - \alpha |W_t - W_\tau| \right\}$$

satisfies $S_n^{(\alpha)} \stackrel{d}{=} S^{(\alpha)}$, for all $\alpha \in [0, 1]$, $n \in \mathbb{N}$.

Equation (3.2) implies that the family of functionals $(v_n^{(\alpha)})_{\alpha \in [0,1]}$ is, up to a scaling factor, coupled with the family of limit experiments $(S_n^{(\alpha)})_{\alpha \in [0,1]}$, whose distribution does not depend on n and is identical to $S^{(\alpha)}$ that we discussed in the previous section. The spot volatility σ_τ arises as the scale parameter of the limit experiment due to the linear design of the functional. This naturally leads to a class of scale-equivariant estimators of σ_τ^p for some $p > 0$ defined as

$\widehat{\sigma}^p(\alpha) := C_p(\alpha)(v_n^{(\alpha)})^p$, where $C_p(\alpha)$ is some positive function of α. By the continuous mapping theorem and the asymptotic equivalence lemma, Theorem 3.1 implies that

$$\widehat{\sigma}^p(\alpha)/\sigma_\tau^p - C_p(\alpha)(S_n^{(\alpha)})^p \xrightarrow{p} 0 \Rightarrow \widehat{\sigma}^p(\alpha)/\sigma_\tau^p \xrightarrow{d} C_p(\alpha)(S^{(\alpha)})^p \quad \text{for all } \alpha \in [0, 1],$$

where the limiting variable is a pivotal quantity that is invariant to the unknown (possibly stochastic) parameter σ_τ. This allows us to construct asymptotic confidence intervals for σ_τ^p. In detail, for a confidence level ρ, a valid confidence interval for σ_τ^p can be constructed by choosing $U_\rho > L_\rho > 0$ that satisfies the coverage constraint

$$\lim_{\Delta_n \to 0} \mathbb{P}(U_\rho^{-1} < \widehat{\sigma}^p(\alpha)/\sigma_\tau^p < L_\rho^{-1}) = \rho \Leftrightarrow \mathbb{P}((C_p(\alpha)L_\rho)^{2/p} < T^{(\alpha)} < (C_p(\alpha)U_\rho)^{2/p}) = \rho,$$

which is based on the distribution of $T^{(\alpha)}$ that can be evaluated from the Laplace transforms in Theorem 2.1. The constants U_ρ and L_ρ can be uniquely determined by minimizing a volume measure of the interval, such as the interval length or the ratio of end-points.

More importantly, we can directly assess the precision of the class of estimators $\widehat{\sigma}^p(\alpha)$ through the properties of its scale-invariant limiting distribution, which leads to optimal choices of $C_p(\alpha)$ and α. Following the classic statistical literature [23, 24, 31] and the procedure in [3], we evaluate the precision of the estimator $\widehat{\sigma}^p(\alpha)$ by its scale-invariant asymptotic mean squared error (AMSE), defined as

$$\text{AMSE}(\widehat{\sigma}^p(\alpha)) := \mathbb{E}\big[(C_p(\alpha)(S^{(\alpha)})^p - 1)^2\big] = C_p(\alpha)^2 \mu_{2p}^{(\alpha)} - 2C_p(\alpha)\mu_p^{(\alpha)} + 1. \tag{3.3}$$

Two choices of $C_p(\alpha)$ are common in this context, which will be our focus here: (1) by imposing an asymptotic unbiasedness (UB) condition, we obtain the estimator $\widehat{\sigma}_{UB}^p(\alpha) := (v_n^{(\alpha)})^p/\mu_p^{(\alpha)}$; (2) by differentiating (3.3) with respect to $C_p(\alpha)$ and minimizing the AMSE, we arrive at $\widehat{\sigma}_{MSE}^p(\alpha) := (v_n^{(\alpha)})^p \mu_p^{(\alpha)}/\mu_{2p}^{(\alpha)}$. The AMSEs of the two estimators have the following simplified expression:

$$\text{AMSE}(\widehat{\sigma}_{UB}^p(\alpha)) = \frac{\mu_{2p}^{(\alpha)}}{(\mu_p^{(\alpha)})^2} - 1, \quad \text{AMSE}(\widehat{\sigma}_{MSE}^p(\alpha)) = 1 - \frac{(\mu_p^{(\alpha)})^2}{\mu_{2p}^{(\alpha)}}, \tag{3.4}$$

which can be directly computed using the results in Corollary 2.2 for every α. It is natural to ask whether we can further minimize the above AMSEs over $\alpha \in [0, 1]$. Conveniently, it is easy to see that $\text{AMSE}(\widehat{\sigma}_{UB}^p(\alpha))$ and $\text{AMSE}(\widehat{\sigma}_{MSE}^p(\alpha))$ share the same first-order optimality condition with respect to α, so the solution to the first-order optimality condition α_p^* will minimize or maximize both AMSEs. We verify graphically in Figure 3 that α_p^* is indeed a global minimizer on $\alpha \in [0, 1]$ for $p \in \{1, 2\}$.

Figure 3 reveals substantial AMSE reductions when one optimally mixes the price range and the price reversal through $v_n^{(\alpha)}$, in comparison to the estimators that only depend on the price range. To quantify the precision gain and compare with the state-of-the-art benchmark estimators, we present the numerical values of the AMSEs for different choices of α in Table 1, along with the AMSEs of the optimal candlestick-based spot volatility estimators of [3], which we shall abbreviate as the BLL estimator.

Table 1 confirms that, compared to the estimators using only price ranges ($\alpha = 0$), the optimal choices of α_p^* are approximately half the AMSEs. This vast AMSE reduction is consistent with our intuition that the reversal part of the price path contains substantial information about
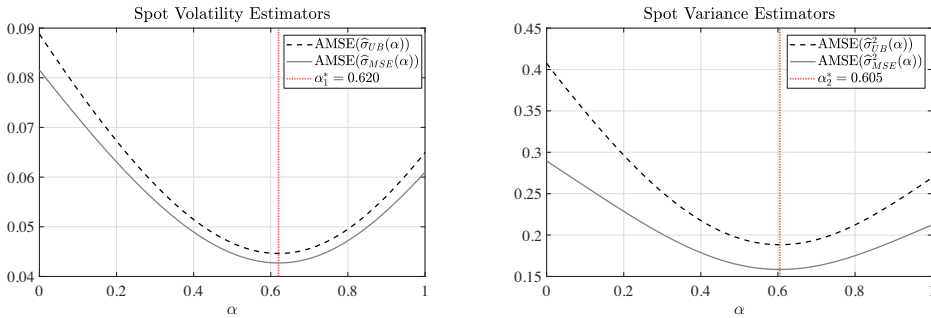
FIGURE 3. AMSEs of $\widehat{\sigma}_{UB}^{p}(\alpha)$ and $\widehat{\sigma}_{MSE}^{p}(\alpha)$ for $p \in \{1, 2\}$ as a function of α. The optimal choices α_{p}^{*} are solved numerically by minimizing (3.4) via MATHEMATICA®.

volatility that is not captured by the price range. For the BLL estimators, their AMSEs are only slightly better than our estimators using solely the maximal price reversal ($\alpha = 1$), but still more than 20% higher than our optimal ones. This suggests that $v_{n}^{(1)}$ alone conveys roughly the same amount of information about the spot volatility as the candlestick data observed in the same interval. From our previous discussion, we can view the BLL estimators as optimal combinations of the price range $w_{\tau+\Delta_{n}}$ and the price reversal $w_{\tau+\Delta_{n}} - |r_{\tau+\Delta_{n}}|$ measured at the end-point of I_{n}. This is enhanced by $v_{n}^{(\alpha)}$, which explores optimal combination for the entire price trajectory on I_{n} through the outer supremum operator.

To reflect on the results in Figure 3 and Table 1, we argue that, in a thought experiment where retail investors can choose to acquire either $v_{n}^{(\alpha)}$ or the candlestick data to learn about the spot volatility of the asset, $v_{n}^{(\alpha)}$ would be the preferred choice due to the lower AMSEs and the simpler functional form of the estimator. In practice, this precision gain may not be easily achievable by retail investors, as $v_{n}^{(\alpha)}$ requires tick-level transaction data which is costly to obtain, while the high-frequency candlestick charts, despite also being constructed from tick-level data, are widely available from trading platforms and apps by convention. However, it is important to point out that the candlestick charts, which have been a prevailing technical analysis tool over the world for centuries, can still be easily improved by the relatively simple and analytically tractable functional $v_{n}^{(\alpha)}$. Further precision gains may be possible by optimally combining the candlestick data with $v_{n}^{(\alpha)}$. Unfortunately this requires knowledge about the joint law $(S^{(\alpha)}, W_{1}, M_{1}, m_{1})$, which seems out of reach by the methods developed in this paper.

We conclude this paper on an intriguing information design problem about high-frequency data: on a given interval Δ_{n}, which functional is the most efficient at summarizing the information about σ_{τ} from the potentially noisy price trajectory? The answer to this question allows data vendors to provide more effective summary statistics for retail investors to monitor the market movements and risks. We are unable to provide a definite answer here, but we hope that the new functional $S^{(\alpha)}$ presents a promising path towards this ultimate goal.

4. Proofs

Proof of Theorem 2.1. We start with the observation that, for all $\alpha \in [0, 1]$, $s_{t}^{(\alpha)} \leq R_{t}$, for all t. Therefore R_{t} must hit 1 before $s_{t}^{(\alpha)}$, which allows us to decompose $T_{1}(s^{(\alpha)})$ as

$$T_{1}(s^{(\alpha)}) = T_{1}(R) + \tau^{(\alpha)}, \tag{4.1}$$

TABLE 1. AMSEs of spot volatility and spot variance estimators

Inference target	Asymptotically unbiased				Optimal AMSE			
	$\alpha = 0$	$\alpha = \alpha_p^*$	$\alpha = 1$	BLL	$\alpha = 0$	$\alpha = \alpha_p^*$	$\alpha = 1$	BLL
Spot volatility ($p = 1$)	0.089	0.045	0.065	0.061	0.082	0.043	0.061	0.058
Spot variance ($p = 2$)	0.407	0.188	0.269	0.259	0.289	0.158	0.212	0.207

Note. For each $p \in \{0, 1\}$ and $\alpha \in \{0, \alpha_p^*, 1\}$ the table reports the AMSEs of the estimators $\widehat{\sigma}_{UB}^p(\alpha)$ (Asymptotically unbiased) and $\widehat{\sigma}_{MSE}^p(\alpha)$ (Optimal AMSE), where the values of α_p^* are reported in Figure 3. The columns headed BLL report the AMSEs of the optimal candlestick-based spot volatility estimators of [3], where the asymptotically unbiased and the optimal AMSE estimators correspond to their optimal estimators under Stein's loss and the quadratic loss, respectively. The AMSEs of these estimators are taken from Tables 1 and 2 of [3].

where $\tau^{(\alpha)} \geq 0$ is another Brownian stopping time which is identically zero if $\alpha = 0$ (since by definition $T_1(s^{(0)}) \equiv T_1(R)$), and its properties for $\alpha \in (0, 1]$ will be elaborated in what follows. Let $W_{T_1(R)}$ denote the value of W at the time $T_1(R)$. We shall use the following Laplace transform of the pair $(T_1(R), W_{T_1(R)})$, which can be found in [5, page 242]:

$$\mathbb{E}\big[e^{-\lambda T_1(R)}; W_{T_1(R)} \in dw\big] = \frac{\sqrt{2\lambda} \sinh\left[\sqrt{2\lambda}|w|\right]}{\sinh\left[\sqrt{2\lambda}\right]^2} \, dw, \quad |w| \leq 1.$$

Furthermore, the Markov property of the Brownian motion implies that $\tau^{(\alpha)}$ is independent of $T_1(R)$ conditional on $W_{T_1(R)}$. Therefore the key to deriving the density of $T_1(s^{(\alpha)})$ is to study the density of $\tau^{(\alpha)}$ conditional on $W_{T_1(R)}$, and without loss of generality we may take $W_{T_1(R)} \geq 0$ by the symmetry of the Brownian motion.

To study the conditional density of $\tau^{(\alpha)}$, let us consider the path of W_t on $t \in [T_1(R), T_1(s^{(\alpha)})]$. Define the re-centred Brownian motion $\tilde{W}_h := W_{T_1(R)+h} - W_{T_1(R)}$ on $h \in [0, \tau^{(\alpha)}]$ with the time $T_1(R)$ normalized to 0, i.e. $h = t - T_1(R)$. Since $T_1(s^{(\alpha)}) \leq \inf\{t \geq T_1(R): W_t \leq 0\}$ or equivalently $\tau^{(\alpha)} \leq \inf\{h \geq 0: \tilde{W}_h \leq -W_{T_1(R)}\}$, it is not hard to see that $\tilde{W}_0 = 0$ and $\tilde{W}_h \geq -W_{T_1(R)}$ on $h \in [0, \tau^{(\alpha)}]$. Denoting $\tilde{M}_h := \sup_{s \in [0,h]} \tilde{W}_s$, one can show that, for all $t \in [T_1(R), T_1(s^{(\alpha)})]$ or $h \in [0, \tau^{(\alpha)}]$,

$$s_t^{(\alpha)} = R_t - \alpha|W_t| = 1 + \tilde{M}_{t-T_1(R)} - \alpha(\tilde{W}_{t-T_1(R)} + W_{T_1(R)}) = 1 + \tilde{M}_h - \alpha(\tilde{W}_h + W_{T_1(R)}).$$

Therefore, with $W_{T_1(R)} \geq 0$, we see that

$$\tau^{(\alpha)} = \inf\{h \geq 0: \tilde{M}_h - \alpha\tilde{W}_h \geq \alpha W_{T_1(R)}\}.$$

Thus $\tau^{(\alpha)}$ is the first hitting time of the non-negative process $\tilde{M}_h - \alpha\tilde{W}_h$ to the level $\alpha W_{T_1(R)}$.

To derive the Laplace transform of $\tau^{(\alpha)}$, we start with a well-known identity due to Lévy (see e.g. [22, eq. (6.34)]), which holds almost surely:

$$(\tilde{M}_h - \tilde{W}_h, \tilde{M}_h) = (|B_h|, L_h) \quad \text{for all } h > 0,$$

where B_h is a standard Brownian motion starting at zero and L_h is its associated Brownian local time process at level 0. This implies that almost surely we have

$$\tilde{M}_h - \alpha \tilde{W}_h = \alpha |B_h| + (1-\alpha)L_h \quad \text{for all } h > 0,$$

which further leads to the observation that

$$\tau^{(\alpha)} | W_{T_1(R)} \overset{d}{=} T_{W_{T_1(R)}}(\Sigma^{(\bar{\alpha})}),$$

where

$$\Sigma_h^{(\bar{\alpha})} := |B_h| + \bar{\alpha}^{-1} L_h, \quad \bar{\alpha} = \frac{\alpha}{1-\alpha}.$$

As $\alpha \in (0, 1] \Leftrightarrow \bar{\alpha} \in (0, \infty]$, the process $\Sigma_h^{(\bar{\alpha})}$ is well-defined. Conditional on $W_{T_1(R)}$, the Laplace transform of $T_{W_{T_1(R)}}(\Sigma^{(\bar{\alpha})})$ for $\alpha < 1$ can be found in [10, page 235] based on the calculations in [1] (see also [21]):

$$\mathbb{E}\big[e^{-\lambda \tau^{(\alpha)}} \mid W_{T_1(R)} = w\big] = \mathbb{E}\big[e^{-\lambda T_{W_{T_1(R)}}(\Sigma^{(\bar{\alpha})})} \mid W_{T_1(R)} = w\big]$$

$$= \frac{\bar{\alpha}\sqrt{2\lambda}}{\sinh[w\sqrt{2\lambda}]^{\bar{\alpha}}} \int_0^w \sinh[y\sqrt{2\lambda}]^{\bar{\alpha}-1} \, \mathrm{d}y$$

$$= \frac{\bar{\alpha}}{2} \int_0^1 \frac{t^{\bar{\alpha}/2-1}}{(1+\sinh[w\sqrt{2\lambda}]^2 t)^{1/2}} \, \mathrm{d}t$$

$$= {}_2F_1\left(\frac{1}{2}, \frac{\bar{\alpha}}{2}; 1+\frac{\bar{\alpha}}{2}; -\sinh[w\sqrt{2\lambda}]^2\right),$$

where the third equality is derived by a change of variable $t = \sinh[y\sqrt{2\lambda}]^2 / \sinh[w\sqrt{2\lambda}]^2$, and the final result follows from the integral representation [26, eq. (15.6.1)] of the Gauss hypergeometric function. Now, by the conditional independence of $\tau^{(\alpha)}$ and $T_1(R)$, we arrive at the Laplace transform of $T_1(s^{(\alpha)})$ by integrating out $W_{T_1(R)}$:

$$\mathcal{L}_\alpha(\lambda) = \int_{-1}^1 \mathbb{E}\big[e^{-\lambda T_1(R)}; W_{T_1(R)} \in \mathrm{d}w\big] \mathbb{E}\big[e^{-\lambda \tau^{(\alpha)}} \mid W_{T_1(R)} = w\big]$$

$$= \int_0^1 \frac{2\sqrt{2\lambda}\sinh[w\sqrt{2\lambda}]}{\sinh[\sqrt{2\lambda}]^2} {}_2F_1\left(\frac{1}{2}, \frac{\bar{\alpha}}{2}; 1+\frac{\bar{\alpha}}{2}; -\sinh[w\sqrt{2\lambda}]^2\right) \mathrm{d}w$$

$$= \int_0^1 \frac{2\sqrt{2\lambda}\tanh[w\sqrt{2\lambda}]}{\sinh[\sqrt{2\lambda}]^2} {}_2F_1\left(\frac{1}{2}, 1; 1+\frac{\bar{\alpha}}{2}; \tanh[w\sqrt{2\lambda}]^2\right) \mathrm{d}w$$

$$= \sinh[\sqrt{2\lambda}]^2 \int_0^{\tanh[\sqrt{2\lambda}]^2} (1-y)^{-1} {}_2F_1\left(\frac{1}{2}, 1; 1+\frac{\bar{\alpha}}{2}; y\right) \mathrm{d}y, \qquad (4.2)$$

where a factor of 2 is added to account for the case $W_{T_1(R)} \in [-1, 0]$, the second equality can be derived from ${}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1(a, c-b; c; z/(1-z))$ in [26, eq. (15.8.1)], and the last equality follows from the change of variable $y = \tanh[w\sqrt{2\lambda}]^2$. This proves (2.1).

We now verify (2.2) and show that $\mathcal{L}_1(\lambda)$ can be obtained from $\mathcal{L}_\alpha(\lambda)$ by letting $\alpha \to 1$. We shall start with a direct derivation of $\mathcal{L}_1(\lambda)$. Notice that when $\alpha = 1$, we have $\tau^{(1)} | W_{T_1(R)} \overset{d}{=}$

$T_{W_{T_1(R)}}(|B|)$. In this case, the conditional Laplace transform of $T_{W_{T_1(R)}}(|B|)$ takes a simple form [5, page 355]:

$$\mathbb{E}\big[e^{-\lambda \tau^{(1)}} \mid W_{T_1(R)} = w\big] = \cosh\,[|w|\sqrt{2\lambda}]^{-1}.$$

Plugging the above into (4.2) and simplifying the integral yields the same Laplace transform as in (2.2). To show that $\mathcal{L}_1(\lambda) \equiv \lim_{\alpha \to 1} \mathcal{L}_\alpha(\lambda)$, we note that

$$_2F_1\left(\frac{1}{2}, 1; 1 + \frac{\bar{\alpha}}{2}; y\right)$$

is monotonically increasing on [0, 1] and satisfies, for all real $\bar{\alpha} > 1$,

$$1 = {}_2F_1\left(\frac{1}{2}, 1; 1 + \frac{\bar{\alpha}}{2}; 0\right) < {}_2F_1\left(\frac{1}{2}, 1; 1 + \frac{\bar{\alpha}}{2}; y\right) < {}_2F_1\left(\frac{1}{2}, 1; 1 + \frac{\bar{\alpha}}{2}; 1\right) = \frac{\bar{\alpha}}{\bar{\alpha} - 1},$$

by the Gauss summation theorem. Since $\alpha \to 1 \Leftrightarrow \bar{\alpha} \to \infty$, we deduce that

$$\lim_{\alpha \to 1} {}_2F_1\left(\frac{1}{2}, 1; 1 + \frac{\bar{\alpha}}{2}; y\right) = 1$$

uniformly on $y \in [0, 1]$. Thus one can interchange the limit with the integration in the last line of (4.2) to derive

$$\lim_{\alpha \to 1} \mathcal{L}_\alpha(\lambda) = \sinh\,[\sqrt{2\lambda}]^{-2} \int_0^{\tanh\,[\sqrt{2\lambda}]^2} (1 - y)^{-1}\,\mathrm{d}y = 2\sinh\,[\sqrt{2\lambda}]^{-2} \ln \cosh\,[\sqrt{2\lambda}] \equiv \mathcal{L}_1(\lambda),$$

as desired. Finally, to verify (2.3), note that by (4.1) we have $T^{(\alpha)} \geq T_1(R)$ pathwise for all $\alpha \in [0, 1]$, which implies that for all $\lambda \geq 0$ and $\alpha \in [0, 1]$, $e^{-\lambda T^{(\alpha)}} \leq e^{-\lambda T_1(R)}$ and hence $\mathcal{L}_\alpha(\lambda) \leq \mathcal{L}_0(\lambda)$ by taking expectation. The analytical expression of $\mathcal{L}_0(\lambda)$ is well-known (see e.g. [6], [18]), from which one immediately concludes that $\mathcal{L}_0(\lambda) \leq \mathcal{L}_0(0) = 1$ for all $\lambda \geq 0$. This completes the proof. □

Proof of Proposition 2.1. We begin with the observation that $T^{(\alpha)} \leq T^{(1)}$ pathwise, and thus $\kappa_k^{(\alpha)} \leq \kappa_k^{(1)}$ for all $k \in \mathbb{N}$. Therefore all moments of $T^{(\alpha)}$ exist if $\kappa_k^{(1)} < \infty$ for all $k \in \mathbb{N}$. As $\mathcal{L}_1(\lambda) < \infty$ for all $\lambda \geq 0$ from Theorem 2.1, by differentiating under the integral sign, the dominated convergence theorem implies that

$$\kappa_k^{(1)} = \lim_{\lambda \to 0} \mathbb{E}[(T^{(1)})^k e^{-\lambda T^{(1)}}] = \lim_{\lambda \to 0} (-1)^k \mathcal{L}_1^{(k)}(\lambda), \tag{4.3}$$

where $\mathcal{L}_\alpha^{(k)}(\lambda)$ is the kth derivative of $\mathcal{L}_\alpha(\lambda)$ with respect to λ. Therefore it suffices to show that

$$\lim_{\lambda \to 0} (-1)^k \mathcal{L}_1^{(k)}(\lambda) = (-1)^k \mathcal{L}_1^{(k)}(0) < \infty,$$

and the above quantity is guaranteed to be non-negative by the Bernstein–Widder theorem [32, Theorem 12a]. To prove this, we shall write

$$\mathcal{L}_1(\lambda) = \mathcal{L}_{1/2}(\lambda)\mathcal{G}_1(\lambda), \tag{4.4}$$

where $\mathcal{L}_{1/2}(\lambda)$ is given in (2.4), and $\mathcal{G}_1(\lambda) = \lambda^{-1} \ln \cosh\,[\sqrt{2\lambda}]$. We thus have, by the general Leibniz rule,

$$\lim_{\lambda \to 0} (-1)^k \mathcal{L}_1^{(k)}(\lambda) = \sum_{n=0}^{k} \binom{k}{n} \kappa_n^{(1/2)} (-1)^{k-n} \lim_{\lambda \to 0} \mathcal{G}_1^{(k-n)}(\lambda),$$

where $\mathcal{L}_{1/2}^{(n)}(0) = \lim_{\lambda \to 0} \mathcal{L}_{1/2}^{(n)}(\lambda) = (-1)^{-n}\kappa_n^{(1/2)}$ holds since $\mathcal{L}_{1/2}(\lambda)$ is the Laplace transform of $T^{(1/2)}$ with all moments finite (see equation (2.5) and [2]). Therefore the above quantity is finite for all $k \in \mathbb{N}$ if

$$(-1)^k \lim_{\lambda \to 0} \mathcal{G}_1^{(k)}(\lambda) = (-1)^k \mathcal{G}_1^{(k)}(0) < \infty.$$

The Maclaurin series of $\mathcal{G}_1(\lambda)$ is known [34]:

$$\mathcal{G}_1(\lambda) = \sum_{k=0}^{\infty} \frac{2^{3k+2}(4^{k+1} - 1)B_{2(k+1)}}{(k+1)(2(k+1))!}\lambda^k = \sum_{k=0}^{\infty} \frac{(-1)^{-k}}{k!}\gamma_k^{(1)}\lambda^k, \qquad (4.5)$$

where $\gamma_k^{(1)}$ is defined in (2.10). From the above, we immediately see that $(-1)^k \mathcal{G}_1^{(k)}(0) = \gamma_k^{(1)} < \infty$, and in fact also $\gamma_k^{(1)} > 0$ as $B_n(-1)^n > 0$, for all n. We thus conclude that all moments of $T^{(1)}$, and hence $T^{(\alpha)}$ for $\alpha \in [0, 1]$, are finite.

The above result allows us to deduce $\kappa_k^{(\alpha)} = (-1)^k \mathcal{L}_1^{(k)}(0)$ in view of (4.3) and the existence of the limit. To derive an explicit formula for $\kappa_k^{(\alpha)}$, we consider the following decomposition of $\mathcal{L}_\alpha(\lambda)$ similar to (4.4):

$$\mathcal{L}_\alpha(\lambda) = \mathcal{L}_{1/2}(\lambda)\mathcal{G}_\alpha(\lambda), \qquad (4.6)$$

where $\mathcal{G}_\alpha(\lambda)$ takes the form

$$\mathcal{G}_\alpha(\lambda) = (2\lambda)^{-1} \int_0^{\tanh[\sqrt{2\lambda}]^2} (1-y)^{-1} {}_2F_1\left(\frac{1}{2}, 1; 1 + \frac{\bar{\alpha}}{2}; y\right) dy.$$

By the general Leibniz rule again, and since we know that the moments exist,

$$\kappa_k^{(\alpha)} = (-1)^k \mathcal{L}_\alpha^{(k)}(0) = (-1)^k \sum_{n=0}^{k} \binom{k}{n} \mathcal{L}_{1/2}^{(n)}(0)\mathcal{G}_\alpha^{(k-n)}(0).$$

Comparing the above to (2.8), we only need to show that $\gamma_k^{(\alpha)}$ in (2.9) satisfies $\gamma_k^{(\alpha)} \equiv (-1)^k \mathcal{G}_\alpha^{(k)}(0)$ for all $k \in \mathbb{N}$ and $\alpha \in [0, 1]$. It suffices to show that $\mathcal{G}_\alpha(\lambda)$ has the following Maclaurin series:

$$\mathcal{G}_\alpha(\lambda) = \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k^{(\alpha)}}{k!}\lambda^k. \qquad (4.7)$$

The case with $\alpha = 1$ is already proved in (4.5), and the case $\alpha = 1/2$ is trivial since $\mathcal{G}_{1/2}(\lambda) \equiv 1$, so we simply have $\gamma_k^{(1/2)} \equiv \mathbb{1}_{\{k=0\}}$. For $\alpha = 0$, we have $\mathcal{G}_0(\lambda) = \lambda^{-1}(\cosh[\sqrt{2\lambda}] - 1)$ with the following Maclaurin series:

$$\mathcal{G}_0(\lambda) = \sum_{k=0}^{\infty} \frac{2^{k+1}}{(2(k+1))!}\lambda^k = \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k^{(0)}}{k!}\lambda^k,$$

where $\gamma_k^{(0)}$ and $\gamma_k^{(1/2)}$ correspond to those provided in (2.10), as desired.

For the general case $\alpha \in (0, 1) \setminus \{\frac{1}{2}\}$, we start with the claim that

$$\tanh[\sqrt{2x}]^{2k} = x^k \sum_{n=0}^{\infty} C_{n,k}x^n, \quad k \in \mathbb{N}.$$

This easily follows by induction from the Maclaurin series of $\tanh[\sqrt{2x}]^2$ below and the recursive relation of $C_{n,k}$ by a Cauchy sum argument,

$$\tanh[\sqrt{2x}]^2 = x \sum_{n=0}^{\infty} C_{n,1} x^n,$$

which can be derived from the Maclaurin series of $\tanh(x)$ provided in [35].

We are now ready to derive the Maclaurin series of $\mathcal{G}_\alpha(\lambda)$. Using the geometric series expansion and the Maclaurin series of the Gauss hypergeometric function, we deduce

$$\mathcal{G}_\alpha(\lambda) = (2\lambda)^{-1} \int_0^{\tanh[\sqrt{2\lambda}]^2} \sum_{s=0}^{\infty} y^s {}_2F_1\left(\frac{1}{2}, 1; 1 + \frac{\bar{\alpha}}{2}; y\right) dy$$

$$= (2\lambda)^{-1} \int_0^{\tanh[\sqrt{2\lambda}]^2} \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} \frac{(1/2)_n}{(1 + \bar{\alpha}/2)_n} y^{n+s} \, dy$$

$$= (2\lambda)^{-1} \int_0^{\tanh[\sqrt{2\lambda}]^2} \sum_{k=0}^{\infty} D_k^{(\alpha)} y^k \, dy$$

$$= (2\lambda)^{-1} \sum_{k=0}^{\infty} D_k^{(\alpha)} \frac{\tanh[\sqrt{2\lambda}]^{2(k+1)}}{k+1}, \tag{4.8}$$

where the third equality follows from a change of index and the identity

$$D_k^{(\alpha)} := \frac{\bar{\alpha}}{\bar{\alpha} - 1}\left(1 - \frac{(1/2)_{k+1}}{(\bar{\alpha}/2)_{k+1}}\right) = \sum_{n=0}^{k} \frac{(1/2)_n}{(1 + \bar{\alpha}/2)_n},$$

which can be easily proved via induction, and is well-defined for all $\bar{\alpha} \in (0, \infty) \setminus \{1\}$ (the limiting cases with $\bar{\alpha} \in \{0, 1, \infty\}$ given in (2.7) can be easily verified). The interchange of sum and limit in the last equality of (4.8) is justified by the fact that all the summands are positive. Plugging in the Maclaurin series of $\tanh[\sqrt{2x}]^{2k}$ and reordering, we arrive at the Maclaurin series of $\mathcal{G}_\alpha(\lambda)$:

$$\mathcal{G}_\alpha(\lambda) = \sum_{k=0}^{\infty} D_k^{(\alpha)} \sum_{n=0}^{\infty} \frac{C_{n,k+1} \lambda^{k+n}}{2(k+1)} = \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{D_n^{(\alpha)} C_{k-n,n+1}}{2(n+1)} \lambda^k.$$

It is now clear that by setting

$$\gamma_k^{(\alpha)} = (-1)^k k! \sum_{n=0}^{k} \frac{D_n^{(\alpha)} C_{k-n,n+1}}{2(n+1)},$$

we recover (4.7) as desired. This completes the proof.   □

Proof of Corollary 2.1. In view of Proposition 2.1 and equation (2.5), the corollary follows from the observation

$$\gamma_n^{(1)} = (-1)^n n! \frac{2^{3n+2}(4^{n+1} - 1)B_{2(n+1)}}{(n+1)(2(n+1))!} = -\frac{(-2)^{2n+1} \kappa_n^{(0)}}{(2n+1)(2n+2)},$$

and the final expression in (2.11) can be obtained by

$$\kappa_k^{(1)} = \sum_{n=0}^{k} \binom{k}{n} \kappa_{k-n}^{(1/2)} \gamma_n^{(1)},$$

which is equivalent to (2.8). This completes the proof. $\qquad\square$

Proof of Proposition 2.2. We start with the following infinite series representation of $\mathcal{L}_\alpha(\lambda)$:

$$\mathcal{L}_\alpha(\lambda) = \sum_{n=0}^{\infty} \frac{D_n^{(\alpha)}}{n+1} \tanh[\sqrt{2\lambda}]^{2n}(1 - \tanh[\sqrt{2\lambda}]^2), \tag{4.9}$$

which follows from (2.4), (4.6), and (4.8) after some straightforward simplification. Since $D_n^{(\alpha)}$ is defined for all $\alpha \in [0, 1]$ by (2.6) and (2.7), there are no ambiguities in the above definition for $\alpha \in \{0, \frac{1}{2}, 1\}$. By Theorem 2.1, (4.9) is convergent on $\lambda \geq 0$ for all $\alpha \in [0, 1]$. We use (2.13) to deduce that

$$\mu_k^{(\alpha)} = \frac{1}{\Gamma(k/2)} \sum_{n=0}^{\infty} \frac{D_n^{(\alpha)}}{n+1} \int_0^\infty \tanh[\sqrt{2\lambda}]^{2n}(1 - \tanh[\sqrt{2\lambda}]^2)\lambda^{k/2-1}\,\mathrm{d}\lambda,$$

$$= \frac{2^{1-k/2}}{\Gamma(k/2)} \sum_{n=0}^{\infty} \frac{D_n^{(\alpha)}}{n+1} \int_0^1 \mathrm{arctanh}[y]^{k-1}y^{2n}\,\mathrm{d}y, \tag{4.10}$$

where the exchange of sum and integration is permitted in the first line due to the fact that all integrands are positive, and the second line follows by change of variables $x = \tanh[\sqrt{2\lambda}]^2$ and then $y = \sqrt{x}$. This completes the proof. $\qquad\square$

Proof of Corollary 2.2. When $\alpha = 1$, from (2.7) we learn that $D_n^{(1)} \equiv 1$. Plugging this into (4.10) and exchanging the order of integration with the infinite sum, (2.14) follows from the identity

$$\sum_{n=0}^{\infty} \frac{y^{2n}}{n+1} = -y^{-2}\ln(1 - y^2).$$

Now, for (2.15), we note that the integral in (2.13) can be solved in closed form,

$$\int_0^1 y^{2n}\,\mathrm{d}y = \frac{1}{2n+1}, \quad \int_0^1 \mathrm{arctanh}[y]y^{2n}\,\mathrm{d}y = \frac{H_n + 2\ln 2}{2(2n+1)},$$

and recall that H_n is the nth harmonic number. The expressions in (2.15) are derived by substituting the above into (4.10) and simplifying the infinite sums using MATHEMATICA®. This completes the proof. $\qquad\square$

Proof of Theorem 3.1. By a standard localization procedure (see [19, Section 4.4.1]), we can without loss of generality work under a strengthened version of Assumption 3.1 by assuming that the bounds in Assumption 3.1(i) hold with $T_1 = \infty$. Also, we can assume that P_t is continuous on I_n, following the discussion in the proof of Theorem 1 of [25].

To prove (3.2), we start with the following normalized statistics defined for $t \in I_n$:

$$w_{n,t} := \Delta_n^{-1/2} w_t, \quad r_{n,t} := \Delta_n^{-1/2} r_t,$$

$$R_{n,t} := \Delta_n^{-1/2} \left(\sup_{s \in [\tau,t]} W_s - \inf_{s \in [\tau,t]} W_s \right), \quad W_{n,t} := \Delta_n^{-1/2} (W_t - W_\tau).$$

Thus it is understood that $v_n^{(\alpha)} \equiv \sup_{t \in I_n} \{w_{n,t} - \alpha |r_{n,t}|\}$ and similarly for $S_n^{(\alpha)} \equiv \sup_{t \in I_n} \{R_{n,t} - \alpha |W_{n,t}|\}$. Intuitively, $(R_{n,t}, W_{n,t})$ is the counterpart of $(w_{n,t}, r_{n,t})$ constructed from W instead of P. We have the following observation:

$$\sup_{\alpha \in [0,1]} |v_n^{(\alpha)} - \sigma_\tau S_n^{(\alpha)}| \le \sup_{\alpha \in [0,1]} \left| \sup_{t \in I_n} \{w_{n,t} - \alpha |r_{n,t}|\} - \sigma_\tau \sup_{t \in I_n} \{R_{n,t} - \alpha |W_{n,t}|\} \right|$$

$$\le \sup_{\alpha \in [0,1]} \sup_{t \in I_n} \left| w_{n,t} - \alpha |r_{n,t}| - \sigma_\tau R_{n,t} + \sigma_\tau \alpha |W_{n,t}| \right|$$

$$\le \sup_{\alpha \in [0,1]} \left\{ \sup_{t \in I_n} |w_{n,t} - \sigma_\tau R_{n,t}| + \alpha \sup_{t \in I_n} \left| |r_{n,t}| - \sigma_\tau |W_{n,t}| \right| \right\}$$

$$\le \sup_{t \in I_n} |w_{n,t} - \sigma_\tau R_{n,t}| + \sup_{t \in I_n} \left| |r_{n,t}| - \sigma_\tau |W_{n,t}| \right|. \tag{4.11}$$

Thus it suffices to prove that both terms in the last line above are of order $o_p(1)$.

Recall that $J \equiv 0$ on I_n. For the first term,

$$\sup_{t \in I_n} |w_{n,t} - \sigma_\tau R_{n,t}| = \Delta_n^{-1/2} \sup_{t \in I_n} \left| \sup_{s \in [\tau,t]} P_s - \sigma_\tau \sup_{s \in [\tau,t]} W_s - \inf_{s \in [\tau,t]} P_s + \sigma_\tau \inf_{s \in [\tau,t]} W_s \right|$$

$$\le \Delta_n^{-1/2} \sup_{t \in I_n} \left| 2 \sup_{s \in [\tau,t]} |P_s - P_\tau - \sigma_\tau (W_s - W_\tau)| \right|$$

$$\le 2 \Delta_n^{-1/2} \sup_{t \in I_n} |P_t - P_\tau - \sigma_\tau (W_t - W_\tau)|$$

$$= 2 \Delta_n^{-1/2} \sup_{t \in I_n} \left| \int_\tau^t \mu_s \, ds + \int_\tau^t (\sigma_s - \sigma_\tau) \, dW_s \right|$$

$$\le 2 \Delta_n^{-1/2} \int_\tau^{\tau + \Delta_n} |\mu_s| \, ds + 2 \Delta_n^{-1/2} \sup_{t \in I_n} \left| \int_\tau^t (\sigma_s - \sigma_\tau) \, dW_s \right|$$

$$= O_p(\Delta_n^{1/2}) + O_p(\Delta_n^\kappa) = o_p(1),$$

where the first inequality is due to a reverse triangle inequality applied to the supremum norm, and note that the infimum can be interpreted as the supremum norm applied to the negative process. The last two estimates follow from Assumption 3.1 which are proved in [25]. Similarly, for the second term in (4.11), we have

$$\sup_{t \in I_n} \left| |r_{n,t}| - \sigma_\tau |W_{n,t}| \right| = \Delta_n^{-1/2} \sup_{t \in I_n} \left| |P_t - P_\tau| - \sigma_\tau |W_t - W_\tau| \right|$$

$$\le \Delta_n^{-1/2} \sup_{t \in I_n} \left| P_t - P_\tau - \sigma_\tau (W_t - W_\tau)| \right|$$

$$= O_p(\Delta_n^{1/2}) + O_p(\Delta_n^\kappa) = o_p(1),$$

as desired. Finally, by the Brownian scaling law, it is not hard to see that $(W_{n,t})_{t \in I_n} \overset{d}{=} (W_t)_{t \in [0,1]}$ for arbitrary choice of $\Delta_n > 0$. Since $[\tau, \tau + \Delta_n] \subset [0, T]$ for all n by assumption, we have $S_n^{(\alpha)} \overset{d}{=} S^{(\alpha)}$ for all $\alpha \in [0, 1]$ and n, and the proof is complete. $\square$

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Competing interests

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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