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AN IMPROVEMENT TO A THEOREM OF LEONETTI AND LUCA

TRAN NGUYEN THANH DANH, HOANG TUAN DUNG, PHAM VIET HUNG, NGUYEN DINH KIEN, NGUYEN AN THINH, KHUC DINH TOAN and NGUYEN XUAN THO

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Abstract

Leonetti and Luca ['On the iterates of the shifted Euler's function', *Bull. Aust. Math. Soc.*, to appear] have shown that the integer sequence $(x_n)_{n\geq 1}$ defined by $x_{n+2} = \phi(x_{n+1}) + \phi(x_n) + k$, where $x_1, x_2 \geq 1$, $k \geq 0$ and $2 \mid k$, is bounded by $4^{X^{3^{k+1}}}$, where $X = (3x_1 + 5x_2 + 7k)/2$. We improve this result by showing that the sequence (x_n) is bounded by 2^{2X^2+X-3}, where $X = x_1 + x_2 + 2k$.

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1. Introduction

In a recent study of integer sequences generated by Euler's totient function, Leonetti and Luca [3] proved the following theorem.

THEOREM 1.1. *Fix an even integer $k \geq 0$. The integer sequence $(x_n)_{n\geq 1}$ defined by $x_{n+2} = \phi(x_{n+1}) + \phi(x_n) + k$, where $x_1, x_2 \geq 1$, is bounded by $4^{X^{3^{k+1}}}$, where $X = (3x_1 + 5x_2 + 7k)/2$.*

It is natural to ask whether the size of the upper bound for the sequence (x_n) in Theorem 1.1 could be reduced. In this paper, we will provide such an improvement. The main result of this paper is the following theorem.

THEOREM 1.2. *Fix an even integer $k \geq 0$. The integer sequence $(x_n)_{n\geq 1}$ defined by $x_{n+2} = \phi(x_{n+1}) + \phi(x_n) + k$, where $x_1, x_2 \geq 1$, is bounded by 2^{2X^2+X-3}, where $X = x_1 + x_2 + 2k$.*

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Note that the bound in Theorem 1.2 is exponentially smaller than the bound in Theorem 1.1. To prove Theorem 1.2, we will use the Chinese remainder theorem in combination with some estimates on prime numbers due to Erdős (Lemma 2.4) and Rosser (Lemma 2.3).

2. Proof of Theorem 1.2

Case 1: $x_3 = 2$. Since $k \geq 0$, $2 \mid k$ and $\phi(x_1) + \phi(x_2) + k = x_3 = 2$, we must have $k = 0$ and $\phi(x_1) = \phi(x_2) = 1$. Hence, $x_1, x_2 \in \{1, 2\}$. Note that if $\phi(x_n) = \phi(x_{n-1}) = 1$, then $x_{n+1} = \phi(x_n) + \phi(x_{n-1}) = 2$. By induction, $x_n = 2$ for all $n \geq 3$. Since $X = x_1 + x_2 + 2k \geq 2$, we have $2^{2X^2+X-3} > 2$. Hence, $x_n < 2^{2X^2+X-3}$ for all $n \geq 1$.

Case 2: $x_3 \geq 3$. Then $x_4 = \phi(x_3) + \phi(x_2) + k \geq 3$. Note that if $x_n \geq 3$ with $n \geq 3$, then

$$x_{n+1} = \phi(x_n) + \phi(x_{n-1}) + k \geq 3 + k \geq 3.$$

By induction, $x_n \geq 3$ for all $n \geq 3$, so that $2 \mid \phi(x_n)$ for all $n \geq 3$. Therefore, $2 \mid \phi(x_{n-1}) + \phi(x_{n-2}) + k = x_n$ for all $n \geq 5$ and so

$$\phi(x_n) \leq \frac{n}{2} \quad \text{for all } n \geq 5. \tag{2.1}$$

LEMMA 2.1. *For* $n = 1, 2, \ldots, 6$,

$$x_n < 2^X. \tag{2.2}$$

PROOF. We consider each value of n in turn.
$n = 1$ *or* $n = 2$. Then (2.2) holds because $\max\{x_1, x_2\} < x_1 + x_2 \leq X < 2^X$.
$n = 3$. Then (2.2) holds because $3 \leq x_3 = \phi(x_1) + \phi(x_2) + k \leq x_1 + x_2 + k \leq X < 2^X$.
$n = 4$. Since $\phi(x_3) \leq x_3 - 1 \leq x_1 + x_2 + k - 1$ and $\phi(x_2) \leq x_2$,

$$x_4 = \phi(x_3) + \phi(x_2) + k \leq x_1 + 2x_2 + 2k - 1 < 2X \leq 2^X \quad (\text{since } 2^X \geq 2X \text{ for } X \geq 2).$$

$n = 5$. Since $\phi(x_4) \leq x_4 - 1 \leq x_1 + 2x_2 + 2k - 2$ and $\phi(x_3) \leq x_1 + x_2 + k - 1$,

$$\begin{aligned} x_5 = \phi(x_4) + \phi(x_3) + k &\leq 2x_1 + 3x_2 + 4k - 3 \\ &= 3(x_1 + x_2 + 2k) - x_1 - 2k - 3 \\ &\leq 3X - 4 \quad (\text{since } x_1 + 2k + 3 \geq 4) \\ &< 2^X \quad (\text{since } 2^X > 3X - 4 \text{ for } X \geq 2). \end{aligned} \tag{2.3}$$

$n = 6$. By (2.1) and (2.3), $\phi(x_5) \leq x_5/2 \leq (2x_1 + 3x_2 + 4k - 3)/2$. Combining this with the estimate $\phi(x_4) \leq x_1 + 2x_2 + 2k - 2$ gives

$$\begin{aligned} x_6 = \phi(x_5) + \phi(x_4) + k &\leq \frac{2x_1 + 3x_2 + 4k - 3}{2} + (x_1 + 2x_2 + 2k - 2) + k \\ &= \frac{4x_1 + 7x_2 + 10k - 7}{2} = \frac{7(x_1 + x_2 + 2k)}{2} - \frac{3x_1 + 4k + 7}{2} \end{aligned}$$

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$$\leq \frac{7}{2}X - 5 \quad (\text{since } (3x_1 + 4k + 7)/2 \geq 5$$
$$< 2^X \quad (\text{since } 2^X > 7X/2 - 5 \text{ for } X \geq 2). \qquad \square$$

We now return to the proof of Theorem 1.2 when $x_3 \geq 3$ (Case 2).

Case 2.1: $k = 0$. Then $x_n = \phi(x_{n-1}) + \phi(x_{n-2})$ for all $n \geq 3$. By (2.1),

$$x_{n+2} = \phi(x_{n+1}) + \phi(x_n) \leq \frac{x_{n+1}}{2} + \frac{x_n}{2} \quad \text{for all } n \geq 5. \tag{2.4}$$

An induction, using Lemma 2.1 and (2.4), shows that $x_n < 2^X$ for all $n \geq 1$. Hence, $x_n < 2^X < 2^{2X^2+X-3}$ for all $n \geq 1$.

Case 2.2: $k \geq 1$. Let $p_1 < p_2 < \cdots < p_{2k+2}$ be the first $2k + 2$ primes. By the Chinese remainder theorem, there exist infinitely many positive integers x such that

$$x \equiv -i \pmod{p_i} \quad \text{for all } i = 1, 2, \ldots, 2k + 2. \tag{2.5}$$

LEMMA 2.2. *Let M be a positive integer satisfying the congruences (2.5) and such that $M \geq \max\{2^X - 2k - 2, kp_{2k+2} - k - 2\}$. Then, for all positive integers n,*

$$x_n \leq M + 2k + 2. \tag{2.6}$$

PROOF. Since $M + 2k + 2 \geq 2^X$, by Lemma 2.1, (2.6) holds for $n = 1, 2, \ldots, 6$. Suppose $n \geq 7$ and assume that Lemma 2.2 is not true. Let m be the smallest positive integer such that $x_m > M + 2k + 2$. Then $x_{m-1}, x_{m-2} \leq M + 2k + 2$ and $m \geq 7$. Thus, $2 \mid x_{m-1}, x_{m-2}$. Therefore,

$$M + 2k + 2 < x_m = \phi(x_{m-1}) + \phi(x_{m-2}) + k \leq \frac{x_{m-1}}{2} + \frac{x_{m-2}}{2} + k$$

and so $x_{m-1} + x_{m-2} > 2M + 2k + 4$. It follows that $x_{m-1}, x_{m-2} > M + 2$ since $x_{m-1}, x_{m-2} \leq M + 2k + 2$. Let $x_{m-1} = M + 2 + r$ and $x_{m-2} = M + 2 + s$, where $r, s \in \mathbb{Z}$ and $0 < r, s \leq 2k$. Since M satisfies (2.5), there exist *odd* primes $p, q \leq p_{2k+2}$ such that $p \mid M + 2 + r = x_{m-1}$ and $q \mid M + 2 + s = x_{m-2}$. Therefore, $2p \mid x_{m-1}$ and $2q \mid x_{m-2}$. Recalling that $x_{m-1}, x_{m-2} \leq M + 2k + 2$ and $p, q \leq p_{2k+2}$,

$$\begin{aligned} x_m = \phi(x_{m-1}) + \phi(x_{m-2}) + k &\leq \frac{x_{m-1}}{2}\Big(1 - \frac{1}{p}\Big) + \frac{x_{m-2}}{2}\Big(1 - \frac{1}{q}\Big) + k \\ &\leq 2 \cdot \frac{M + 2k + 2}{2}\Big(1 - \frac{1}{p_{2k+2}}\Big) + k \\ &= (M + 2k + 2)\Big(1 - \frac{1}{p_{2k+2}}\Big) + k \\ &\leq M + 2k + 2 \quad (\text{since } M + 2k + 2 \geq p_{2k+2}k), \end{aligned}$$

which contradicts the assumption that $x_m > M + 2k + 2$. Therefore, (2.6) holds for all positive integers n. Lemma 2.2 is proved. $\square$

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LEMMA 2.3 (Rosser [4, Theorem 2]). *For all positive integers $n \geq 4$,*

$$p_n < n(\log n + 2 \log\log n),$$

where p_n is the nth prime and log *denotes the natural logarithm.*

LEMMA 2.4 (Erdős [2]; see Aigner and Ziegler [1, Ch. 2, page 10]). *For all positive integers $n \geq 2$,*

$$\prod_{p \leq n} p \leq 4^{n-1},$$

where the product is taken over all primes $p \leq n$.

LEMMA 2.5. *For all positive integers $n \geq 4$,*

$$p_n < n^2, \tag{2.7}$$

$$p_1 p_2 \cdots p_n < 4^{n^2-1}. \tag{2.8}$$

PROOF. It is a routine verification that $\log x + 2 \log\log x < x$ for all $x > 1$. Hence,

$$n(\log n + 2 \log\log n) < n^2. \tag{2.9}$$

Then (2.7) follows from Lemma 2.3 and inequality (2.9).

Inequality (2.8) is a consequence of Lemma 2.4 and (2.7). Indeed,

$$p_1 p_2 \cdots p_n \leq 4^{p_n - 1} < 4^{n^2-1}.$$ □

We return to the proof of Theorem 1.2. Let α be the smallest positive integer satisfying (2.5). Then $\alpha \leq p_1 p_2 \cdots p_{2k+2}$. Note that $4 \leq 2k + 2 \leq 2k + x_1 + x_2 = X$. It follows from (2.8) that

$$\alpha \leq p_1 p_2 \cdots p_{2k+2} < 4^{(2k+2)^2-1} \leq 4^{X^2-1}. \tag{2.10}$$

Let $M = \alpha + 2^{x_1+x_2} p_1 p_2 \cdots p_{2k+2}$. Then M satisfies (2.5). Since $p_1 p_2 \cdots p_{2k+2} > 2^{2k}$ and $p_1 p_2 \cdots p_{2k+2} > p_k p_{2k+2} > k p_{2k+2}$,

$$M > 2^{x_1+x_2} p_1 p_2 \cdots p_{2k+2} > 2^{x_1+x_2} 2^{2k} = 2^X$$

and

$$M > p_1 p_2 \cdots p_{2k+2} > k p_{2k+2}.$$

Thus, M satisfies all the conditions in Lemma 2.2. Hence,

$$x_n \leq M + 2k + 2 \quad \text{for all } n \geq 1. \tag{2.11}$$

Since $x_1 + x_2 = X - 2k \leq X - 2$, by (2.10),

$$M = \alpha + 2^{x_1+x_2} p_1 p_2 \cdots p_{2k+2} < (1 + 2^{X-2}) \cdot 4^{X^2-1}. \tag{2.12}$$

Note that $X \leq 2^{X-2}$ since $X \geq 4$. Thus,

$$2k + 2 \leq 2k + x_1 + x_2 = X \leq 2^{X-2}. \tag{2.13}$$

Combining (2.12) and (2.13) gives

$$\begin{aligned} M + 2k + 2 \leq M + 2^{X-2} &< (1 + 2^{X-2}) \cdot 4^{X^2-1} + 2^{X-2} \\ &< 2^{X-2} \cdot 4^{X^2-1} + 2^{X-2} \cdot 4^{X^2-1} = 2^{2X^2+X-3}. \end{aligned} \tag{2.14}$$

It follows from (2.11) and (2.14) that $x_n < 2^{2X^2+X-3}$ for all $n \geq 1$. This completes the proof of Theorem 1.2.

REMARK 2.6. Leonetti and Luca [3] also proved that the integer sequence $(x_n)_{n\geq 1}$ defined by $x_{n+1} = \phi(x_n) + k$, where $x_1 \geq 1$ and $k \geq 0$, is bounded by $\max\{x_1, k^4\} + (k+1)^2$. A similar argument to that in the proof of Theorem 1.2 shows that the sequence (x_n) is bounded but with a worse upper bound.

REMARK 2.7. It is an open question to find the best possible bound (in terms of x_1, x_2 and k) for the sequence (x_n) in Theorems 1.1 and 1.2.

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TRAN NGUYEN THANH DANH, VNU-HCM High School for the Gifted Students, Ho Chi Minh City, Vietnam
e-mail: danhtran17022005@gmail.com

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HOANG TUAN DUNG, Hanoi National University of Education High School for the Gifted Students, Hanoi, Vietnam
e-mail: tuandunghg01@gmail.com

PHAM VIET HUNG, HUS High School for Gifted Students,
Hanoi, Vietnam
e-mail: vhpro2005@gmail.com

NGUYEN DINH KIEN, Tran Phu High School for the Gifted Students,
Haiphong, Vietnam
e-mail: kien08022006@gmail.com

NGUYEN AN THINH, Tran Phu High School for the Gifted Students,
Haiphong, Vietnam
e-mail: thinhyte0x0@gmail.com

KHUC DINH TOAN, Bac Ninh High School for the Gifted Students,
Bắc Ninh, Vietnam
e-mail: khucdinhtoan985@gmail.com

NGUYEN XUAN THO, School of Applied Mathematics and Informatics,
Hanoi University of Science and Technology, Hanoi, Vietnam
e-mail: tho.nguyenxuan1@hust.edu.vn