

WEAK CAYLEY TABLE GROUPS OF SOME CRYSTALLOGRAPHIC GROUPS

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Abstract. For a group G , a *weak Cayley table isomorphism* is a bijection $f : G \rightarrow G$ such that $f(g_1g_2)$ is conjugate to $f(g_1)f(g_2)$ for all $g_1, g_2 \in G$. The set of all weak Cayley table isomorphisms forms a group $\mathcal{W}(G)$ that is the group of symmetries of the weak Cayley table of G . We determine $\mathcal{W}(G)$ for each of the 17 wallpaper groups G , and for some other crystallographic groups.

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1. Introduction. For a group G , the *weak Cayley table group* $\mathcal{W}(G)$ is the set of bijections $f : G \rightarrow G$ such that $f(g_1g_2) \sim f(g_1)f(g_2)$ for all $g_1, g_2 \in G$. Here, \sim denotes conjugacy in G . An element of $\mathcal{W}(G)$ is called a *weak Cayley table isomorphism* (of G).

It is easy to see that $\mathcal{W}(G)$ is a group that contains $\text{Aut}(G)$ and the inverse map $\iota = \iota_G : G \rightarrow G : \iota(g) = g^{-1}$. We let $\mathcal{W}_0(G) = \langle \text{Aut}(G), \iota \rangle$, the subgroup of *trivial* weak Cayley table isomorphisms. We note that $\mathcal{W}_0(G) = \text{Aut}(G) \times \langle \iota_G \rangle$.

If $G = \{g_1 = 1, g_2, \dots\}$, then the group $\mathcal{W}(G)$ acts naturally on the *weak Cayley table*, this being the $|G| \times |G|$ matrix whose ij entry is the conjugacy class of $g_i g_j$. It is well-known that two finite groups have the same weak Cayley table if and only if they have the same 1- and 2-characters, in the sense of Frobenius [10]. Here, for a character χ , the corresponding 1-character is $\chi^{(1)} = \chi$, and the 2-character is $\chi^{(2)} : G^2 \rightarrow \mathbb{C}$, $\chi^{(2)}(x, y) = \chi(x)\chi(y) - \chi(xy)$.

We note that the notion of k -character only makes sense in the situation where G is finite; however, the weak Cayley table is defined for any countable group, and the group $\mathcal{W}(G)$ makes sense for any group.

In previous papers [5–7], we have determined classes of groups that have the property $\mathcal{W}(G) = \mathcal{W}_0(G)$; these include finite symmetric groups, some free groups and free products, some Coxeter groups, $\text{PSL}(2, p^n)$ and some sporadic groups. Groups with this property are termed *trivial*. Elements of $\mathcal{W}(G) \setminus \mathcal{W}_0(G)$ are called *non-trivial*.

In this paper, we consider the 17 wallpaper groups G , and determine the group $\mathcal{W}(G)$ explicitly; while doing so we determine which are trivial. We recall the standard notation and definitions for these crystallographic groups [8, 9]: a *wallpaper group* is a discrete group of isometries of the plane whose subgroup, A , of translations is isomorphic to \mathbb{Z}^2 . Two such are considered equivalent if they are conjugate by an affine transformation. Such a group contains only translations, rotations, reflections, and glide reflections as symmetries. We use the Hermann–Mauguin notation for wallpaper groups, also called the IUC notation. We note that the automorphism group $\text{Aut}(G)$ of each wallpaper group G has been determined in [4], however we do not make use

of their result in this paper, and so our classification will give an independent proof of their result; in fact in each case we determine generators for $\mathcal{W}_0(G) = \text{Aut}(G) \times \langle t \rangle$, and note that $\text{Aut}(G)$ can then be easily determined.

The trivial cases are (see Section 2 for notation) as follows:

**p1, p2, c1m1, p1m1, p1g1, c2mm, p2mg, p2gg, p3m1, p31m,
p4mg, p4mm, p6mm.**

For the non-trivial cases, we have the following theorem.

THEOREM 1.1. *Let G be one of the 17 wallpaper groups. Then, we have $\mathcal{W}(G) \neq \mathcal{W}_0(G)$ if and only if either*

- (i) G is a direct product of non-abelian groups (i.e., G is of type **p2mm**, which is isomorphic to $D_\infty \times D_\infty$), or
- (ii) G contains only translations and rotations, with some rotation of order greater than 2 (i.e., G is of type **p3, p4, p6**).

If G has rotations of order at least 4, then there is a subgroup $\mathcal{N}(G) \triangleleft \mathcal{W}(G)$, all of whose non-identity elements are non-trivial, such that

$$\mathcal{W}(G) = \mathcal{N}(G) \rtimes \mathcal{W}_0(G).$$

*(This latter case consists of groups of types **p4, p6**.)*

More detailed descriptions of the groups $\mathcal{W}(G)$ will be given later.

Let \mathcal{C}_m denote the cyclic group of order m . Many crystallographic groups in higher dimensions have the form $\mathbb{Z}^n \rtimes \mathcal{C}_m$; for groups of this type, we have the following theorems.

THEOREM 1.2. *Any semi-direct product $\mathbb{Z}^n \rtimes \mathcal{C}_2$, $n \geq 0$, has trivial weak Cayley table group.*

THEOREM 1.3. *Let p be an odd prime and let $G = \mathbb{Z}^n \rtimes \mathcal{C}_p$, $n \geq 1$, be a non-abelian semi-direct product. Then, G has a non-trivial weak Cayley table isomorphism.*

In Section 2, we recall some facts about elements of $\mathcal{W}(G)$, describe properties of wallpaper groups, and explain the strategy for finding $\mathcal{W}(G)$ for each wallpaper group G . In Sections 3–8, we prove the various steps outlined in the strategy. In Section 9, we prove Theorems 1.2 and 1.3.

2. Weak Cayley table maps and wallpaper groups. We collect together some results on weak Cayley table isomorphisms taken from [10]. For groups G, H a *weak Cayley table map* is a bijection $\varphi : G \rightarrow H$ such that $f(g_1g_2) \sim f(g_1)f(g_2)$, for all $g_1, g_2 \in G$, where \sim denotes conjugacy in H .

LEMMA 2.1 [10]. *Let $\varphi : G \rightarrow G$ be a weak Cayley table isomorphism and let $g \in G$, $N \triangleleft G$. Then,*

- (1) $\varphi(e) = e$ and $\varphi(g^{-1}) = \varphi(g)^{-1}$, where e denotes the identity element of G . Further, if $g^2 = e$, then $\varphi(g)^2 = e$.
- (2) $\varphi(N) \triangleleft G$.
- (3) $\varphi(gN) = \varphi(g)\varphi(N)$.
- (4) φ induces a weak Cayley table map $\tilde{\varphi} : G/N \rightarrow G/\varphi(N)$.
- (5) $\varphi^{-1} : G \rightarrow G$ is a weak Cayley table isomorphism.
- (6) If $a, b \in G$, then $a \sim b$ if and only if $\varphi(a) \sim \varphi(b)$.

We have already remarked that a wallpaper group G contains a translation subgroup A . Also, $G = A$ if and only if G is abelian. If $G = A$, then certainly $\mathcal{W}(G) = \mathcal{W}_0(G)$, so we will assume that G is not abelian from now on. The following is well-known.

LEMMA 2.2. *Let G be a wallpaper group. Then,*

(i) $[G : A] < \infty$;

(ii) $g \in G$ has a finite conjugacy class if and only if $g \in A$.

Here, we introduce the 17 types of wallpaper groups and give a presentation for each group type. We also, for each of the 17 wallpaper groups, identify a set F of coset representatives for G/A :

p1 $\langle x, y | (x, y); F = \{1\}$.

p2 $\langle x, y, \rho | (x, y), x^\rho = x^{-1}, y^\rho = y^{-1}, \rho^2 \rangle; F = \langle \rho \rangle$.

p3 $\langle x, y, \rho | (x, y), x^\rho = x^{-1}y, y^\rho = x^{-1}, \rho^3 \rangle; F = \langle \rho \rangle$.

p4 $\langle x, y, \rho | (x, y), x^\rho = y, y^\rho = x^{-1}, \rho^4 \rangle; F = \langle \rho \rangle$.

p6 $\langle x, y, \rho | (x, y), x^\rho = y, y^\rho = x^{-1}y, \rho^6 \rangle; F = \langle \rho \rangle$.

c1m1 $\langle x, y, \sigma | (x, y), x^\sigma = y, y^\sigma = x, \sigma^2 \rangle; F = \langle \sigma \rangle$.

p1m1 $\langle x, y, \sigma | (x, y), x^\sigma = x, y^\sigma = y^{-1}, \sigma^2 \rangle; F = \langle \sigma \rangle$.

p1g1 $\langle x, y, \gamma | (x, y), x^\gamma = x, y^\gamma = y^{-1}, \gamma^2 = x \rangle; F = \{1, \gamma\}$.

c2mm $\langle x, y, \rho, \sigma | (x, y), \rho^2, \sigma^2, x^\rho = x^{-1}, y^\rho = y^{-1}, x^\sigma = y, y^\sigma = x, (\rho\sigma)^2 \rangle;$
 $F = \langle \rho, \sigma \rangle$.

p2mm $\langle x, y, \rho, \sigma | (x, y), \rho^2, \sigma^2, (\rho, \sigma), x^\rho = x^{-1}, y^\rho = y^{-1}, x^\sigma = x, y^\sigma = y^{-1} \rangle;$
 $F = \langle \rho, \sigma \rangle$.

p2mg $\langle x, y, \rho, \sigma | (x, y), \rho^2, \sigma^2, x^\rho = x^{-1}, y^\rho = y^{-1}, x^\sigma = x, y^\sigma = y^{-1}, (\rho\sigma)^2 = y \rangle;$
 $F = \{1, \rho, \sigma, \rho\sigma\}$.

p2gg $\langle x, y, \rho, \gamma | (x, y), \rho^2, \gamma^2 = x, x^\rho = x^{-1}, y^\rho = y^{-1}, x^\gamma = x, y^\gamma = y^{-1}, (\rho\gamma)^2 = y \rangle;$
 $F = \{1, \rho, \gamma, \rho\gamma\}$.

p3m1 $\langle x, y, \rho, \sigma | (x, y), \rho^3, \sigma^2, (\rho\sigma)^2, x^\rho = x^{-1}y, y^\rho = x^{-1}, x^\sigma = y, y^\sigma = x \rangle;$
 $F = \langle \rho, \sigma \rangle$.

p31m $\langle x, y, \rho, \sigma | (xy), \rho^3, \sigma^2, (\rho\sigma)^2, x^\rho = x^{-1}y, y^\rho = x^{-1}, x^\sigma = x, y^\sigma = xy^{-1} \rangle;$
 $F = \langle \rho, \sigma \rangle$.

p4mg $\langle x, y, \rho, \gamma | (x, y), \rho^4, \gamma^2 = x, x^\rho = y, y^\rho = x^{-1}, x^\gamma = x, y^\gamma = y^{-1}, (\rho\gamma)^2 \rangle;$
 $F = \{1, \rho, \rho^2, \rho^{-1}, \gamma, \rho\gamma, \rho^2\gamma, \rho^{-1}\gamma\}$.

p4mm $\langle x, y, \rho, \sigma | (x, y), \rho^4, \sigma^2, x^\rho = y, y^\rho = x^{-1}, x^\sigma = x, y^\sigma = y^{-1}, (\rho\sigma)^2 \rangle;$
 $F = \langle \rho, \sigma \rangle$.

p6mm $\langle x, y, \rho, \sigma | (x, y), \rho^6, \sigma^2, x^\rho = y, y^\rho = x^{-1}y, x^\sigma = x, y^\sigma = xy^{-1}, (\rho\sigma)^2 \rangle;$
 $F = \langle \rho, \sigma \rangle$.

In these presentations, translation generators are denoted by x, y , rotation generators by $\rho\sigma$, reflection generators by σ , and glide reflection generators by $\gamma\sigma$.

Fundamental to what we do is the following, which is the first step in the strategy for determining $\mathcal{W}(G)$.

LEMMA 2.3. *If $\varphi \in \mathcal{W}(G)$, then $\varphi(A) = A$. In particular, for $g \in G$ we have $\varphi(gA) = \varphi(g)A$.*

Proof. Since φ preserves the size of a conjugacy class (Lemma 2.1 (6)), the first statement follows from Lemma 2.2 (ii). The rest follows from Lemma 2.1 (3). \square

The following indicates the rest of the strategy for determining $\mathcal{W}(G)$ (we call these items the *Steps* of the proof):

- (1) Show that $\varphi|_A : A \rightarrow A$ is an automorphism.
- (2) Show that we can compose φ with an element of $\mathcal{W}_0(G)$ so that the resulting element of $\mathcal{W}(G)$ is the identity on A .
- (3) Show that we can then compose φ with an element of $\mathcal{W}_0(G)$ so that the resulting element of $\mathcal{W}(G)$ (which we still call φ) satisfies $\varphi(Ag) = Ag$ for all $g \in G$. This is the same as showing that $\tilde{\varphi} : G/A \rightarrow G/A$ is the identity.
- (4) Show that we can assume, after composing with some specified generators (not necessarily trivial elements), that the resulting φ fixes each of the elements in F .
- (5) Show that for every $t \in F$ there is some $f = f_t \in F$ such that $\varphi(at) = a^f t$ for all $a \in A$,
- (6) Show that φ can be adjusted so as to be the identity on each coset of A . We find that this step is only necessary in the non-trivial cases.

The wallpaper group G acts by isometries on the Euclidean space \mathbb{E}^2 ; we think of a wallpaper pattern corresponding to G as a subset of \mathbb{E}^2 (containing $(0, 0)$) whose set of symmetries is G . In the given presentation of G , we have generators x, y of A . Further, for any $a \in A$ there is a translation of \mathbb{E}^2 by a vector $v_a \in \mathbb{E}^2$ (say) corresponding to a . Thus, v_x, v_y span a lattice $\mathcal{L} \subset \mathbb{E}^2$. We will think of A as identified with the lattice \mathcal{L} : $a = x^i y^j \in A$ corresponds to $v_a = iv_x + jv_y \in \mathbb{E}$. The natural action of G on \mathbb{E}^2 satisfies $(v_a)g = v_{ag}$ for $a \in A, g \in G$.

Now, relative to the metric on \mathbb{E}^2 , we have (closed) balls B_r of radius $r \geq 0$ centred at $(0, 0)$. Further, each $f \in F$ determines an element of the orthogonal group $O(\mathbb{E}^2)$, where $a \mapsto a^f$, and so each conjugacy class $a^F, a \in A$, is contained in the boundary of the ball $B_{|a|}$.

For any reflection or glide reflection $r \in G$, there is a *line of reflection* $L(r) \leq A, L(r) = \{a \in A | a^r = a\}$. Given G , we let

$$H = H(G) = \{a \in A | |a^G| \neq [G : A]\}.$$

It is easy to see that H is the union of the lines of reflection for all reflections and glide reflections in G . Since $\varphi \in \mathcal{W}(G)$ respects the sizes of conjugacy classes we have the following lemma.

LEMMA 2.4. *For any $\varphi \in \mathcal{W}(G)$, we have $\varphi(H) = H$.*

Suppose that $r \in G$ is a reflection or a glide reflection. Let $L^\perp(r) = \{a \in A | a^r = a^{-1}\}$. We have the following lemma.

LEMMA 2.5. *Suppose $r \in G$ is a reflection or a glide reflection. Let $\beta = r^2 \in A$. Then, for $a \in A$ we have $(ar)^2 = \beta$ if and only if $a \in L^\perp(r)$. In particular, if $\sigma \in G$ is a reflection, then $(a\sigma)^2 = 1$ if and only if $a \in L^\perp(\sigma)$.*

Proof. We have $(ar)^2 = \beta$ if and only if $ar^2 \cdot r^{-1}ar = a\beta a^r = \beta$ if and only if $aa^r = 1$ if and only if $a \in L^\perp(r)$. \square

For G of type **p2mg** let $\beta_1 = \sigma^2 = 1$, $\beta_2 = (\rho\sigma)^2 = y$.

For G of type **p2gg** let $\beta_1 = \gamma^2 = x$, $\beta_2 = (\rho\gamma)^2 = y$.

For G of type **p4mg** let $\beta_1 = \gamma^2 = x$, $\beta_2 = (\rho^2\gamma)^2 = y$.

For G of any other type let $\beta_1 = \beta_2 = 1$.

Let ρ_θ denote rotation by θ about $(0, 0)$.

Our convention for the commutator (g, h) is $g^{-1}h^{-1}gh$. For $f \in G$, we define

$$K_f = \{(a, f) : a \in A\}.$$

Since $(ab, f) = (a, f)(b, f)$, for $a, b \in A$, it is easy to see that K_f is a subgroup of A .

LEMMA 2.6. *Let $f \in F$. Then,*

(i) $K_f = \langle (x, f), (y, f) \rangle$.

(ii) $[G : K_f]$ is finite if and only if f is a non-trivial rotation. Also, $K_f = \{1\}$ if and only if $f = 1$.

(iii) K_f is a non-trivial cyclic group if and only if f is a reflection or glide reflection. If f, f' are reflections or glide reflections, then $K_f = K_{f'}$ if and only if $f = f'$.

(iv) For all $a \in A$, we have

$$(af)^G \cap Af = \bigcup_{f' \in F'} K_f(af)^{f'},$$

where $F' = \{f' \in F \mid (f, f') \in A\}$. In the cases where $G = A \rtimes F$, this is equivalent to $(af)^G \cap Af = \bigcup_{f' \in C} K_f a^{f'} f$ where C is the centralizer of f in F .

(v) If f is a rotation, then (iv) gives the following:

$$(af)^G \cap Af = \begin{cases} aK_f f \cup a^\rho K_f f & \text{if } G \text{ contains } \rho_{\pi/2}; \\ aK_f f \cup (a\beta_1\beta_2)^{-1}K_f f & \text{if } G \text{ is of type } \mathbf{p2mm}, \mathbf{p2mg} \text{ or } \mathbf{p2gg}; \\ aK_f f \cup a^\sigma K_f f & \text{if } G \text{ is of type } \mathbf{c2mm}; \\ aK_f f \cup a\beta_1\beta_1^\rho K_f f & \text{if } f = \rho_\pi \text{ in } G \text{ of type } \mathbf{p4mg}; \\ aK_f f & \text{otherwise.} \end{cases}$$

If f is glide reflection or reflection, (iv) implies that

$$(af)^G \cap Af = \begin{cases} aK_f f \cup (a\beta_1\beta_2)^{-1}K_f f & \text{if } G \text{ contains } \rho_\pi; \\ aK_f f & \text{otherwise.} \end{cases}$$

If G has no glide reflections, the above becomes

$$(af)^G \cap Af = \begin{cases} aK_f f \cup a^{-1}K_f f & \text{if } G \text{ contains } \rho_\pi; \\ aK_f f & \text{otherwise.} \end{cases}$$

Proof. Since A is abelian the Witt–Hall identities [12, p. 290] show that if $a, b \in A$, then $(ab, f) = (a, f)(b, f)$, from which (i) follows. Now, (ii) and (iii) follow from consideration of the specific groups and the choice of F .

Then, (iv) follows because conjugates of the element af result from conjugating by something in A or something in F . Conjugating by elements in A gives us $K_f af$. If

$(f', f) \notin A$, then $(af')' \notin Af$. Thus, it only remains to conjugate by those f' that satisfy $(f', f) \in A$.

Most of the rest is left to the reader. We will do two cases. First, consider the group of type **p4mm**, which contains $\rho_{\pi/2}$. The conjugacy class of $a\rho$ contains $(a\rho)^x = a(x, \rho^{-1})\rho$ and $(a\rho)^y = a(y, \rho^{-1})\rho$, as well as $(a\rho)^\rho = a^\rho\rho$. Since $(a\rho)^\sigma \notin A\rho$, we see $(a\rho)^G \cap A\rho = aK_\rho\rho \cup a^\rho K_\rho\rho$. Likewise, the conjugacy class of $a\rho^2$ contains $(a\rho^2)^x = a(x, \rho^2)\rho^2$ and $(a\rho^2)^y = a(y, \rho^2)\rho^2$, as well as $(a\rho^2)^\rho = a^\rho\rho^2$. Since $(a\rho^2)^\sigma \in aK_{\rho^2}\rho^2$, we have $(a\rho^2)^G \cap A\rho^2 = aK_{\rho^2}\rho^2 \cup a^{\rho^2}K_{\rho^2}\rho^2$, proving the first line of (v).

For a second example, suppose γ is a glide reflection in a group with a generator $\rho = \rho_\pi$. Recall that we defined $\beta_1 = \gamma^2, \beta_2 = (\rho\gamma)^2$. Then, $(a\gamma)^G$ will include $(a\gamma)^x = a(x, \gamma^{-1})\gamma$ and $(a\gamma)^y = a(y, \gamma^{-1})\gamma$, thus it includes $aK_\gamma\gamma$. Note that $(a, \gamma) \in K_\gamma$ implies that $(a\gamma)^\rho \in aK_\gamma\gamma$. Last, we consider $(a\gamma)^\rho$. Since $\gamma^\rho = \beta_1^{-1}\beta_2\gamma$, we have $(a\gamma)^\rho = (a\beta_1\beta_2^{-1})^{-1}\gamma$. But since $\beta_2 \in K_\gamma$ we see that $(a\beta_1\beta_2)^{-1}K_\gamma\gamma \subset (a\gamma)^G$. Thus, $(a\gamma)^G = aK_\gamma\gamma \cup (a\beta_1\beta_2)^{-1}K_\gamma\gamma$. □

Some automorphisms of wallpaper groups. Here, we list some automorphisms that we will use in what follows. Inner automorphisms, which will be denoted $I_g, g \in G$, are not listed here unless we compose with them in Step (2).

p3, p4, p6 $\psi_x : x \mapsto x, y \mapsto y, r \mapsto xr; \psi_y : x \mapsto x, y \mapsto y, r \mapsto yr$.

c1m1 $\psi : (x, y, \sigma) \mapsto (x^{-1}, y^{-1}, \sigma)$; also I_σ and $(x, y, \sigma) \mapsto (x, y, a\sigma)$, for $a \in A$ where $(a\sigma)^2 = 1$.

p1m1 $\psi : (x, y, \sigma) \mapsto (x^{-1}, y^{-1}, \sigma)$; also I_σ and $(x, y, \sigma) \mapsto (x, y, a\sigma)$, for $a \in A$ where $(a\sigma)^2 = 1$.

p1g1 $\psi_y : (x, y, \gamma) \mapsto (x, y, y\gamma)$; also $\psi : (x, y, \gamma) \mapsto (x^{-1}, y^{-1}, \gamma^{-1})$ and I_γ .

c2mm $\psi_1 : (x, y, \rho, \sigma) \mapsto (x, y^{-1}, \rho, \rho\sigma)$ and $\psi_{u,v,i} : (x, y, \rho, \sigma) \mapsto (x, y, x^u y^v \rho, x^i y^{-i} \sigma)$ where $u = v + 2i$; also I_σ and I_ρ .

p2mm $\psi_{u,v} : (x, y, \rho, \sigma) \mapsto (x, y, x^u y^v \rho, y^v \sigma)$; also $\psi : (x, y, \rho, \sigma) \mapsto (y, x, \rho, \rho\sigma)$.

p2mg $\psi_x : (x, y, \rho, \sigma) \mapsto (x, y, x\rho, \sigma), \psi_y : (x, y, \rho, \sigma) \mapsto (x, y, y\rho, y\sigma)$, and $\psi : (x, y, \rho, \sigma) \mapsto (x^{-1}, y^{-1}, y^{-1}\rho, \sigma)$; also I_ρ and I_σ .

p2gg $\psi_1 : (x, y, \rho, \gamma) \mapsto (y, x, \rho, \rho\gamma)$ as well as $\psi_x : (x, y, \rho, \gamma) \mapsto (x, y, x\rho, \gamma)$ and $\psi_y : (x, y, \rho, \gamma) \mapsto (x, y, y\rho, y\gamma)$; also I_ρ and I_γ .

p3m1 $\psi_1 : (x, y, \rho, \sigma) \mapsto (x, y, y^{-1}\rho, \sigma)$ and $\psi_2 : (x, y, \rho, \sigma) \mapsto (x^{-1}, y^{-1}, \rho, \sigma)$; also I_ρ and I_σ .

p31m I_σ , and $\psi_1 : (x, y, \rho, \sigma) \mapsto (y, x^{-1}y, \rho, \rho^2\sigma)$.

p4mg $\psi_1 : (x, y, \rho, \gamma) \mapsto (x, y, x\rho, y^{-1}\gamma); \psi_2 : (x, y, \rho, \gamma) \mapsto (x^{-1}, y^{-1}, y^{-1}\rho, x^{-1}\gamma)$; also I_ρ and I_γ .

p4mm $\psi_1 : (x, y, \rho, \sigma) \mapsto (x, y, y\rho, y\sigma)$; also, I_σ and I_ρ .

p6mm I_σ, I_ρ .

Some non-trivial weak Cayley table isomorphisms of wallpaper groups.

p2mm Define τ by $\tau : \begin{cases} g \mapsto g & \text{for } g \in A \cup A\rho\sigma; \\ g \mapsto g^\sigma & \text{for } g \in A\rho \cup A\sigma \end{cases}$

p3 Define $\tau : \begin{cases} g \mapsto g & \text{for } g \notin A\rho \cup A\rho^2; \\ g \mapsto g^\rho & \text{for } g \in A\rho \cup A\rho^2 \end{cases}$

p4 For $h \in \{x, y, \rho^2\}$ define $\tau_h : \begin{cases} g \mapsto g & \text{for } g \notin A\rho^2; \\ g \mapsto g^h & \text{for } g \in A\rho^2 \end{cases}$

For $h \in \{x, y, \rho\}$ define $\mu_h : \begin{cases} g \mapsto g & \text{for } g \in A \cup A\rho^2; \\ g \mapsto g^h & \text{for } g \notin A \cup A\rho^2. \end{cases}$

p6 For $h \in \{xy, xy^{-2}, \rho^2\}$ define $\tau_h : \begin{cases} g \mapsto g & \text{for } g \in A\rho \cup A\rho^3 \cup A\rho^5; \\ g \mapsto g^h & \text{for } g \notin A\rho \cup A\rho^3 \cup A\rho^5. \end{cases}$

For $h \in \{x^2, y^2, \rho^3\}$ define $\mu_h : \begin{cases} g \mapsto g & \text{for } g \notin A \cup A\rho^3; \\ g \mapsto g^h & \text{for } g \in A \cup A\rho^3. \end{cases}$

3. Steps (1) and (2). Let $\varphi \in \mathcal{W}(G)$ where G is a wallpaper group. In this section, we will denote A additively, so that the action of $g \in G$ on A by conjugation is denoted $(a)g$, $a \in A$. Then, for all $a, b \in A$ there is some $g_{a,b} \in F$ such that

$$\varphi(a+b) = (\varphi(a) + \varphi(b))g_{a,b}. \quad (3.1)$$

Thus,

$$\begin{aligned} \varphi(b) &= \varphi(a+b-a) = (\varphi(a+b) + \varphi(-a))g_{a+b,-a} \\ &= (\varphi(a+b) - \varphi(a))g_{a+b,-a}. \end{aligned} \quad (3.2)$$

LEMMA 3.1. *For all $a, b \in A$, there is $f \in F$ such that*

$$\varphi(a+b) = \varphi(a) + \varphi(b)f.$$

Proof. From equation (3.2), we see that $f = g_{a+b,-a}^{-1}$ will work. \square

Let $a, b \in A$. We wish to show that $\varphi(a+b) = \varphi(a) + \varphi(b)$. From equation (3.1) and Lemma 3.1, we see that

$$\varphi(a+b) \in (\varphi(a) + \varphi(b))F \cap (\varphi(a) + \varphi(b)F) \cap (\varphi(a)F + \varphi(b)). \quad (3.3)$$

Since F acts by orthogonal matrices we see that

- (i) any point of the form $(\varphi(a) + \varphi(b))F$ is on the circle C_1 of radius $|\varphi(a) + \varphi(b)|$ centred at $(0, 0)$;
- (ii) any point of the form $\varphi(a) + \varphi(b)F$ is on the circle C_2 of radius $|\varphi(b)|$ centred at $\varphi(a)$;
- (iii) any point of the form $\varphi(a)F + \varphi(b)$ is on the circle C_3 of radius $|\varphi(a)|$ centred at $\varphi(b)$.

The intersection $C_1 \cap C_2 \cap C_3$ certainly contains the point $p_1 = \varphi(a) + \varphi(b)$, and any other possibility for $\varphi(a+b)$. If p_1 is the only such point of $C_1 \cap C_2 \cap C_3$, then by equation (3.3), we must have $\varphi(a+b) = \varphi(a) + \varphi(b)$, and we are done.

So, now assume that $C_1 \cap C_2 \cap C_3$ also contains $p_2 \neq p_1$; then the arc p_1p_2 has a perpendicular bisector, L say, that contains the centres of each of C_1, C_2 and C_3 . Thus, $\varphi(a), \varphi(b)$ and $(0, 0)$ are on this line, and so $p_1 = \varphi(a) + \varphi(b)$ is also on L .

But p_1 was on a line perpendicular to L that also contains p_2 , where the distance from p_1 to L was the same as the distance of p_2 to L . It follows that $p_1 = p_2$, and we have a contradiction. Thus, $\varphi|_A$ is a homomorphism, however φ is a bijection, and so we have shown Step (1):

LEMMA 3.2. *For any $\varphi \in \mathcal{W}(G)$, we have $\varphi|_A : A \rightarrow A$ is an automorphism.*

So, we may now assume that $\varphi|_A : A \rightarrow A$ is an automorphism. Relative to the basis v_x, v_y the matrix representing $\varphi|_A$ is an integer matrix, $M(\varphi)$ say, and since it is invertible $M(\varphi)$ must have determinant ± 1 . We first show:

LEMMA 3.3. *If G does not have type **p1,p2**, then $\varphi|_A$ has finite order.*

Proof. First, assume that there is some $a \in A$ such that $(a)F$ does not span a cyclic subgroup of A . Let $a_1, a_2 \in (a)F$ generate a \mathbb{Z}^2 subgroup. If $\varphi|_A$ has infinite order, then for every $N > 0$ there is $k \in \mathbb{N}$ such that some point of $\varphi^k((a)F)$ is outside the ball B_N . But all the elements of $(a)F$ are conjugate, and so all the elements of $\varphi^k((a)F)$ are conjugate by Lemma 2.1. Thus, if one of the points of $\varphi^k((a)F)$ is outside B_N , then they are all outside of B_N . Thus, $\varphi(a_1), \varphi(a_2)$ are both on the boundary of the ball $B_{|\varphi(a_1)|} \supset B_N$. Now, the triangle with vertices $(0, 0), a_1, a_2$ is sent to the triangle with vertices $(0, 0), \varphi(a_1), \varphi(a_2)$; however (as the distance between $\varphi(a_1), \varphi(a_2)$ is at least 1, and we can choose N arbitrarily large), the latter triangle clearly has larger area than the former, contradicting the fact that $M(\varphi)$ has determinant ± 1 .

Now, one can check (i) and (ii) of the following result.

LEMMA 3.4. (i) *Every wallpaper group G other than **p1** and **p2** has some $a \in A$, where $(a)F$ does not span a cyclic group.*

(ii) *If τ is some isometry of the wallpaper paper pattern in \mathbb{E}^2 corresponding to the wallpaper group G and $\tau((0, 0)) = (0, 0)$, then there is some element of $\mathcal{W}_0(G)$ whose action on A is equal to that of τ on \mathbb{E}^2 .*

The elements referred to in Lemma 3.4 (ii) are compositions of ι and the automorphisms that were defined in Section 2.

Lemma 3.3 now follows from Lemma 3.4 (i). □

Thus, by Lemma 3.3, we see that if G is not of type **p1,p2**, then $M(\varphi)$ is an integer matrix of finite order. This order is well-known to be 1, 2, 3, 4, or 6, and the element is an orthogonal rotation or a reflection. This matrix also preserves the lattice \mathcal{L} . It thus corresponds to the action of some element of $\mathcal{W}_0(G)$, by Lemma 3.4 (ii). Composing φ with the inverse of this element gives a new φ where $\varphi|_A = \text{Id}_A$, as required for Step (2).

Thus, we now assume that G has type **p2**. In this case, it is easy to see that whenever $a, b, c, d \in \mathbb{Z}$ satisfy $ad - bc = \pm 1$, then the assignments

$$x \mapsto x^a y^b, \quad y \mapsto x^c y^d, \quad r \mapsto r,$$

determine an automorphism of G . Thus, we can compose φ with some element of $\mathcal{W}_0(G)$ to obtain a new φ satisfying $\varphi|_A = \text{Id}_A$. This gives Step (2):

PROPOSITION 3.5. *For $\varphi \in \mathcal{W}(G)$, there is $\varphi' \in \mathcal{W}_0(G)$ with $(\varphi' \circ \varphi)|_A = \text{Id}_A$.*

4. Step (3). Since $A \triangleleft G$, we see from Lemma 2.1 that φ induces a weak Cayley table isomorphism $\tilde{\varphi} : G/A \rightarrow G/A$. Now, G/A is one of the groups $C_1, C_2, C_3, C_4, C_6, D_4 = C_2 \times C_2, D_6, D_8, D_{12}$. It is known [5] that each of these groups has trivial weak Cayley table group.

LEMMA 4.1. *If G is a wallpaper group having no glide reflections in F , with presentation as given in Section 2 with generators x, y, ρ, σ , (respectively x, y, ρ), then the assignments*

$$x \mapsto x^{-1}, y \mapsto y^{-1}, \rho \mapsto \rho, \sigma \mapsto \sigma,$$

(respectively $x \mapsto x^{-1}, y \mapsto y^{-1}, \rho \mapsto \rho$) determine an automorphism of G that we denote by ι_A .

Proof. This follows from the fact that each relation involving x, y has the form (x, y) or $x^\rho = w = w(x, y) \in A$, this latter type being equivalent to $(x^{-1})^\rho = w(x, y)^{-1} = w(x^{-1}, y^{-1})$. \square

Since $\tilde{\varphi} : G/A \rightarrow G/A$ is an automorphism or antiautomorphism, we see that if $\rho \in G$ is a rotation that generates the subgroup of rotations fixing $(0, 0)$, then we must have $\tilde{\varphi}(A\rho) = A\rho^{\pm 1}$, and so $\varphi(\rho) \in A\rho^{\pm 1}$. If $\varphi(\rho) \in A\rho^{-1}$, then we compose φ with ι and then with ι_A , to give a new φ which is still the identity on A , but which satisfies $\varphi(\rho) \in A\rho$.

If $|G/A| = 2$, then $\tilde{\varphi}$ is the identity. So, assume that $|G/A| > 2$.

If G does not have reflections or glide reflections, then the above shows that $\tilde{\varphi}$ is the identity.

So, now we assume that G has reflections, but does not have glide reflections. Then, we have $G/A \cong D_4 = C_2 \times C_2, D_6, D_8, D_{12}$. In particular, G has at least two reflection cosets: there are $\sigma_1, \sigma_2 \in G, A\sigma_1 \neq A\sigma_2$, where $\sigma_i, i = 1, 2$, are reflections. Assume that $\varphi(\sigma_1) = a\sigma_2$. Since σ_1 is a reflection, it follows that $a \in L^\perp(\sigma_2)$ by Lemmas 2.5 and 2.1 (1). Also, $L^\perp(\sigma_1) \neq L^\perp(\sigma_2)$, since $A\sigma_1 \neq A\sigma_2$. Then, there is $b \in L^\perp(\sigma_1)$ such that $ab \notin L^\perp(\sigma_2)$. But $b\sigma_1$ has order 2, however $\varphi(b\sigma_1) \sim \varphi(b)\varphi(\sigma_1) = ba\sigma_2$; since $ab \notin L^\perp(\sigma_2)$, we see that $ab\sigma_2$ does not have order 2, a contradiction. Thus, we have proved Step (3) for groups without glide reflections:

LEMMA 4.2. *If G does not have glide reflections in F , then we can assume (by composing with a trivial weak Cayley table map if necessary) that $\varphi|_A = \text{Id}_A$ and that $\tilde{\varphi} : G/A \rightarrow G/A$ is the identity.*

So, now assume that G has a glide reflection in F . There are four wallpaper groups of this type that we now consider individually.

If G is of type **p1g1**, then $|G/A| = 2$ and so we certainly have $\tilde{\varphi} = \text{Id}_{G/A}$.

If G is of type **p2mg**, then $G/A \cong C_2^2$, and the cosets are $A, A\rho, A\sigma, A\rho\sigma$. Here, $A\rho$ consists entirely of rotations; if $x^i y^j \sigma \in A\sigma$, then $(x^i y^j \sigma)^2 = x^{2i}$; and if $x^i y^j \rho \sigma \in A\rho\sigma$, then $(x^i y^j \rho \sigma)^2 = y^{2j+1}$. This shows that $\varphi(A\rho) = A\rho$. If we have $\varphi(\sigma) = x^i y^j \rho \sigma$, then $\sigma^2 = 1$ gives $(x^i y^j \rho \sigma)^2 = 1$, so that $y^{2j+1} = 1$, a contradiction. Thus, we have $\tilde{\varphi} = \text{Id}_{G/A}$.

If G is of type **p2gg**, then $G/A \cong C_2^2$, and the cosets are $A, A\rho, A\gamma, A\rho\gamma$. Again $A\rho$ consists entirely of order 2 rotations, while each element of $A\gamma, A\rho\gamma$ is a glide reflection and so not of order 2. Thus, we have $\varphi(A\rho) = A\rho$. If $\varphi(\gamma) = x^i y^j \rho \gamma$, then

$$\begin{aligned} x &= \varphi(x) = \varphi(\gamma^2) \sim (x^i y^j \rho \gamma)^2 = x^i y^j \rho \gamma x^i y^j \rho \gamma \\ &= x^i y^j \rho \gamma \rho x^{-i} y^{-j} \gamma = x^i y^j y \gamma^{-1} x^{-i} y^{-j} \gamma = y^{2j+1}. \end{aligned}$$

Thus, $y^{2j+1} \in x^G = \{x, x^{-1}\}$, a contradiction. Thus, we have $\tilde{\varphi} = \text{Id}_{G/A}$.

If G is of type **p4mg**, then $G/A \cong D_8$. We describe $\mathcal{W}(G/A)$. Order the cosets of G/A as

$$A, A\rho^2, A\rho^2\gamma, A\gamma, A\rho\gamma, A\rho^3\gamma, A\rho^3, A\rho.$$

Then, relative to this ordering the action of $\mathcal{W}(G/A)$ is given by the permutation group $\langle (3, 5, 4, 6), (3, 4), (7, 8) \rangle \cong D_8 \times C_2$.

Thus, the possibilities for $\tilde{\varphi}(A\rho)$ are $A\rho, A\rho^3$; we also see that $\varphi(A\rho^2) = A\rho^2$, and that $\{A\gamma, A\rho\gamma, A\rho^2\gamma, A\rho^3\gamma\}$ are permuted by $\tilde{\varphi}$. If $\varphi(A\rho) = A\rho^3$, then composing with ι and then ψ_2 shows that we can assume $\varphi|_A = \text{Id}_A, \tilde{\varphi}(A\rho) = A\rho, \tilde{\varphi}(A\rho^2) = A\rho^2, \tilde{\varphi}(A\rho^3) = A\rho^3$.

Considering the other cosets, we have

$$\begin{aligned} (x^i y^j \gamma)^2 &= x^{2i+1}; & (x^i y^j \rho \gamma)^2 &= x^{i-j} y^{j-i}; \\ (x^i y^j \rho^2 \gamma)^2 &= y^{2j+1}; & (x^i y^j \rho^3 \gamma)^2 &= (xy)^{i+j+1}. \end{aligned}$$

This shows that $A\rho\gamma, A\rho^3\gamma$ are the only cosets that contain involutions, so that we must have $\{\varphi(A\rho\gamma), \varphi(A\rho^3\gamma)\} = \{A\rho\gamma, A\rho^3\gamma\}$. This shows that the action of $\tilde{\varphi}$ is given by some element of the group $\langle (3, 4), (5, 6) \rangle$. If $\tilde{\varphi} \neq 1$, then we must either have $\varphi(A\gamma) = A\rho^2\gamma$ or $\varphi(A\rho\gamma) = A\rho^3\gamma$.

CASE 1. We first show that we cannot have $\varphi(A\gamma) = A\rho^2\gamma$. So, assume that this is the case, and that $\varphi(\gamma) = x^u y^v \rho^2 \gamma$. Since $\gamma^2 = x$, we have

$$\begin{aligned} x &= \varphi(x) = \varphi(\gamma^2) \sim (x^u y^v \rho^2 \gamma)^2 = x^u y^v \rho^2 \gamma \cdot x^u y^v \rho^2 \gamma \\ &= x^u y^v (\rho^2 \gamma \rho^2) x^{-u} y^{-v} \gamma = x^u y^v (y \gamma^{-1}) x^{-u} y^{-v} \gamma = x^u y^v y x^{-u} y^v = y^{2v+1}. \end{aligned}$$

Since $x^G = \{x, y, x^{-1}, y^{-1}\}$, we see that $2v + 1 \in \{1, -1\}$, so that $v \in \{0, -1\}$.

The next result follows from the presentation for G of type **p4mg**:

LEMMA 4.3. *Let G have type **p4mg**. Let $X = \langle x^2 \rangle, Y = \langle y^2 \rangle$. For $a \in A$, we have*

$$(a\rho^2\gamma)^G = yYa^{\rho^3}\gamma \cup x^{-1}Ya^{\rho}\gamma \cup Xa\rho^2\gamma \cup xy^{-1}Xa^{\rho}\rho^2\gamma.$$

Now, $(x\gamma)^2 = x^3$, giving $x^3 = \varphi(x^3) = \varphi((x\gamma)^2) \sim \varphi(x\gamma)^2 \sim (\varphi(x)\varphi(\gamma))^2$, so that $x^3 \sim (x^{u+1}y^v\rho^2\gamma)^2$. Lemma 4.3 shows that $(x^{u+1}y^v\rho^2\gamma)^G$ has one of the following forms, for some $k \in \mathbb{Z}$:

- (i) $y^{2k+1}y^{-u-1}x^v\gamma;$ (ii) $x^{-1}y^{2k}y^{u+1}x^{-v}\gamma;$
- (iii) $x^{2k}x^u y^v \rho^2 \gamma;$ (iv) $x^{2k+1}y^{-1}x^{u+1}y^{-v}\rho^2\gamma.$

If we have (i), then

$$x^3 \sim (y^{2k+1}y^{-u-1}x^v\gamma)^2 = y^{2k-u}x^v\gamma \cdot y^{2k-u}x^v\gamma = y^{2k-u}x^v\gamma^2 y^{-2k+u}x^v = x^{2v+1},$$

which is a contradiction since $v \in \{0, -1\}$.

If we have (ii), then $x^3 \sim (y^{2k+u+1}x^{-v-1}\gamma)^2$, where

$$\begin{aligned}(y^{2k+u+1}x^{-v-1}\gamma)^2 &= y^{2k+u+1}x^{-v-1}\gamma \cdot y^{2k+u+1}x^{-v-1}\gamma \\ &= y^{2k+u+1}x^{-v-1}\gamma^2 \cdot y^{-2k-u-1}x^{-v-1} = x^{-2v-1},\end{aligned}$$

which is a contradiction since $v \in \{0, -1\}$.

If we have (iii), then $x^3 \sim (x^{2k+u}y^v\rho^2\gamma)^2$, where

$$\begin{aligned}(x^{2k+u}y^v\rho^2\gamma)^2 &= x^{2k+u}y^v\rho^2\gamma \cdot x^{2k+u}y^v\rho^2\gamma = x^{2k+u}y^v\rho^2\gamma\rho^2x^{-2k-u}y^{-v}\gamma \\ &= x^{2k+u}y^v\gamma\gamma^{-1}x^{-2k-u}y^{-v}\gamma = y^{2v+1},\end{aligned}$$

which is a contradiction since $v \in \{0, -1\}$.

If we have (iv), then $x^3 \sim (x^{2k+2+u}y^v\rho^2\gamma)^2$, where

$$\begin{aligned}(x^{2k+2+u}y^v\rho^2\gamma)^2 &= x^{2k+2+u}y^v\rho^2\gamma \cdot x^{2k+2+u}y^v\rho^2\gamma = x^{2k+2+u}y^v\rho^2\gamma\rho^2x^{-2k-2-u}y^{-v}\gamma \\ &= x^{2k+2+u}y^v\gamma\gamma^{-1}x^{-2k-2-u}y^{-v}\gamma = y^{2v+1},\end{aligned}$$

which is a contradiction since $v \in \{0, -1\}$. Thus, Case 1 does not happen.

CASE 2. Here, we assume that $\varphi(A\rho\gamma) = A\rho^3\gamma$ and write $\varphi(\rho\gamma) = b\rho^3\gamma$, $b = x^u y^v$. Since $(\rho\gamma)^2 = 1$, we must have $(b\rho^3\gamma)^2 = 1$. But

$$\begin{aligned}(b\rho^3\gamma)^2 &= x^u y^v \rho^3 \gamma \cdot x^u y^v \rho^3 \gamma = x^u y^v \rho^3 \gamma \rho^3 y^{-u} x^v \sigma = x^u y^v \gamma \gamma y^{-u} x^v \sigma \\ &= x^u y^{v+1} s^2 y^u x^v = (xy)^{u+v+1},\end{aligned}$$

so that $u + v + 1 = 0$.

Now, $(xy\rho\gamma)^2 = 1$, and so $1 = \varphi((xy\rho\gamma)^2) \sim \varphi(xy\rho\gamma)^2$, so that $\varphi(xy\rho\gamma)^2 = 1$. But $\varphi(xy\rho\gamma) \sim xy \cdot x^u y^v \rho^3 \sigma$, and so we must have $(x^{u+1}y^{v+1}\rho^3\sigma)^2 = 1$. One finds that $(x^{u+1}y^{v+1}\rho^3\sigma)^2 = (xy)^{u+v+3}$, so that $u + v + 3 = 0$, a contradiction.

Thus, Case 2 does not happen either, and we have proved Step (3).

5. Step (4). We need to show that we can assume (after possibly composing φ with an element of $\mathcal{W}_0(G)$ or a known non-trivial weak Cayley table isomorphism) that φ fixes each element of F . We do this in such a way as to preserve the fact that $\varphi|_A$ is the identity on A , and that $\tilde{\varphi}$ is the identity map on cosets of A . We consider each type of wallpaper group.

p2, p3, p4, p6 cases: Here, we have $\varphi(\rho) = a\rho$, $a \in A$; however, the assignment $(x, y, \rho) \mapsto (x, y, a\rho)$ determines an automorphism of G in this case. We thus compose φ with the inverse of this automorphism to obtain $\varphi(\rho) = \rho$, $\varphi(\rho^{-1}) = \rho^{-1}$. This then does the cases **p2, p3**.

For **p4**, we note that $\varphi(\rho^2) \sim \varphi(\rho)^2 = \rho^2$, which shows that $\varphi(\rho^2) = (\rho^2)^a$, $a = x^i y^j \in A$. But then composing φ with $\tau_x^{-i} \tau_y^{-j}$ will then give $\varphi(\rho^i) = \rho^i$, $0 \leq i < 4$.

For **p6**, we note that $\varphi(\rho^2) \sim \varphi(\rho)^2 = \rho^2$, which shows that $\varphi(\rho^2) = (\rho^2)^a$, $a = x^i y^j \in A$. Thus, $\varphi(\rho^2) = (xy)^k (xy^{-2})^m \rho^2$ for some $k, m \in \mathbb{Z}$. But now we can compose φ with $\tau_{xy}^{-k} \tau_{xy^{-2}}^{-m}$ so as to be able to assume $\varphi(\rho^2) = \rho^2$, $\varphi(\rho^{-2}) = \rho^{-2}$. Now, $\varphi(\rho^3) \sim \varphi(\rho)\varphi(\rho^2) = \rho^3$, so that $\varphi(\rho) = x^{2i} y^{2j} \rho^3$, $i, j \in \mathbb{Z}$. Now, one can use $\mu_{x^2}^{-i} \mu_{y^2}^{-j}$ so as to be able to assume $\varphi(\rho^3) = \rho^3$. This does the **p6** case.

c1m1, p1m1 cases: Here, we must have $\varphi(\sigma) = a\sigma, a \in A$, where $(a\sigma)^2 = 1$ since $\sigma^2 = 1$, however the assignment $(x, y, \sigma) \mapsto (x, y, a\sigma)$ determines an automorphism of G in these cases, and we are done.

p1g1 case: Here, we note that $x = \gamma^2$ is central in G , so that we must have $\varphi(\gamma)^2 = x$; one finds that if $\varphi(\gamma) = x^u y^v \gamma$, then $u = 0$. Thus, we can compose with a power of the automorphism ψ_y of G so as to get $\varphi|_A = Id$ and $\varphi(\gamma) = \gamma$.

We will find the following useful in what follows:

LEMMA 5.1. *Suppose that $\varphi|_A = Id_A$ and that $\varphi(\rho) = \rho$ for a rotation ρ . Suppose that $\sigma, \sigma\rho$ are distinct reflections in G with $\varphi(A\sigma) = A\sigma$. Then, $\varphi(\sigma) = \sigma$.*

Proof. Since $\varphi(A\sigma) = A\sigma$, we can write $\varphi(\sigma) = a\sigma, a \in A$. Since $\sigma^2 = 1$, we must have $a \in L^\perp(\sigma)$. Now, $\varphi(\rho \cdot \sigma) \sim \rho a \sigma = a^{\rho^{-1}} \rho \sigma$ and so $a^{\rho^{-1}} \in L^\perp(\rho\sigma)$. But then $a \in L^\perp(\sigma\rho)$. Since $\sigma \neq \sigma\rho$, we have $a \in L^\perp(\sigma) \cap L^\perp(\sigma\rho) = \{1\}$, and we are done. \square

LEMMA 5.2. *Let $\varphi|_A = Id|_A$ and assume $\varphi(A\gamma) = A\gamma$ for some glide reflection coset $A\gamma$. Then, $\varphi(\gamma) = a\gamma$ implies that $a \in L^\perp(\gamma)$.*

Proof. Let $\beta = \gamma^2$, so that $\beta \in L(\gamma), \beta \neq 1$. Squaring both sides of $\varphi(\gamma) = a\gamma$ gives us $\beta \sim \beta a a^\gamma$. Since β and $a a^\gamma$ both lie on $L(\gamma)$, this implies that $\beta a a^\gamma = \beta^{\pm 1}$. Then, we have two possibilities: either $a a^\gamma = 1$ which implies that $a \in L^\perp(\gamma)$, or $a a^\gamma = \beta^{-2}$ in which case $a \in \beta^{-1} L^\perp(\gamma)$. Now, let $L^\perp(\gamma) = \langle \alpha \rangle, \alpha \in A$, and suppose we have $a = \beta^{-1} \alpha^k$. Then, $\varphi(\beta \cdot \gamma) \sim \beta \cdot a\gamma = \alpha^k \gamma$. Since $(\beta\gamma)^2 = \beta^3$, squaring both sides gives $\beta^3 \sim \beta$, which is a contradiction. Thus, we must have $a \in L^\perp(\gamma)$. \square

LEMMA 5.3. *Let G be a wallpaper group not of type **p4mg**, with a reflection or glide reflection coset Ar . Assume $\varphi|_A = Id_A$ and $\varphi(Ar) = Ar$. Suppose that for $ar \in Ar$, we have $\varphi(ar) \sim ar$. Then, $\varphi(ar) \in K_r ar$. In particular, $ar = \varphi(r) \sim r$ implies that $a \in K_r$.*

Proof. If $\rho_\pi \notin G$, then by Lemma 2.6, we have $(ar)^G \cap Ar = K_r ar$ and we are done, so suppose $\rho_\pi \in G$. Let $\lambda = (\rho_\pi, r^{-1})$, so that $\lambda \in A$ and $r^{\rho_\pi} = \lambda r$. Let $\beta = r^2$ so that $r = \beta r^{-1}$. (If r is a reflection, then $\beta = 1$.) Thus, again by Lemma 2.6, we have

$$(ar)^G \cap Ar = K_r ar \cup K_r (ar)^{\rho_\pi} = K_r ar \cup K_r a^{-1} \lambda r.$$

First, we note that if $\lambda = 1$ and $a^2 \in K_r$, then we have nothing to prove; so suppose that this is not the case. Since $\varphi(Ar) = Ar$ it suffices to show that $\varphi(ar) \notin K_r (ar)^{\rho_\pi}$. Now, K_r is a cyclic group: $K_r = \langle \alpha \rangle$, where $\alpha \in A$. Also, in this case one sees that $K_r \leq L^\perp(r)$. Suppose to the contrary that $\varphi(ar) = \alpha^i (ar)^{\rho_\pi}$ for some $\alpha^i \in K_r$. Then, for any $b \in A$, we have

$$\varphi(b \cdot ar) \sim b \alpha^i a^{-1} \lambda r.$$

Squaring both sides, we have

$$\varphi(bar)^2 \sim \varphi((bar)^2) = \varphi(ba\beta(ba)^r) = ba\beta(ba)^r \sim (b\alpha^i a^{-1} \lambda \beta)(b\alpha^i a^{-1} \lambda)^r.$$

Reordering the elements this becomes

$$\beta a a^r b b^r \sim \beta \alpha^i (\alpha^i)^r a^{-1} (a^{-1})^r b b^r \lambda \lambda^r. \tag{5.1}$$

Recall that $L(r)$ is the line fixed by the action of r . Note that aa^r is in $L(r)$ (since $(aa^r)^r = a^r a = aa^r$.) Similarly, $bb^r, \lambda\lambda^r \in L(r)$. Recall that $\alpha \in K_r \leq L^\perp(r)$, so that $\alpha^r = \alpha^{-1}$. So, we know $\alpha^i(\alpha^i)^r = 1$. Also, $\beta \in L(r)$ (since $\beta^r = (r^2)^r = r^2 = \beta$). Thus, Equation (5.1) is stating that two elements of $L(r)$ are conjugate to each other. This implies that they are equal or they are inverses. If they are inverses of each other, Equation (5.1) becomes $b^{-2}(b^{-2})^r = \beta^2\lambda\lambda^r$ which can't be true for all $b \in A$. Thus, they are equal to each other and we have $a^2(a^2)^r = \lambda\lambda^r$. We can assume λ is not trivial in this case because if $\lambda = 1$, then $a^2 \in K_r$, and we assumed both of those cannot be true together.

So, assume $\lambda \neq 1$. Now, the fact that G has a reflection or glide reflection r , contains ρ_π , and $\lambda = (\rho_\pi, r^{-1}) \neq 1$, restricts us to the groups of type **p2mg**, **p2gg**. In these cases, one checks that $a^2(a^2)^r \in \langle x^4, y^4 \rangle$. However (looking at the possible λ s in each of these three groups), we find that $\lambda\lambda^r \notin \langle x^4, y^4 \rangle$. This is a contradiction. Thus, $\varphi(ar) \notin K_r(ar)^{\rho_\pi}$. \square

c2mm case: Suppose that $\varphi(\rho) = x^u y^v \rho$, $\varphi(\sigma) = x^i y^j \sigma$. We must have $j = -i$ since $(\sigma)^2 = 1$. Similarly, the fact that $(\rho\sigma)^2 = 1$ implies that $1 = \varphi((\rho\sigma)^2) = \varphi(\rho\sigma)^2 = (\varphi(\rho)\varphi(\sigma))^2 = (x^u y^v \rho x^i y^{-i} \sigma)^2$, which is only true when $u = v + 2i$, so that we can compose φ with the inverse of $\psi_{u,v,i}$ to get $\varphi(\rho) = \rho$, $\varphi(\sigma) = \sigma$. To get $\varphi(\rho\sigma) = \rho\sigma$, we now apply Lemma 5.1, and we are done.

p2mm case: Suppose that $\varphi(\rho) = x^u y^v \rho$, $\varphi(\sigma) = y^j \sigma$. We can compose φ with the inverse of $\psi_{u,v}$ to get $\varphi(\rho) = \rho$. To get $\varphi(\sigma) = \sigma$ and $\varphi(\rho\sigma) = \rho\sigma$, we now apply Lemma 5.1, and we are done.

p2mg case: Suppose that $\varphi(\rho) = x^u y^v \rho$. Composing with a power of ψ_y and then a power of ψ_x we can arrange that $v = 0$ and $u = 0$. Let $\varphi(\sigma) = a\sigma$. The fact that $\sigma^2 = 1$ gives $a \in L^\perp(\sigma)$. In this group, this implies $\varphi(\sigma) \sim \sigma$ and we may apply Lemma 5.3 to give $a \in K_\sigma$. If $\varphi(\rho\sigma) = b\rho\sigma$, then Lemma 5.2 gives $b \in L^\perp(\rho\sigma)$. Since $(c\rho\sigma)^G = \langle x^2 \rangle a c \rho \sigma \cup \langle x^2 \rangle (c\gamma)^{-1} \rho \sigma$, $L^\perp(\rho\sigma) = \langle x^2 \rangle$ and $b\rho\sigma = \varphi(\rho \cdot \sigma) \sim a^\rho \rho\sigma$, we see a^ρ lies on $L^\perp(\rho\sigma) \cup y^{-1}L^\perp(\rho\sigma)$. Thus, a lies on $(L^\perp(\rho\sigma) \cup yL^\perp(\rho\sigma)) \cap K_\sigma$. Hence, $a = 1$, which gives us $\varphi(\rho\sigma) \sim \rho\sigma$, and so by Lemma 5.3, we know $b \in K_{\rho\sigma}$. Now, $\sigma = \varphi(\rho \cdot \rho\sigma) \sim b^\rho \sigma$ which tells us $b \in L^\perp(\sigma)$, and so b is also trivial.

p2gg case: Suppose that $\varphi(\rho) = x^u y^v \rho$. By using ψ_x, ψ_y , we can arrange that $u = v = 0$. Let $\varphi(\gamma) = a\gamma$, $\varphi(\rho\gamma) = b\rho\gamma$. By Lemma (5.2), $a \in L^\perp(\gamma)$. Note that $L^\perp(\gamma)$ contains elements from only two conjugacy classes, namely γ^G and $(y\gamma)^G$. If $a\gamma \sim y\gamma$, then we may compose with $I_\rho \circ \iota$. So, now we have $\varphi(\gamma) \sim \gamma$ and so by Lemma 5.3 $a \in K_\gamma$. Lemma 5.2 gives us $b \in L^\perp(\rho\gamma)$ and thus $b\rho\gamma = \varphi(\rho \cdot \gamma) \sim \rho a \gamma = a^{-1} \rho \gamma$ implies that $a \in L^\perp(\rho\gamma) \cup yL^\perp(\rho\gamma)$. This intersects K_γ trivially hence $\varphi(\gamma) = \gamma$.

Now, $\varphi(\rho \cdot \gamma) \sim \rho\gamma$ so we may apply Lemma 5.3 which gives us $b \in K_{\rho\gamma}$. We also know $\gamma = \varphi(\rho \cdot \rho\gamma) \sim b^{-1}\gamma$ which tells us $b \in K_\gamma \cup x\gamma K_\gamma$. The only possibility then is that $b = 1$.

p3m1 case: Suppose that $\varphi(\rho) = x^u y^v \rho$. Composing with an element of $\langle \psi_1, I_x \rangle$ we can arrange that $\varphi(\rho) = \rho$. Thus, $\varphi(\rho) = \rho$, $\varphi(\rho^2) = \rho^2$. We now apply Lemma 5.1 to conclude that $\varphi(\sigma) = \sigma$, $\varphi(\rho\sigma) = \rho\sigma$, $\varphi(\rho^2\sigma) = \rho^2\sigma$, and we are done.

p31m case: Suppose that $\varphi(\rho) = x^u y^v \rho$, $\varphi(\sigma) = x^i y^j \sigma$. Here, we note that by composing with an element of $\langle I_x, I_y \rangle$ we can arrange that $\varphi(\rho) = \rho$, so that $\varphi(\rho^2) = \rho^2$. From Lemma 5.1, we obtain $\varphi(\rho^k \sigma) = \rho^k \sigma$, $k = 0, 1, 2$, as required.

p4mm case: Suppose that $\varphi(\rho) = x^u y^v \rho$, $\varphi(\sigma) = x^i y^j \gamma$. Then, acting by some I_b , $b \in A$, we can assume $\varphi(\rho) \in \{\rho, y\rho\}$. If we have $\varphi(\rho) = y\rho$, then acting by the automorphism

ψ_1^{-1} we see that $\varphi(\rho) = \rho, \varphi(\rho^3) = \rho^3$. Now, Lemma 5.1 gives $\varphi(\rho^k\sigma) = \rho^k\sigma, 0 \leq k \leq 3$.

Let $\varphi(\rho^2) = x^i y^j \rho^2$. Then, for any reflection $t \in \{\sigma, \rho\sigma, \rho^2\sigma\rho^3\sigma\}$, we have $\rho^2 t = \varphi(\rho^2 t) \sim \varphi(\rho^2)\varphi(t) = x^i y^j \rho^2 t$, from which we see that $x^i y^j \in L^\perp(\rho^2 t)$ for all t . This gives $i = j = 0$, so we have $\varphi(\rho^2) = \rho^2$ and this concludes this case.

p6mm case: Suppose that $\varphi(\rho) = x^u y^v \rho, \varphi(\sigma) = x^i y^j \gamma$. Then, acting by some $I_b, b \in A$, we can assume that $\varphi(\rho) = \rho$. Now, Lemma 5.1 gives $\varphi(\rho^k\sigma) = \rho^k\sigma, 0 \leq k \leq 5$, and the rest of the proof follows as in the previous case. This completes this case and concludes the proof of Step (4). □

6. Steps (5) and (6). LEMMA 6.1. *Suppose that $\varphi|_A = \text{Id}_A$ and $t \in G$ where $\varphi(t) = t$. If $a \in A$ where $\varphi(at) = bt$, then $a \sim b$.*

Proof. Given the hypotheses, we have

$$a \sim a' = \varphi(a') = \varphi(t^{-1} \cdot at) \sim \varphi(t^{-1}) \cdot \varphi(at) = t^{-1}bt \sim b. \quad \square$$

Recall that F is a set of coset representatives for G/A . In the situation of Lemma 6.1, we will write $\varphi(at) = a^r t$, where $r_a \in F$.

PROPOSITION 6.2. *Suppose that $\varphi|_A = \text{Id}_A$ and $t \in F$ where $\varphi(t) = t$. Then, there is $f \in F$ such that $\varphi(at) = a^f t$ for all $a \in A$.*

Proof. From Lemma 2.1, we see that $\varphi(At) = At$. Now, for $a, b \in A$ we have

$$ab^{-1} = at(bt)^{-1} = \varphi(at \cdot (bt)^{-1}) \sim a^r t \cdot t^{-1}(b^{r_b})^{-1} = a^r (b^{r_b})^{-1}.$$

Thus, there is some $f \in F$ such that $ab^{-1} = (a^r (b^{r_b})^{-1})^f$, so that letting $\alpha = r_a f, \beta = r_b f$, we have

$$a(a^{-1})^\alpha = b(b^{-1})^\beta. \tag{6.1}$$

For $a \in A$, we let C_a denote the circle in \mathbb{E}^2 that contains the origin and that is centred at v_a , and let $S_a = \{v_{a(a^{-1})^f} : f \in F\}$. Then, S_a consists of $|a^G|$ points that lie on C_a . Note that $(0, 0) \in S_a$ for all $a \in A$. Since two distinct circles can meet in at most two points equation (6.1) gives $|S_a \cap S_b| \in \{1, 2\}$.

LEMMA 6.3.

- (i) *If $|S_a \cap S_b| = 1$, then there is $\delta \in F$ such that $a^r a = a^\delta, b^{r_b} = b^\delta$.*
- (ii) *If $(0, 0), v_a, v_b$ are collinear, then there is a $\delta \in F$ such that $a^r a = a^\delta, b^{r_b} = b^\delta$.*

Proof.

- (i) If $|S_a \cap S_b| = 1$, then $S_a \cap S_b = \{(0, 0)\}$ and so from equation (6.1), we get $a(a^{-1})^\alpha = b(b^{-1})^\beta = 1$, so that $a = a^\alpha, b = b^\beta$. From this, we obtain $a^{f^{-1}} = a^r, b^{f^{-1}} = b^{r_b}$, so that we can let $\delta = f^{-1}$.
- (ii) We may assume that $a \neq b$. Since $(0, 0), v_a, v_b$ are collinear the centres of the circles C_a, C_b are on the line through these points. Since $a \neq b$ we have $C_a \neq C_b$;

since $(0, 0)$ is common to C_a and C_b we see that $C_a \neq C_b$ implies $C_a \cap C_b = \{(0, 0)\}$. This gives $|S_a \cap S_b| = 1$, as in (i), which then gives the result. \square

LEMMA 6.4. *Let $b \in A$, $b \notin H$. Then, there is a unique $f = f_b \in F$ such that for all $a \in A$ we have $\varphi(at) = a^f t$, $\varphi(bt) = b^f t$.*

Proof. Recall that $b \notin H$ means that $|b^G| = |F|$. If $(0, 0)$, v_a, v_b are collinear, then the existence of such a $\delta \in F$ follows from Lemma 6.3 (ii), while the uniqueness follows from the fact that $b \notin H$.

Now, assume that $(0, 0)$, v_a, v_b are not collinear. If $|S_a \cap S_b| = 1$, then by Lemma 6.3 (i), there is $\delta \in F$ such that $a^{\delta a} = a^\delta$, $b^{\delta b} = b^\delta$, and the fact that δ is unique follows from $b \notin H$. So, now assume that $|S_a \cap S_b| = 2$. Then, by Lemma 6.3 (ii), there is some $f \in F$ such that for all $k \in \mathbb{N}$ we have $(b^k)^f = (b^k)^{r_{fk}}$. Since $b \notin H$ this element f is unique. Since S_a is finite there is some $k \in \mathbb{N}$ such that $|S_a \cap S_{b^k}| = 1$; then we have $h, h' \in F$ with $b^{r_b} = b^h$, $b^{r_{b^k}} = (b^k)^h$ and $b^{r_{b^k}} = (b^k)^{h'}$, $a^{r_a} = a^{h'}$. Since $b \notin H$ we again see that h, h' are unique, so that $h = h'$. Thus, we have $a^{r_a} = a^h$ and $b^{r_b} = b^h$, as required. \square

Now, let $a, b, c \in A$ where $b \notin H$. Then, by Lemma 6.4 there are unique $f, h \in F$ such that $\varphi(at) = a^f t$, $\varphi(bt) = b^f t$ and $\varphi(bt) = b^h t$, $\varphi(ct) = c^h t$. Since f, h are unique and $\varphi(bt) = b^f t$, $\varphi(bt) = b^h t$ we must have $f = h$; it follows (by fixing a and varying c) that for all $d \in A$ we must have $\varphi(dt) = d^f t$ for this value of $f \in F$ that is completely determined by $b \notin H$. This concludes the proof of Proposition 6.2, and so completes Step (5). \square

For the remainder of this section, we assume that $\varphi|_A = \text{Id}_A$ and that $\varphi(f) = f$ for all $f \in F$. We also assume that for all $f \in F$ there is r_f such that $\varphi(af) = a^{r_f} f$ for all $a \in A$. Our goal will be to reduce to the situation where $\varphi(af) = af$ for all $f \in F$, $a \in A$. We start with the following lemma.

LEMMA 6.5. *For all $a \in A$ and all reflections or glide reflections $r \in F$, we have $\varphi(ar) \in \{ar, a^r r\}$.*

Proof. Since $ar \sim \varphi(a \cdot r)$ we may apply Lemma 5.3. This gives us $\varphi(ar) = a^t r \in K_r ar$ for some $t \in F$. This implies that $a^{-1} a^t \in K_r$ for all $a \in A$. But $a^{-1} a^t = (a, t)$ and $K_t = \{(a, t) : a \in A\}$ shows that $K_t \subset K_r$. From Lemma 2.6 (iii), it follows that either $K_t = \{1\}$ or $K_t = K_r$, from which we see that either $t = 1$ or $t = r$. \square

The following applies to the groups of type **p4mm**, **p4mg**, **p31m**, **p3m1**, **p6mm**.

THEOREM 6.6. *Let G be a wallpaper group having a non-trivial rotation $\rho \neq \rho_\pi$, and distinct reflection or glide reflection cosets $Ar, Ar\rho$. Suppose we have $\varphi|_A = \text{Id}_A$, $\varphi(r) = r$, $\varphi(r\rho) = r\rho$, and $\varphi(\rho) = \rho$. Then, $\varphi|_{Ar} = \text{Id}_{Ar}$.*

Proof. Recall from Lemma 2.6 that for any of these five groups, we have

$$(ar)^G \cap Ar \subseteq K_r ar \cup K_r (a\beta_1\beta_2)^{-1} r,$$

where $\beta_1\beta_2 = xy$ when G is of type **p4mg**, but is trivial otherwise. Suppose to the contrary that $\varphi|_{Ar} \neq \text{Id}_{Ar}$. Then, by Lemma 6.5, we must have $\varphi(ar) = a^r r$ for all $a \in A$. Now, $r\rho$ is also a reflection or a glide reflection, and so we have

$$a^r r \cdot \rho \sim \varphi(ar \cdot \rho) = \varphi(a \cdot r\rho) \sim ar\rho.$$

Thus, for all $a \in A$ we must have (i) $a^r a^{-1} \in K_{r\rho}$, or (ii) $a^r a \in K_{r\rho}(\beta_1\beta_2)^{-1}$. Since r and $r\rho$ are distinct reflections, case (i) is only true for a which satisfy $a^r a^{-1} = 1$. Any such $a \in A$ must lie on $L(r)$.

Since $\rho \neq \rho_\pi$, $L(r)$, and $K_{r\rho}$ are not parallel. Hence, $L(r) \cap K_{r\rho}(\beta_1\beta_2)^{-1}$ is exactly one point. (If G is of type **p4mg** and $r = \rho\gamma$ this point is xy^{-1} , when $r = \rho^2\gamma$ it is y^{-2} , and when $r = \rho^3\gamma$ it is $x^{-1}y^{-1}$. In all other cases, this point is trivial.) Thus, case (ii) is only true for $a \in L^\perp(r)y^{-1}$ or $a \in L^\perp(r)$. This shows that the only $a \in A$ that satisfy $a^r r\rho \sim ar\rho$ are in the union of two lines (either $L(r) \cup L^\perp(r)$ or $L(r) \cup L^\perp(r)y^{-1}$) which is not all of A . Thus, $\varphi(ar) = a^r r$ is not possible. We conclude that $\varphi|_{Ar} = Id$. \square

The following applies to the eight groups **c2mm**, **p2mm**, **p2mg**, **p2gg**, **p4mm**, **p4mg**, **p31m**, **p3m1**, and **p6mm**.

THEOREM 6.7. *Let G be a wallpaper group having at least two cosets that are reflection or glide reflection cosets. Suppose we have $\varphi|_A = Id_A$ and $\varphi|_{A\sigma} = Id$ for all reflection cosets $A\sigma$ as well as $\varphi|_{A\gamma} = Id$ for all glide reflection cosets $A\gamma$. Assume $\varphi(t) = t$ for all $t \in F$. Then, $\varphi = Id$.*

Proof. Let Ar denote a reflection or glide reflection coset. Let $A\rho^k$ be a rotation coset. We know by Proposition 6.2 that there is some $t \in F$ such that $\varphi(a\rho^k) = a^t \rho^k$ for all $a \in A$. By hypothesis, we have $\varphi(ra^{-1}) = ra^{-1}$ which gives

$$r \cdot \rho^k \sim \varphi(r \cdot \rho^k) = \varphi(ra^{-1} \cdot a\rho^k) \sim ra^{-1} \cdot a^t \rho^k = (a^{-1}a^t)r^{-1} \cdot r\rho^k.$$

This gives $(a^{-1}a^t)r^{-1} \in K_{r\rho^k} \cup K_{r\rho^k}(\beta_1\beta_2)^{-1}$ for all $a \in A$. Write $\beta_r = ((\beta_1\beta_2)^{-1})^r$ so that $(a, t) \in K_{\rho^{k_r}} \cup K_{\rho^{k_r}\beta_r}$ for all $a \in A$. Now, the element $t \in F$ depends upon k and not on r . Then, the fact that we have at least two reflection or glide reflection cosets, say $r, r' \in F$ means that $(a, t) \in (K_{\rho^{k_r}} \cup K_{\rho^{k_r}\beta_r}) \cap (K_{\rho^{k_{r'}}} \cup K_{\rho^{k_{r'}}\beta_{r'}})$. This intersection is trivial if neither r nor r' are glide reflections. Otherwise, this intersection is finite and therefore bounded. Thus, there exists a $t \in F$ such that for all $a \in A$, we must have (a, t) in this small intersection. This can only be true if t is trivial. So, $\varphi(a\rho^k) = a\rho^k$. This completes the proof. \square

The group of type c2mm By Lemma 6.5, there are $s \in \{1, \sigma\}, u \in \{1, \rho\sigma\}$, where $\varphi(a\sigma) = a^s\sigma, \varphi(a\rho\sigma) = a^u\rho\sigma$ for all $a \in A$. If we have $u = \rho\sigma$, we may compose with $I_\rho \circ \iota$ so that we have $u = 1$. If $s = \sigma$, then we have

$$xy\rho \sim \varphi(xy \cdot \rho) = \varphi(x\sigma \cdot x\rho\sigma) \sim y\sigma x\rho\sigma = y^2\rho.$$

Since $xy\rho \approx y^2\rho$, this is not possible so we must have $s = 1$. Theorem 6.7 now shows that $\varphi(a\rho) = a\rho$ for all $a \in A$, and so G has trivial weak Cayley table group. \square

The group of type p2mm

By Lemma 6.5, there are $s \in \{1, \sigma\}, u \in \{1, \rho\sigma\}$, where $\varphi(a\sigma) = a^s\sigma, \varphi(a\rho\sigma) = a^u\rho\sigma$ for all $a \in A$. If we have $s = \sigma$, we may compose with τ so that we have $s = 1$.

Now, note that τ^ψ is the identity on $A \cup A\sigma$ and conjugates elements of $A\rho \cup A\rho\sigma$ by $\rho\sigma$. Thus, if we now have $u = \rho\sigma$, then we compose with τ^ψ to obtain $s = 1, u = 1$. Theorem 6.7 now shows that $\varphi(a\rho) = a\rho$ for all $a \in A$. This shows that groups G of type **p2mm** have weak Cayley table group generated by $\mathcal{W}_0(G)$ and τ . \square

The group of type **p2mg** By Lemma 6.5, we know that $\varphi(a\sigma) = a^s\sigma$, and $\varphi(a\rho\sigma) = a^u\rho\sigma$ for some $s \in \{1, \sigma\}$, $u \in \{1, \rho\sigma\}$, for all $a \in A$. Now, if $s = \sigma$, then

$$\rho\sigma = \varphi(\rho\sigma) = \varphi(y\sigma \cdot \rho) \sim y^s\sigma\rho = y^s y^{-1}\rho\sigma = y^{-2}\rho\sigma,$$

which contradicts Lemma 2.6. Thus, $s = 1$.

Next, if $u = \rho\sigma$, then

$$x\sigma = \varphi(x\sigma) = \varphi(x\sigma\rho \cdot \rho) = \varphi(xy^{-1}\rho\sigma \cdot \rho) \sim (xy^{-1})^u\rho\sigma\rho = (xy^{-1})^u y\sigma,$$

so that $(xy^{-1})^u y = x^{-1}y^{-1}y = x^{-1}$ has the form xy^{2k} or $x^{-1}y^{2k}y^{-1}$, both of which are not possible. Thus, $u = 1$. Theorem 6.7 now shows that $\varphi(a\rho) = a\rho$ for all $a \in A$, and so G has trivial weak Cayley table group. \square

The group **p2gg** By Proposition 6.2, there are $s, u \in F = \{1, \rho, \gamma, \rho\gamma\}$ such that $\varphi(a\gamma) = a^s\gamma$, and $\varphi(a\rho\gamma) = a^u\rho\gamma$ for all $a \in A$. By Lemma 6.5, $s \in \{1, \gamma\}$, $u \in \{1, \rho\gamma\}$. Let $X = \langle x^2 \rangle$, $Y = \langle y^2 \rangle$. Thus, for G a group of type **p2gg** we have

LEMMA 6.8. Let $a, b \in A$. Then,

- (i) $a\gamma \sim b\gamma$ if and only if $ab^{-1} \in Y$ or $ab \in x^{-1}yY$.
- (ii) $a\rho\gamma \sim b\rho\gamma$ if and only if $ab^{-1} \in X$ or $ab \in xy^{-1}X$.
- (iii) If $v \in F$ and $a \in A$, then $aa^v \notin x^{-1}yY \cup xy^{-1}X$.

Proof. (i) and (ii) follow from the presentation (see Lemma 2.6). For (iii), note that for $a = x^i y^j$ we have $a^\rho = a^{-1}$, $a^\gamma = x^i y^{-j}$, $a^{\rho\gamma} = x^{-i} y^j$, which gives (iii). \square

Now, for all $a \in A$, we have

$$\begin{aligned} (a^{-1})^{\gamma u} \rho\gamma &= a^{\rho\gamma u} \rho\gamma = \varphi(a^{\rho\gamma} \rho\gamma) = \varphi((a\rho\gamma)^{\rho\gamma}) \sim \varphi(a\rho\gamma) \sim \varphi(\rho\gamma \cdot a) \\ &= \varphi(\rho a^{\gamma^{-1}} \gamma) = \varphi(\rho \cdot a^\gamma \gamma) \sim \rho(a^\gamma)^s \gamma = a^{\gamma s \rho} \rho\gamma = (a^{-1})^{\gamma s} \rho\gamma. \end{aligned}$$

From Lemma 6.8 (ii), (iii), we thus get $a^s(a^{-1})^u \in X$ for all $a \in A$. Let $a = x^i y^j$, and recall that $u \in \{1, \rho\gamma\}$. We show that this implies $s = 1$, for if $s = \gamma$, then the possibility $u = 1$ gives $a^s(a^{-1})^u = (x^i y^{-j})(x^{-i} y^{-j}) = y^{-2j} \notin X$; while $u = \rho\gamma$ gives $a^s(a^{-1})^u = (x^i y^{-j})(x^i y^{-j}) = x^{2i} y^{-2j} \notin X$. Thus, we have $s = 1$.

Similarly, for all $a \in A$ we have

$$a^\gamma \gamma = \varphi(a^\gamma \gamma) = \varphi(\rho \cdot a^{\rho\gamma} \rho\gamma) \sim \rho a^{\rho\gamma u} \rho\gamma = a^{\gamma u} \gamma,$$

from which we obtain $(a^{-1})^\gamma a^{\gamma u} \in Y$ for all $a \in A$; or $a^{-1} a^u \in Y$ for all $a \in A$. However, if $u = \rho\gamma$, then $x^{-1} x^{\rho\gamma} = x^{-2}$, a contradiction. Thus, $u = 1$. \square

Theorem 6.7 now shows that $\varphi(a\rho) = a\rho$ for all $a \in A$, and so any G of type **p2gg** has trivial weak Cayley table group. \square

Using the above cases, Theorems 6.6 and 6.7, we see that

COROLLARY 6.9. Each group of type **c2mm**, **p2mg**, **p2gg**, **p4mm**, **p31m**, **p3m1**, **p6mm** has trivial weak Cayley table group. \square

The group **p2**

We have $\varphi|_A = \text{Id}_A$ and that there is some $t \in F$ such that $\varphi(a\rho) = a^t\rho$ for all $a \in A$. If $t = 1$, then $\varphi = \text{id}$, while if $t = \rho$, then $\varphi = I_\rho t$, and we have proved

COROLLARY 6.10. The group of type **p2** has trivial weak Cayley table group.

LEMMA 6.11. *Let G be one of the groups of type **p3**, **p4**, **p6** (respectively), and let $d = 3, 4, 6$ (respectively). Suppose that $\rho \in F$ is a rotation of order d , and that there are $s, t \in F$ with $\varphi(a\rho) = a^t \rho$, $\varphi(a\rho^{-1}) = a^s \rho^{-1}$ for all $a \in A$. Then, $s = t$.*

Proof. Given $a \in A$, let $a\rho^{d-1} = (b\rho)^{-1}$, $b \in A$. Then, solving gives $b = (a^{-1})^{\rho^{d-1}}$. Since $(t, \rho) = 1$ we have

$$\varphi(a\rho^{d-1}) = \varphi(b\rho)^{-1} = (b^t \rho)^{-1} = \rho^{-1}(b^{-1})^t = \rho^{-1}(a^{\rho^{-1}})^t = a^t \rho^{d-1}. \quad \square$$

*The group **p3***

We know that $\varphi|_A = \text{Id}_A$ and that there is $t \in F = \{1, \rho, \rho^2\}$ such that $\varphi(a\rho) = a^t \rho$ for all $a \in A$. By Lemma 6.11, we have $\varphi(a\rho^2) = a^t \rho^2$ for all $a \in A$. Thus, φ is a power of the weak Cayley table isomorphism τ defined in Section 2, and we have proved

COROLLARY 6.12. *The group of type **p3** has weak Cayley table group generated by $\mathcal{W}_0(G)$ and τ .*

*The group **p4***

We know that $\varphi|_A = \text{Id}_A$ and that there is some $t \in F = \langle \rho \rangle$ such that $\varphi(a\rho) = a^t \rho$ for all $a \in A$. By Lemma 6.11, we have $\varphi(a\rho^3) = a^t \rho^3$ for all $a \in A$. If $t = 1$, then $\varphi|_{A\rho \cup A\rho^3} = \text{Id}$. If $t = \rho^k$, then composing with μ_ρ^{-k} reduces to the case where $\varphi|_{A\rho \cup A\rho^3} = \text{Id}$, also.

Now, there is $u \in F$ such that $\varphi(a\rho^2) = a^u \rho^2$ for all $a \in A$. If $u = \rho^2$, then we can compose with τ_{ρ^2} to obtain $\varphi = \text{Id}$. Thus, the remaining case reduces to $u = \rho$. In this situation, one has $\varphi(x^2 \rho^2) = (x^2)^\rho \rho^2 = y^2 \rho^2$, while $\varphi(x \cdot x\rho^2) \sim x \cdot x^\rho \rho^2 = x \cdot y\rho^2$, giving $y^2 \rho^2 \sim xy\rho^2$, a contradiction. This gives

COROLLARY 6.13. *The group of type **p4** has weak Cayley table group generated by $\mathcal{W}_0(G)$ and $\tau_x, \tau_y, \tau_{\rho^2}, \mu_x, \mu_y, \mu_\rho$.*

*The group **p6***

We know that $\varphi|_A = \text{Id}_A$ and that there is some $t \in F = \langle \rho \rangle$ such that $\varphi(a\rho) = a^t \rho$ for all $a \in A$. By Lemma 6.11, we have $\varphi(a\rho^5) = a^t \rho^5$ for all $a \in A$. Similarly, there is $s \in F$ such that if $\varphi(a\rho^2) = a^s \rho^2$ for all $a \in A$, then we also have $\varphi(a\rho^4) = a^s \rho^4$ for all $a \in A$. Last, assume that $\varphi(a\rho^3) = a^u \rho^3$ for all $a \in A$.

Now, using an element of $\langle \tau_{\rho^2} I_{\rho^2}, \mu_{\rho^3} I_{\rho^3} \rangle$, we can reduce to the case where $t = 1$ while preserving the conditions $\varphi|_A = \text{Id}_A$, $\varphi(\rho^k) = \rho^k$, $0 \leq k < 6$.

LEMMA 6.14. *Assume that for some $t \in \langle \rho \rangle$, we have $a^{-1}a^t \in K_{\rho^2} \cap K_{\rho^3}$ for all $a \in A$. Then, $t = 1$.*

Proof. If $a = y$, then $\{a^{-1}a^v : v \in \langle \rho \rangle\} = \{1, x^{-1}, x^{-1}y^{-1}, xy^{-2}, xy^{-1}, y^{-2}\}$. Now,

$$K_{\rho^2} = \langle (x, \rho^2), (y, \rho^2) \rangle = \langle x^{-2}y, xy \rangle \text{ and } K_{\rho^3} = \langle x^2, y^2 \rangle,$$

so that $K_{\rho^2} \cap K_{\rho^3} = \langle x^2y^2, x^2y^{-4} \rangle$. But the only element of $\{1, x^{-1}, x^{-1}y^{-1}, xy^{-2}, xy^{-1}, y^{-2}\}$ that is also in $K_{\rho^2} \cap K_{\rho^3}$ is the identity, and so we are done. \square

Now, we have the following:

- (i) $a^s \rho^2 = \varphi(a\rho^2) = \varphi(a\rho \cdot \rho) \sim a\rho^2$ gives $a^{-1}a^s \in K_{\rho^2}$;
- (ii) $a^s \rho^4 = \varphi(a\rho^3 \cdot \rho) \sim a^u \rho^4$ gives $(a^{-1})^s a^u \in K_{\rho^2}$;

- (iii) $a^u \rho^3 = \varphi(a\rho^3) = \varphi(a\rho \cdot \rho^2) \sim a\rho^3$ gives $a^{-1}a^u \in K_{\rho^3}$;
- (iv) $a^u \rho^3 = \varphi(a\rho^2 \cdot \rho) \sim a^s \rho^3$ give $a^u(a^{-1})^s \in K_{\rho^3}$.

Since K_{ρ^2}, K_{ρ^3} are subgroups (i) and (ii) give $a^{-1}a^u \in K_{\rho^2}$ for all $a \in A$. Thus, by (iii), we have $a^{-1}a^u \in K_{\rho^2} \cap K_{\rho^3}$, so that Lemma 6.14 tells us that $u = 1$.

Similarly, (iii) and (iv) give $a^{-1}a^s \in K_{\rho^3}$, which combined with (i) gives $a^{-1}a^s \in K_{\rho^2} \cap K_{\rho^3}$, from which we obtain $s = 1$, using Lemma 6.14. Thus, we now have $\varphi = \text{Id}_G$. This gives:

COROLLARY 6.15. *The group of type **p6** has weak Cayley table group generated by $\mathcal{W}_0(G)$ and $\tau_\alpha, \tau_\beta, \tau_{\rho^2}, \mu_{\rho^3}, \mu_{x^2}, \mu_{y^2}$.*

The groups **clm1, plm1, plg1**

Here, we have $t \in F$ such that $\varphi(a\sigma) = a'\sigma$. If $t = 1$, then we are done. So, assume that $t \neq 1, t \in F$. In each such group, we have the automorphism $\psi : (x, y, r) \mapsto (x^{-1}, y^{-1}, r^{-1})$, where r is σ, σ, γ , respectively, for the three groups. Then, composing ψ with ι gives φ , and we have proved:

COROLLARY 6.16. *The groups of type **clm1, plm1, plg1** have trivial weak Cayley table groups.*

7. The groups of type **p4mg.** We note that much of what we have done in other cases does not apply to a group of type **p4mg**.

Here, we start by assuming $\varphi|_A = \text{Id}_A$ and $\varphi(At) = At$ for all $t \in F$. Let

$$U = \langle xy, x^2 \rangle, \quad X = \langle x^2 \rangle, \quad Y = \langle y^2 \rangle, \quad V = \langle xy \rangle, \quad W = \langle xy^{-1} \rangle.$$

In the next two results, we list the conjugacy classes and involutions:

LEMMA 7.1. *Let $a = x^i y^j \in A$. Then,*

$$\begin{aligned} (x^i y^j)^G &= \{x^{\pm i} y^{\pm j}, x^{\pm i} y^{\pm j}\}; \\ (a\rho)^G &= Ua\rho \cup Uax\rho^3; \\ (a\rho^2)^G &= Ua\rho^2; \\ (a\rho^3)^G &= Ua\rho^3 \cup Uax\rho; \\ (a\gamma)^G &= Ya\gamma \cup Ya^{-1}x^{-1}y\gamma \cup Xa^\rho x\rho^2\gamma \cup Xa^\rho y^{-1}\rho^2\gamma; \\ (a\rho\gamma)^G &= Va\rho\gamma \cup Va^{-1}\rho\gamma \cup Wy^{-1}a^\rho\rho^3\gamma \cup Wx^{-1}(a^{-1})^\rho\rho^3\gamma; \\ (a\rho^2\gamma)^G &= Xa\rho^2\gamma \cup Xa^{-1}xy^{-1}\rho^2\gamma \cup Ya^\rho y\gamma \cup Ya^\rho x^{-1}\gamma; \\ (a\rho^3\gamma)^G &= Wa\rho^3\gamma \cup Wx^{-1}y^{-1}a^{-1}\rho^3\gamma \cup Vxa^\rho\rho^3\gamma \cup Vx^{-1}a^\rho\rho\gamma. \end{aligned}$$

LEMMA 7.2. *Any involution in G is in one of the cosets:*

$$(a) A\rho^2; \quad (b) V\rho\gamma; \quad (c) x^{-1}W\rho^3\gamma.$$

Now, assume that $\varphi(\rho) = a\rho, a \in A$. Note that $I_x(\rho) = (xy)^{-1}\rho, I_y(\rho) = xy^{-1}\rho$, so that by acting by some $I_b, b \in A$, we can assume $\varphi(\rho) \in \{\rho, x\rho\}$. If we have $\varphi(\rho) = x\rho$, then composing with the automorphism ψ_1 we see that $\varphi(\rho) = \rho$. Now, let $\varphi(\gamma) = a\gamma$.

Lemma 5.2 gives $a \in \langle y \rangle$. Note that $(\rho\gamma)^2 = 1$ and Lemma 7.2 implies that $\varphi(\rho\gamma) \sim \rho\gamma$. So, Lemma 7.2 (b) with $\varphi(\rho \cdot \gamma) \sim \rho a\gamma$ tells us $a^{\rho^3} \in V$, so $a \in V^\rho = W$. We conclude that $a \in \langle y \rangle \cap W$ is trivial. Thus, we now have $\varphi(\rho) = \rho$, $\varphi(\gamma) = \gamma$, so that $\varphi(\rho^3) = \rho^3$.

Now, let $\varphi(\rho^2\gamma) = b\rho^2\gamma$, $b \in A$. Lemma 5.2 gives $b \in \langle x \rangle$. Then, using Lemma 7.2 (b) again, $(\rho\gamma)^2 = 1$ and $\varphi(\rho\gamma) = \varphi(\rho^{-1} \cdot \rho^2\gamma) \sim \rho^{-1}b\rho^2\gamma = b^\rho\rho\gamma$ gives $b^\rho \in V$, hence $b \in W \cap \langle x \rangle$. Thus, $b = 1$ and $\varphi(\rho^2\gamma) = \rho^2\gamma$.

Assume that $\varphi(\rho^3\gamma) = c\rho^3\gamma$, $c \in A$. By Lemma 5.2, we have $c \in W$. Note that $\gamma = \varphi(\rho \cdot \rho^3\gamma) \sim \rho c\rho^3\gamma = c^{\rho^3}\gamma$ implies that $c^{\rho^3} \in Y \cup Yx^{-1}y$, and so $c \in X \cup Xxy^{-1}$. But also, $\rho^2\gamma = \varphi(\rho^3 \cdot \rho^3\gamma) \sim c^\rho\rho^2\gamma$ implies that $c^\rho \in X \cup Xxy^{-1}$, so that $c \in Y \cup Yx^{-1}y$. As $c \in W \cap (X \cup Xxy^{-1}) \cap (Y \cup Yx^{-1}y)$, we see that $c = 1$, and so $\varphi(\rho^3\gamma) = \rho^3\gamma$.

The above shows that $\varphi(v) = v$ for all $v \in F \setminus \{\rho^2, \rho\gamma\}$. Thus, by Proposition 6.2, for $v \in F \setminus \{\rho^2, \rho\gamma\}$ there is $t_v \in F \setminus \{\rho^2, \rho\gamma\}$ such that for all $a \in A$ we have $\varphi(av) = a^{t_v}v$. Let $t, u, w \in F$ satisfy $\varphi(a\rho) = a^t\rho$, $\varphi(a\gamma) = a^u\gamma$, and $\varphi(a\rho^{-1}\gamma) = a^w\rho^{-1}\gamma$. By Lemma 6.5, we have $u \in \{1, \gamma\}$ and $w \in \{1, \rho^{-1}\gamma\}$.

Now, let $\varphi(\rho\gamma) = a\rho\gamma$, $a \in A$. Then, Lemma 7.2 (b) gives $a \in V$, while $\varphi(\rho \cdot \rho\gamma) \sim a^{\rho^{-1}}\rho^2\gamma$ tells us $a \in Y \cup Yxy$. Therefore, we know $a \in \{1, xy\}$. However, if we write $t^{\rho\gamma} = s$, and assume $\varphi(\rho\gamma) = xy\rho\gamma$, then

$$\begin{aligned} x^2\gamma &\sim \varphi(x^3 \cdot x^{-1}\gamma) = \varphi(x^3\rho\gamma\rho) = \varphi(\rho\gamma \cdot (x^3)^{\rho\gamma}\rho) \\ &\sim xy\rho\gamma(x^3)^{\rho\gamma t} \rho = xy(x^3)^s\rho\gamma\rho = xy(x^3)^s\gamma^{-1} = y(x^3)^s\gamma. \end{aligned}$$

This gives a contradiction unless $s \in \{\rho^2, \rho^2\gamma\}$ which corresponds to $t \in \{\rho^2, \gamma\}$. But since $(y^{-1})^{\rho^2} = (y^{-1})^\gamma = y$ and $(x^{-1})^\gamma = x^{-1}$ for any choice of $t \in \{\rho^2, \gamma\}$ and $u \in \{1, \gamma\}$, we have

$$\rho\gamma \sim \varphi(\rho\gamma) = \varphi(y^{-1}\rho \cdot x^{-1}\gamma) \sim (y^{-1})^t\rho(x^{-1})^u\gamma = y\rho(x^{-1})\gamma = y^2\rho\gamma,$$

which is a contradiction. Thus, $\varphi(\rho\gamma) = \rho\gamma$.

Now, we have $\varphi(\rho^k\gamma) = \rho^k\gamma$ for $k \in \{0, 1, 2, 3\}$, and we may apply Theorem 6.6 four times to get $\varphi = Id$ on both reflection cosets and both glide reflection cosets.

Let $\varphi(\rho^2) = b\rho^2$, $b \in A$. Since $\varphi(\rho^2 \cdot \gamma) \sim b\rho^2\gamma$ we see $b \in X \cup Xxy^{-1}$. By considering $\varphi(\rho^2 \cdot \rho^2\gamma) \sim b\gamma$, we know $b \in Y \cup Yx^{-1}y$. Thus, $b \in \{1, x^{-1}y^{-1}\}$. Assume $b = x^{-1}y^{-1}$. Since $\rho^2\gamma = x^{-1}y\gamma\rho^2$ this gives

$$\rho^2\gamma = \varphi(x^{-1}y\gamma \cdot \rho^2) \sim x^{-1}y\gamma x^{-1}y^{-1}\rho^2 = x^{-2}y^2\gamma\rho = x^{-1}y\rho^2\gamma,$$

which is a contradiction according to Lemma 7.1. Thus, $\varphi(\rho^2) = \rho^2$. We may now apply Theorem 6.7 to obtain $\varphi = Id_G$. Thus, the groups of type **p4mg** have trivial weak Cayley table group. □

8. Non-trivial cases. Thus, we have found generators for $\mathcal{W}(G)$ for each wallpaper group G . Presentations for $\mathcal{W}(G)$ can now be obtained; however, in the trivial cases one can find such a presentation in [4]. We, thus, only give presentations in the non-trivial cases:

Groups G of type p3 Here, we have an order 6 automorphism $\psi_1 : x \mapsto y, y \mapsto x^{-1}y, \rho \mapsto \rho$ and an order 2 automorphism $\psi_2 : x \mapsto x, y \mapsto xy^{-1}, \rho \mapsto \rho^{-1}$. Then,

$$\begin{aligned} \mathcal{W} &= ((\langle \psi_x, \psi_y \rangle \times \langle \tau \rangle) \times (\langle \psi_1 \rangle \times \langle \psi_2 \rangle)) \times \langle \iota \rangle \\ &\cong ((\mathbb{Z}^2 \rtimes C_3) \rtimes D_{12}) \times C_2. \end{aligned} \quad (8.1)$$

Note that $I_\rho = \psi_1^2, I_x = \psi_x^{-1}\psi_y^{-1}, I_y = \psi_x\psi_y^{-2}$. Here, the actions for the semi-direct products are determined by

$$\psi_x^\tau = \psi_y^{-1}, \psi_y^\tau = \psi_x\psi_y^{-1}$$

and

$$\psi_x^{\psi_1} = \psi_x\psi_y^{-1}, \psi_x^{\psi_2} = \psi_y, \psi_y^{\psi_1} = \psi_y^{\psi_2} = \psi_x, \tau^{\psi_2} = \tau^{-1}.$$

Here, we note that $\mathcal{W}_0(G)$ is generated by the $\psi_x, \psi_y, \psi_1, \psi_2, \iota$.

Groups G of type p4 Let $\psi_1 : x \mapsto x, y \mapsto y^{-1}, \rho \mapsto \rho^{-1}$. Then, we have

$$\begin{aligned} \mathcal{W}(G) &= (\langle \tau_x, \tau_y, \tau_{\rho^2}, \mu_x, \mu_y, \mu_\rho \rangle \times (\langle \psi_x, \psi_y \rangle \times \langle I_\rho, \psi_1 \rangle)) \times \langle \iota \rangle \\ &\cong [(\mathbb{Z}^2 \rtimes C_2) \times (\mathbb{Z}^2 \rtimes C_4)] \times (\mathbb{Z}^2 \rtimes D_8) \times C_2 \end{aligned} \quad (8.2)$$

Here, $\mathbb{Z}^2 \rtimes C_2 = \langle \tau_x, \tau_y, \tau_{\rho^2} \rangle$ where $\tau_{\rho^2}^{\psi_x} = \tau_{\rho^2}\tau_x^{-1}\tau_y, \tau_{\rho^2}^{\psi_y} = \tau_{\rho^2}\tau_x\tau_y^{-1}$, gives this semi-direct product, and $\mathbb{Z}^2 \rtimes C_4 = \langle \mu_x, \mu_y, \mu_\rho \rangle$, where $\mu_x^{\mu_\rho} = \mu_y^{-1}, \mu_y^{\mu_\rho} = \mu_x$ gives this semi-direct product.

Next, the subgroup $\mathbb{Z}^2 \rtimes D_8$ is $\text{Aut}(G) = \langle \psi_x, \psi_y \rangle \times \langle I_\rho, \psi_1 \rangle$. To see that $\mathbb{Z}^2 \rtimes D_8$ acts on $\mathbb{Z}^2 \rtimes C_2$ and $\mathbb{Z}^2 \rtimes C_4$, we note that for $\psi \in \text{Aut}(G)$, we have

$$(\tau_u)^\psi = \tau_{\psi^{-1}(u)}, \quad (\mu_u)^\psi = \mu_{\psi^{-1}(u)}.$$

Thus, for G of type **p4**, we can take $\mathcal{N}(G) = \langle \tau_x, \tau_y, \tau_{\rho^2}, \mu_x, \mu_y, \mu_\rho \rangle$ in Theorem 1.1.

In this case, $\mathcal{W}_0(G) = (\langle \psi_x, \psi_y \rangle \times \langle I_\rho, \psi_1 \rangle) \times \langle \iota \rangle$.

Groups G of type p6 Let $\alpha = xy, \beta = xy^{-2}$ and $\psi : x \mapsto y, y \mapsto x, \rho \mapsto \rho^{-1}$. Here, we have

$$\begin{aligned} \mathcal{W}(G) &= (\langle \mu_{x^2}, \mu_{y^2}, \mu_{\rho^3} \rangle \times \langle \tau_\alpha, \tau_\beta, \tau_{\rho^2} \rangle \times (\langle I_x, I_y \rangle \times \langle I_\rho, \psi \rangle)) \times \langle \iota \rangle \\ &\cong [(\mathbb{Z}^2 \rtimes C_2) \times (\mathbb{Z}^2 \rtimes C_3)] \times (\mathbb{Z}^2 \rtimes D_{12}) \times C_2, \end{aligned} \quad (8.3)$$

with the details being similar to the last case.

Here, we have $\mathcal{W}_0(G) = (\langle I_x, I_y \rangle \times \langle I_\rho, \psi \rangle) \times \langle \iota \rangle$.

Groups G of type p2mm We note that if $G = H \times J$ is a direct product of non-abelian groups, then G always has a non-trivial weak Cayley table isomorphism: just take $\varphi \times \psi$, where $\varphi \in \text{Aut}(H)$ and $\psi \in \mathcal{W}(J)$ is an anti-automorphism. A group of type **p2mm** is a direct product. In this case, we have

$$\begin{aligned} \mathcal{W}(G) &= ((\langle \psi_{1,0}, \psi_{0,1} \rangle \times \langle \tau, \tau^\psi \rangle) \times \langle I_\rho, I_\sigma, \psi \rangle) \times \langle \iota \rangle \\ &\cong ((\mathbb{Z}^2 \rtimes (C_2 \times C_2)) \rtimes D_8) \times C_2. \end{aligned} \quad (8.4)$$

We note that equations (8.1)–(8.4) determine $\mathcal{W}_0(G)$ for each of the groups with non-trivial $\mathcal{W}(G)$.

Here, we have $\mathcal{W}_0(G) = (\langle \psi_{1,0}, \psi_{0,1} \rangle \times \langle I_\rho, I_\sigma, \psi \rangle) \times \langle \iota \rangle$. □

9. The semi-direct product cases: $H = A \rtimes C_p$, where A is abelian. So, let $H = A \rtimes_{\theta} C_2$, where $A = \langle a_1, \dots, a_n \rangle \cong \mathbb{Z}^n$ and $\theta \in \text{Aut}(A)$. Let $C_2 = \langle r \rangle$ and $Z = Z(H)$. We write elements of H as ar^{ε} , $a \in A$, $\varepsilon \in \{0, 1\}$. We also assume that H is not abelian. This allows us to assume that $a_i \notin Z(H)$ for all $1 \leq i \leq n$, since if $a_i \in Z(H)$ and $a_j \notin Z(H)$, then we just replace a_i by $a_i a_j$. We further note that $a \in A$ is in $Z(H)$ if and only if $a^r = a$.

Let $A_i = \langle a_i \rangle$, $1 \leq i \leq n$.

LEMMA 9.1.

- (i) For $a \in A$, we have $a^H = \{a, a^r\}$.
- (ii) For $h = ar \in H \setminus A$, we have $h^H = H'h$.

Proof. (i) is clear since A is abelian of index 2.

For (ii), we note that $r^a = a^{-1}a^r r$, so that for $b \in A$ we have $(br)^{a^m} = b(a^{-1}a^r)^m r$ for all $m \in \mathbb{Z}$. Let

$$\beta_i = a_i^{-1} a_i^r \in H', \quad 1 \leq i \leq n.$$

Let $B_i = \langle \beta_i \rangle \leq H'$, $1 \leq i \leq n$, and note that

$$(a_i^{-1} a_i^r)^r = (a_i^r)^{-1} a_i = (a_i^{-1} a_i^r)^{-1} \in B_i.$$

Also, for $a \in A$, we have $(ar)^{A_i} = a B_i r$ and so $(ar)^A = a \langle B_1, \dots, B_n \rangle r$. Also,

$$\begin{aligned} (ar)^{A^r} &= (\langle B_1, \dots, B_n \rangle ar)^r = \langle B_1, \dots, B_n \rangle^r a^r r = \langle B_1, \dots, B_n \rangle a^r r \\ &= \langle B_1, \dots, B_n \rangle (a^r a^{-1}) \cdot ar. \end{aligned}$$

Here, we note that $a^r a^{-1} \in H'$ and the result will follow upon showing that $H' = \langle B_1, \dots, B_n \rangle$. Now, for $b \in A$, we have

$$(a_i, br) = a_i^{-1} r b^{-1} a_i b r = a_i^{-1} r a_i r = a_i^{-1} a_i^r \in B_i,$$

and

$$(a_i r, a_j r) = r a_i^{-1} r a_j^{-1} a_i r a_j r = (a_i^r)^{-1} a_i \cdot a_j^{-1} a_j^r \in B_i B_j,$$

shows that $H' \leq \langle B_1, \dots, B_n \rangle$, and it follows that $H' = \langle B_1, \dots, B_n \rangle$. □

LEMMA 9.2. Let $\varphi \in \mathcal{W}(H)$. Then, $\varphi(A) = A$ and if $\varphi(a_i) = \alpha_i$, $1 \leq i \leq n$, then $\varphi(a_1^{\lambda_1} \dots a_n^{\lambda_n}) = \alpha_1^{\lambda_1} \dots \alpha_n^{\lambda_n}$ for all $\lambda_i \in \mathbb{Z}$.

Proof. From Lemma 9.1, we see that $(ar)^H$ is infinite for any $a \in A$. Thus, only the elements of A have finite conjugacy classes, and so $\varphi(A) = A$. Now, from $a^H = \{a, a^r\}$, we have $\varphi(a) \sim \varphi(a^r) = \varphi(a)^r$. Thus, $\varphi(\{a, a^r\}) = \{\varphi(a), \varphi(a)^r\}$. □

LEMMA 9.3. For all $n \in \mathbb{Z}$ and $a \in A$, we have $\varphi(a^n) = \varphi(a)^n$.

Proof. We have $a^H = \{a, a^r\}$. Let $\varphi(a) = \alpha$, so that $\varphi(a^r) = \alpha^r$. We then have $\varphi(a^\varepsilon) = \alpha^\varepsilon$ for $\varepsilon = 0, \pm 1$. So, now assume (inductively) that $\varphi(a^i) = \alpha^i$ for $|i| < n$ for some $n \geq 2$ and that $\varphi(a^n) \neq \alpha^n$. Then,

$$\varphi(a^n) = \varphi(a \cdot a^{n-1}) \sim \alpha \alpha^{n-1} \in \{\alpha^n, (\alpha^n)^r\},$$

which shows that $\varphi(a^n) = (\alpha^n)^r$. In particular, we have $\varphi(a^n) = (\alpha^n)^r \neq \alpha^n$.

Now, $\alpha = \varphi(a) = \varphi(a^n \cdot a^{1-n}) \sim (\alpha^r)^n \alpha^{1-n}$. Thus, we must have either

- (a) $(\alpha^r)^n \alpha^{1-n} = \alpha$, or
- (b) $(\alpha^r)^n \alpha^{1-n} = \alpha^r$.

If we have (a), then $(\alpha^r)^n = \alpha^n$, from which we get (since A is a free abelian group) $\alpha^r = \alpha$, giving a contradiction.

If we have (b), then we get $(\alpha^r)^{n-1} = \alpha^{n-1}$, which again gives $\alpha^r = \alpha$, since $n \geq 2$. This gives another contradiction and we have completed the proof. \square

For any $a \in A$, we can write $a = a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n}$, $\lambda_i \in \mathbb{Z}$. We define a length function on A as follows:

$$|a| = |\lambda_1| + |\lambda_2| + \dots + |\lambda_n|.$$

Thus, we now know that $\varphi(a) = \prod_{i=1}^n \alpha_i^{\lambda_i}$ for all $a \in A$ with $|a| \leq 1$.

So, now assume (inductively) that there is $m \geq 2$ such that $\varphi(a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n}) = \alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n}$ for all $a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n} \in A$ with $|a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n}| < m$, and that there is $a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n}$ such that $|a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n}| = m$ and $\varphi(a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n}) \neq \alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n}$. Without loss, we can assume that $\lambda_1 \geq 1$.

Further, since $\varphi(a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n}) \neq \alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n}$, we must have $\varphi(a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n}) = (\alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n})^r$ since

$$\begin{aligned} \varphi(a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n}) &= \varphi(a_1 \cdot a_1^{\lambda_1-1} a_2^{\lambda_2} \dots a_n^{\lambda_n}) \sim \varphi(a_1) \varphi(a_1^{\lambda_1-1} a_2^{\lambda_2} \dots a_n^{\lambda_n}) \\ &= \alpha_1 \cdot \alpha_1^{\lambda_1-1} \alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n} = \alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n}. \end{aligned}$$

By induction, we have $\varphi(a_2^{\lambda_2} \dots a_n^{\lambda_n}) = \alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n}$; however, using Lemma 9.3, we also have

$$\varphi(a_2^{\lambda_2} \dots a_n^{\lambda_n}) = \varphi(a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n} \cdot a_1^{-\lambda_1}) \sim (\alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n})^r \cdot \alpha_1^{-\lambda_1}$$

Thus, we have $\alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n} \sim (\alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n})^r \cdot \alpha_1^{-\lambda_1}$.

This gives two cases:

- (a) $\alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n} = (\alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n})^r \cdot \alpha_1^{-\lambda_1}$, or
- (b) $\alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n} = (\alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n}) \cdot (\alpha_1^{-\lambda_1})^r$.

Here, possibility (a) gives $(\alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n})^r = \alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n}$, a contradiction since $\varphi(a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n}) = (\alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n})^r \neq \alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \dots \alpha_n^{\lambda_n}$.

But (b) gives $(\alpha_1^{\lambda_1})^r = \alpha_1^{\lambda_1}$; this shows that $\varphi(a_1) = \alpha_1 \in Z(H)$, which in turn shows that $a_1 \in Z(H)$, which is a contradiction to the choice of the a_i . This concludes the proof of Lemma 9.2. \square

Thus, we see that $\varphi|_A : A \rightarrow A$ is an automorphism of A .

LEMMA 9.4. (1) For any $b \in A$ such that br has order 2, the homomorphism determined by $a \mapsto a, (a \in A), r \mapsto br$ is an automorphism of H .

(2) Let $\psi \in \text{Aut}(A)$ where ψ commutes with θ . Then, the homomorphism determined by $a \mapsto \psi(a)$, ($a \in A$), $r \mapsto r$ is an automorphism of H .

Proof.

- (1) Is clear since r and br have order 2 and $a^{br} = a^r$.
- (2) The relations in H have the form $a^r = a^\theta$ and $r^2 = 1$. We have $\psi(a^r) = \psi(\theta(a)) = \theta(\psi(a)) = (\psi(a))^r$, and we are done. □

Since $\varphi(Ar) = Ar$ we can assume that $\varphi(r) = br$ for some $b \in A$. Further, we know that br has order 2, since φ is a weak Cayley table map.

By Lemma 9.4, we can assume (by composing φ with the inverse of such an automorphism) that $b = 1$, i.e., that $\varphi(r) = r$.

LEMMA 9.5. *Suppose that $a \in A$. Then, $\varphi(a^r) = \varphi(a)^r$.*

Proof. We know that $\varphi(a^r) \in \{\varphi(a), \varphi(a)^r\}$; however if $\varphi(a^r) = \varphi(a)$, then $a^r = a$ (so that $a = a^r$ and $\varphi(a) = \varphi(a^r)$ are central), and otherwise we have $\varphi(a^r) = \varphi(a)^r$. □

This lemma shows that the automorphism of A determined by the action of φ commutes with the action of θ . Since $\varphi|_A$ is a automorphism of A we can now use Lemmas 9.4 (2) and 9.5 so as to be able to assume that $\varphi|_A = Id|_A$.

Thus, we can now assume: $\varphi|_A = Id|_A$ and $\varphi(r) = r$.

By Lemma 6.1, we know that if $a \in A$ and $\varphi(ar) = br$, $b \in A$, then $b \in \{a, a^r\}$. Thus, if $\varphi(ar) \neq ar$, then $\varphi(ar) = a^r r = ra$.

Now, assume that there are $a, b \in A \setminus \{1\}$ such that $\varphi(ar) = ar \neq a^r r$, $\varphi(br) = b^r r \neq br$. Then,

$$ba^r = \varphi(ba^r) = \varphi(br \cdot ar) \sim \varphi(br)\varphi(ar) = b^r r ar = b^r a^r.$$

It follows that either $ba^r = b^r a^r$ or $ba^r = ab$, each possibility giving a contradiction.

Thus, there is $\varepsilon \in \{0, 1\}$ such that $\varphi(ar) = a^{\varepsilon} r$ for all $a \in A$. If $\varepsilon = 0$, then φ is the identity map on H .

So, now assume that $\varepsilon = 1$, so that φ satisfies

$$\varphi(a) = a, \quad \varphi(ar) = a^r r, \quad \text{for all } a \in A.$$

Then, for all $a, b \in A$, we have: $rab = (ab)^r r = \varphi(a \cdot br) \sim ab^r r = arb$, so that either (i) $rab = arb$; or (ii) $rab = (arb)^r$. But (i) gives $(a, r) = 1$, and (ii) gives $(b, r) = 1$. Since (i) and (ii) are true for all $a, b \in A$ we have a contradiction, and we have proved Theorem 1.2. □

THEOREM 9.6. *Let p be an odd prime, A an abelian group and let $G = A \rtimes_{\theta} C_p$ be a semi-direct product. We assume that θ is not trivial. Then, G has a non-trivial weak Cayley table map.*

Proof. This will also give a proof of Theorem 1.3. Let $C_p = \langle r \rangle$. □

LEMMA 9.7.

- (i) Let $1 \leq k < p$. Then, every $g \in G'$ has the form (a, r^k) , for some $a \in A$.
- (ii) For any $1 \leq k < p$ and $g \in G'$, there is $a \in A$ such that $(r^k)^a = gr^k$.

Proof. The proof of (i) is standard, using the Witt–Hall identities [12, p. 290].
(ii) Now, let $g \in G'$ and fix $1 \leq k < p$. Then, by (i) there is $a \in A$ such that

$$g = (a, r^{-k}) = a^{-1}r^k a r^{-k}, \text{ so that } gr^k = (r^k)^a.$$

This concludes the proof of Lemma 9.7. □

Now, define a bijection $\varphi : G \rightarrow G$ by

$$\varphi(a) = a^r \text{ for } a \in A, \quad \varphi(g) = g \text{ for } g \in G \setminus A.$$

Since θ is not trivial there is some $a \in A$ such that $a^r \neq a$. Thus, $\varphi(a)\varphi(r) = a^r \cdot r \neq ar = \varphi(ar)$, so that φ is not a homomorphism. Similarly, we have, using the fact that $p > 2$,

$$\varphi(ar)\varphi(r) = arr \neq rar = \varphi(r \cdot ar),$$

so that φ is not an antihomomorphism.

We now show that φ is a weak Cayley table isomorphism. We need to show that $\varphi(g_1 g_2) \sim \varphi(g_1)\varphi(g_2)$ for all $g_1, g_2 \in G$, and we do this by considering the following situations:

CASE 1. $g_1, g_2 \in A$. Here, we have $\varphi(g_1)\varphi(g_2) = g_1^r g_2^r = (g_1 g_2)^r = \varphi(g_1 g_2)$.

CASE 2. $g_1 \in A, g_2 \notin A$. Here, we write $g_1 = a, g_2 = br^k, a, b \in k, 1 \leq k < p$. Then,

$$\varphi(g_1 g_2) = \varphi(abr^k) = abr^k, \quad \varphi(g_1)\varphi(g_2) = \varphi(a)\varphi(br^k) = (a^r b) \cdot r^k.$$

Now, by Lemma 9.7, there is $c \in A$ such that

$$(a^r b)(ab)^{-1} = a^r a^{-1} = (c, r^{-k}) = (r^k)^c \cdot r^{-k}.$$

Then, we have

$$\begin{aligned} \varphi(g_1 g_2)^c &= (abr^k)^c = (ab)(r^k)^c = (ab) \cdot (a^r b)(ab)^{-1} r^k \\ &= a^r (br^k) = \varphi(g_1)\varphi(g_2). \end{aligned}$$

This does this case and a similar argument does the case where $g_1 \notin A, g_2 \in A$.

CASE 3. $g_1 \notin A, g_2 \notin A, g_1 g_2 \notin A$. Here, we have $\varphi(g_1) = g_1, \varphi(g_2) = g_2, \varphi(g_1 g_2) = g_1 g_2$ and so $\varphi(g_1 g_2) = g_1 g_2 = \varphi(g_1)\varphi(g_2)$, which does this case.

CASE 4. $g_1 \notin A, g_2 \notin A, g_1 g_2 \in A$. Here, we have $\varphi(g_1 g_2) = (g_1 g_2)^r$ and $\varphi(g_1)\varphi(g_2) = g_1 g_2$, from which it follows that $\varphi(g_1 g_2) = (\varphi(g_1)\varphi(g_2))^r$.

This concludes the proof that φ is a weak Cayley table isomorphism. □

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