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Affine isoperimetric inequalities on flag manifolds

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Abstract. Building on work of Furstenberg and Tzkoni, we introduce **r**-flag affine quermassintegrals and their dual versions. These quantities generalize affine and dual affine quermassintegrals as averages on flag manifolds (where the Grassmannian can be considered as a special case). We establish affine and linear invariance properties and extend fundamental results to this new setting. In particular, we prove several affine isoperimetric inequalities from convex geometry and their approximate reverse forms. We also introduce functional forms of these quantities and establish corresponding inequalities.

1 Introduction

Affine isoperimetric inequalities provide a rich foundation for understanding principles in geometry and analysis that arise in the presence of symmetries. Among the most fundamental examples is the Blaschke–Santaló inequality [48] on the product of volumes of an origin-symmetric convex body L in $\mathbb{R}^n$ and its polar $L^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \le 1,\ \forall y \in L\}$. The latter asserts that this product is maximized for ellipsoids, i.e.,

$$|L|\,|L^\circ| \le \omega_n^2, \tag{1}$$

where ω_n is the volume of the unit Euclidean ball B_2^n, and $|\cdot|$ denotes Lebesgue measure. The Blaschke–Santaló inequality, and its version for non-origin-symmetric bodies, is one of several equivalent forms of the affine isoperimetric inequality; see, e.g., the survey [34]. Moreover, it admits numerous extensions: for example, L_p versions [37], generalizations from convex bodies to functions, e.g., [1, 3, 14] with applications to concentration of measure [1, 27]; further functional affine isoperimetric inequalities, e.g., [2]; stronger versions in which stochastic dominance holds [10].

Another fundamental affine isoperimetric inequality is the Petty projection inequality [44]. This concerns projection bodies, which are special zonoids that play a fundamental role in convex geometry and functional analysis, among other fields, e.g., [17, 49, 50]. The projection body of a convex body $L \subseteq \mathbb{R}^n$ is the convex body ΠL

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defined by its support function in direction $\theta \in S^{n-1}$ as $h_{\Pi L}(\theta) = |P_{\theta^\perp} L|$, where $P_{\theta^\perp}$ is the orthogonal projection onto $\theta^\perp$, the central hyperplane perpendicular to θ. The Petty projection inequality asserts that the affine-invariant quantity $|L|^{n-1}|(\Pi L)^\circ|$ is maximized by ellipsoids, i.e.,

$$|L|^{n-1}|(\Pi L)^\circ| \leq \omega_n^n \omega_{n-1}^{-n}. \tag{2}$$

The Petty projection inequality is the geometric foundation for Zhang's affine Sobolev inequality [53]. Its equivalent forms and extensions have given rise to fundamental inequalities in analysis, geometry, and information theory, e.g., [35, 36].

The affine invariance in inequalities (1) and (2) follows from volumetric considerations. However, as we will review below, the underlying principle goes much deeper and extends to the family of affine quermassintegrals, of which $|L^\circ|$ and $|(\Pi L)^\circ|$ are just two special cases, up to normalization. Formally, the affine quermassintegrals are defined for compact bodies L in $\mathbb{R}^n$ (i.e., compact sets with non-empty interior), and $1 \leq k \leq n$, by

$$\Phi_{[k]}(L) = \left(\int_{G_{n,k}} |P_E L|^{-n} \, d\nu_{n,k}(E) \right)^{-\frac{1}{kn}}, \tag{3}$$

where $G_{n,k}$ is the Grassmannian manifold of k-dimensional linear subspaces equipped with the Haar probability measure $\nu_{n,k}$ (see [30]). Writing $|L^\circ|$ and $|(\Pi L)^\circ|$ in polar coordinates shows a direct connection to $k = 1$ and $k = n-1$ in (3), respectively. As the name suggests, they are affine-invariant, i.e., $\Phi_{[k]}(TL) = \Phi_{[k]}(L)$ for each volume preserving affine transformation T, as proved by Grinberg [20], extending earlier work on ellipsoids by Furstenberg and Tzkoni [16] and Lutwak [32].

The quantities $\Phi_{[k]}(L)$ are affine versions of quermassintegrals or intrinsic volumes, which play a central role in Brunn–Minkowski theory [49]. In particular, the intrinsic volumes $V_1(L), \ldots, V_n(L)$ of a convex body L admit similar representations through Kubota's integral recursion as

$$V_k(L) = c_{n,k} \int_{G_{n,k}} |P_E L| \, d\nu_{n,k}(E),$$

where $c_{n,k}$ is a constant that depends only on n and k. They enjoy many fundamental inequalities, such as

$$V_k(L) \geq V_k(r_L B_2^n), \tag{4}$$

for $k = 1, \ldots, n-1$, where r_L is the radius of a Euclidean ball in $\mathbb{R}^n$ having the same volume as L. Taking $k = 1$ in (4) corresponds to Urysohn's inequality, while $k = n-1$ is the standard isoperimetric inequality. From Jensen's inequality, one sees that (1) and (2) provide stronger affine-invariant analogues of (4) for $k = 1$ and $k = n-1$, respectively. For the intermediary values $1 < k < n-1$, the inequalities in (4) are well-known consequences of Alexandrov–Fenchel inequality (e.g., [49]). It was a long-standing problem posed by Lutwak [32] whether affine quermassintegrals are minimized by ellipsoids. In a recent breakthrough, this has been resolved by E. Milman and Yehudayoff in [38]; namely, for any convex body L, and $1 < k < n-1$,

$$\Phi_{[k]}(L) \geq \Phi_{[k]}(r_L B_2^n). \tag{5}$$

In the last 40 years, a compelling dual theory, initiated by Lutwak in [29], has flourished (see, e.g., [17, 49]). Rather than convex bodies and projections onto lower-dimensional subspaces, this involves star-shaped sets and intersections with subspaces. As above, a key isoperimetric inequality lies at its foundation. The intersection body of a star-shaped body L is the star-shaped body IL with radial function $\rho_{IL}(\theta) := |L \cap \theta^\perp|$. The Busemann intersection inequality [8], proved originally for convex bodies L, states that

$$|IL|\,|L|^{-(n-1)} \leq \omega_{n-1}^n \omega_n^{-(n-2)}. \tag{6}$$

The volume of the intersection body lies at one endpoint of a sequence of SL_n-invariant quantities that are called the dual affine quermassintegrals. These are SL_n-invariant analogues of the dual quermassintegrals introduced by Lutwak [31]. Formally, for a compact body $L \subseteq \mathbb{R}^n$ and $1 \leq k \leq n$, the dual affine quermassintegrals[1] of L are defined by

$$\Psi_{[k]}(L) = \left(\int_{G_{n,k}} |L \cap E|^n \, d\nu_{n,k}(E) \right)^{\frac{1}{kn}}. \tag{7}$$

As above, Grinberg [20], drawing on [16], showed that these enjoy invariance under volume-preserving linear transformations, i.e., $\Psi_{[k]}(TL) = \Psi_{[k]}(L)$ for $T \in SL_n$. They also satisfy the following extension of (6), proved by Busemann and Straus [9] and Grinberg [20]:

$$\Psi_{[k]}(L) \leq \Psi_{[k]}(r_L B_2^n). \tag{8}$$

While the dual theory has been developed for star-shaped bodies, the investigation of these quantities goes deeper and can be extended to bounded Borel sets and non-negative measurable functions [12, 18]. For recent developments on dual Brunn–Minkowski theory, see [17, 22, 49] and the references therein.

Affine and dual affine quermassintegrals have implications well outside of integral geometry, convexity, and isoperimetry. They are essential for understanding phenomena in high-dimensional probability. Indeed, functional versions of (8) lead to sharp asymptotics for small-ball probabilities for marginal densities of probability measures [12]. They also govern key parameters of marginals of log-concave probability measures connected to the Slicing Problem [43]. Furthermore, small-ball probabilities for the volume of random convex sets in [42] also depend on quantifying volumetric bounds for random projections and sections of convex bodies. Each of these applications to high-dimensional probability boils down to understanding affine invariant quantities on Grassmannians.

1.1 A return to flag manifolds

The work of Furstenberg and Tzkoni [16] that established the SL_n-invariance of (7) for ellipsoids on the Grassmannian provided the impetus for the development of affine and dual affine quermassintegrals. However, Furstenberg and Tzkoni went

[1] These quantities are dual to affine quermassintegrals, but we emphasize that they are not translation-invariant.

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well beyond the Grassmannian and derived kindred integral geometric formulas for ellipsoids on flag manifolds. They established deeper connections to representation of spherical functions on symmetric spaces. Their work was motivated by results in ergodic theory of Furstenberg and Keston [15]. Recently, Hanin and the second-named author studied randomness on flag manifolds and implications for high-dimensional frames and products of random matrices. They derived non-asymptotic results in the ergodic theorem [21] that ultimately rely on notions for affine quantities on flag manifolds.

Unlike for affine or dual affine quermassintegrals, the corresponding notions for convex bodies, compact bodies, and functions have not been investigated in the setting of flag manifolds. Our main goal is to initiate such a study in this paper. We extend and develop fundamental notions that are currently only available on Grassmannians to the setting of flag manifolds. We introduce flag versions of (dual) affine quermassintegrals (cf. (3) and (7)). Our investigation includes (i) affine invariance properties, (ii) sharp extremal inequalities, (iii) approximate reverse isoperimetric inequalities, and (iv) functional versions. As mentioned, each of these notions on the Grassmannian has played an important role in high-dimensional convex geometry and probability. New connections discovered in [21] suggest that corresponding flag versions are needed and will have broader applicability.

As flag manifolds are natural generalizations of Grassmannians, they have been studied from several different perspectives. In convex geometry, mixed volumes admit representations in terms of certain flag measures (e.g., [23]). Recently, there is increasing interest in other probabilistic aspects of Grassmannians and flag manifolds such as topological properties of random sets in real algebraic geometry (see [7] and the references therein). Our aim here is to develop a corresponding theory within high-dimensional convex geometry.

1.2 Main results

We start by recalling the setting from work of Furstenberg and Tzkoni [16]. Let $1 \le r \le n-1$, and let $\mathbf{r} := (i_1, i_2, \dots, i_r)$ be a strictly increasing sequence of integers, $1 \le i_1 < i_2 < \cdots < i_r \le n-1$. Let $\xi_{\mathbf{r}} := (F_1, \dots, F_r)$ be a (partial) flag of subspaces; i.e., $F_1 \subset F_2 \subset \cdots \subset F_r$ with each F_j an i_j-dimensional subspace. We denote by $F^n_{\mathbf{r}}$ the flag manifold (with indices $\mathbf{r}$) as the set of all partial flags $\xi_{\mathbf{r}}$. $F^n_{\mathbf{r}}$ is equipped with the unique Haar probability measure that is invariant under the action of SO_n and all integrations on this set in this note are meant with respect to this measure.

In the special case when $r = 1$ and $i_1 = k$, the partial flag manifold $F^n_{\mathbf{r}}$ is just the Grassmann manifold $G_{n,k}$. Hence, (partial) flag-manifolds can be considered as generalizations of Grassmannians. When $r = n-1$, so that $\mathbf{r} := (1, 2, \dots, n-1)$, we write $F^n := F^n_{\mathbf{r}}$ for the complete flag manifold. We follow the convention that $i_0 = 0$ and $i_{r+1} = n$; hence,

$$\sum_{j=1}^{r} i_j (i_{j+1} - i_{j-1}) = i_r n. \tag{9}$$

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Let L be a compact body in $\mathbb{R}^n$, and let $1 \le r \le n-1$ and $\mathbf{r}$ be a set of indices as above. We define the $\mathbf{r}$-*flag affine quermassintegral* of L by

$$\Phi_{\mathbf{r}}(L) := \left(\int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |P_{F_j} L|^{i_{j-1}-i_{j+1}} d\xi_{\mathbf{r}} \right)^{-\frac{1}{i_r n}}. \tag{10}$$

Similarly, we define the *dual* $\mathbf{r}$-*flag affine quermassintegral* of L by

$$\Psi_{\mathbf{r}}(L) := \left(\int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |L \cap F_j|^{i_{j+1}-i_{j-1}} d\xi_{\mathbf{r}} \right)^{\frac{1}{i_r n}}. \tag{11}$$

In [16], it was shown that when $L = \mathcal{E}$ is an ellipsoid, $\Psi_{\mathbf{r}}(\mathcal{E})$ is invariant under SL_n. When $r = 1$, the $\mathbf{r}$-flag affine quermassintegrals are exactly the affine quermassintegrals; similarly for the dual case. Thus, the latter quantities can be considered as extensions of the (dual) affine quermassintegrals to flag manifolds. For complete flag manifolds, we similarly define

(12)

$$\Psi_{\mathbf{F}^n}(L) := \left(\int_{F^n} \prod_{i=1}^{n-1} |L \cap F_i|^2 d\xi \right)^{\frac{1}{n(n-1)}} \quad \text{and} \quad \Phi_{\mathbf{F}^n}(L) := \left(\int_{F^n} \prod_{i=1}^{n-1} |P_{F_i} L|^{-2} d\xi \right)^{-\frac{1}{n(n-1)}}.$$

Clearly, by (9),

$$\Psi_{\mathbf{r}}(\lambda L) = \lambda \Psi_{\mathbf{r}}(L), \qquad \Phi_{\mathbf{r}}(\lambda L) = \lambda \Phi_{\mathbf{r}}(L) \quad (\lambda > 0).$$

Our first result extends the invariance results of Grinberg [20] that give invariance of (3) and (7) under volume-preserving affine and linear transformations, respectively.

Theorem 1.1 *Let L be a compact body in $\mathbb{R}^n$, $1 \le r \le n-1$, and let $\mathbf{r} := (i_1, \dots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. Let A be an affine map that preserves volume and $T \in SL_n$. Then*

$$\Phi_{\mathbf{r}}(AL) = \Phi_{\mathbf{r}}(L) \quad \text{and} \quad \Psi_{\mathbf{r}}(TL) = \Psi_{\mathbf{r}}(L).$$

With such invariance properties, it is natural to seek extremizers of $\Phi_{\mathbf{r}}(L)$ and $\Psi_{\mathbf{r}}(L)$, especially over convex bodies $L \subseteq \mathbb{R}^n$. However, even for the Grassmannian, very few such results are known; cf. (5) and (8), with (5) established only recently in [38]. Previously, (5) was shown to hold at the expense of a universal constant in [42] by the second- and third-named authors. It is easy to construct compact sets $L \subseteq \mathbb{R}^n$ of a given volume such that $\Phi_{[k]}(L)$ is arbitrarily large. This, however, cannot happen when L is convex: in [11], it was shown that up to a logarithmic factor in the dimension n, $\Phi_{[k]}(L)$ does not exceed $\Phi_{[k]}(r_L B_2^n)$ (where, as above, r_L is the radius of a Euclidean ball in $\mathbb{R}^n$ with the same volume as L).

We extend the aforementioned results to the setting of flag manifolds. In this note, $c, c', c_0, \dots$ etc. will denote universal constants (not necessarily the same at each occurrence).

Theorem 1.2 *Let L be a compact body in $\mathbb{R}^n$, $1 \le r \le n-1$ and $\mathbf{r} := (i_1, \dots, i_r)$ an increasing sequence of integers between* 1 *and* $n-1$. *Then*

$$\Psi_{\mathbf{r}}(L) \le \Psi_{\mathbf{r}}(r_L B_2^n). \tag{13}$$

If L is an origin-symmetric convex body, then

$$\Psi_{\mathbf{r}}(L) \ge \frac{c}{\min\left\{\sqrt{\frac{n}{i_r}}, \log n\right\}} \Psi_{\mathbf{r}}(r_L B_2^n). \tag{14}$$

If L is a convex body, then

$$\Phi_{\mathbf{r}}(r_L B_2^n) \le \Phi_{\mathbf{r}}(L) \le c \min\left\{\sqrt{\frac{n}{i_r}}, \log n\right\} \Phi_{\mathbf{r}}(r_L B_2^n). \tag{15}$$

The necessity of the logarithmic factors appearing in (14) and (15) remains an open question. For $\Phi_{[k]}$, the analogous theorem was proved in [11]. For $\Phi_{\mathbf{r}}$ and $\Psi_{\mathbf{r}}$, the result is new and the path that we use to prove Theorem 1.2 is quite different than that of [11]. We use isotropic position and the theory of isotropic convex bodies. In particular, we use the isomorphic solution to the Slicing Problem, due to B. Klartag. We believe this is of its own independent interest.

Further drawing on [16], we also consider variants of (dual) $\mathbf{r}$-flag affine quermassintegrals involving permutations ω of $\{1, \dots, n\}$. We define the *ω-flag quermassintegral* and *dual ω-flag quermassintegral* as follows: for every compact body L in $\mathbb{R}^n$,

$$\Phi_\omega(L) := \begin{cases} \left(\int_{F^n} \prod_{j=1}^{n-1} |P_{F_j} L|^{-\omega(j)+\omega(j+1)-1} d\xi\right)^{-\frac{1}{n(n-\omega(n))}}, & \text{if } \omega(n) \ne n, \\ \int_{F^n} \prod_{j=1}^{n-1} |P_{F_j} L|^{-\omega(j)+\omega(j+1)-1} d\xi, & \text{if } \omega(n) = n, \end{cases} \tag{16}$$

and

$$\Psi_\omega(L) := \begin{cases} \left(\int_{F^n} \prod_{j=1}^{n-1} |L \cap F_j|^{\omega(j)-\omega(j+1)+1} d\xi\right)^{\frac{1}{n(n-\omega(n))}}, & \text{if } \omega(n) \ne n, \\ \int_{F^n} \prod_{j=1}^{n-1} |L \cap F_j|^{\omega(j)-\omega(j+1)+1} d\xi, & \text{if } \omega(n) = n. \end{cases} \tag{17}$$

Furstenberg and Tzkoni showed SL_n-invariance of Ψ_ω for ellipsoids. We investigate the extent to which this invariance carries over to compact bodies. Moreover, in the case of convex bodies, we show that such quantities cannot be too degenerate in the sense that they admit uniform upper and lower bounds, independent of the body. We apply V. D. Milman's M-ellipsoids [40], together with the aforementioned SL_n-invariance of Furstenberg–Tzkoni to establish these bounds (see Corollary 4.4).

In Section 5, we introduce functional analogues of the dual $\mathbf{r}$-flag affine quermassintegrals. We show that more general quantities share the SL_n-invariance properties, and we prove sharp isoperimetric inequalities. In this section, we invoke techniques and results from our previous work [12]. Lastly, in Section 5, we also introduce a functional form of $\mathbf{r}$-flag affine quermassintegrals. There is much recent interest in extending fundamental geometric inequalities from convex bodies to certain classes of functions (e.g., [4, 25, 39]). The latter works have treated variants of inequalities for intrinsic volumes, or even mixed volumes, and other general quantities; for example, they establish functional analogues of (4). Here, we establish a functional generalization of (15), using E. Milman and Yehudayoff's result (5). Invariance properties and bounds for these quantities are treated in Section 5.2.

2 Affine invariance

In this section, we will present the proof of Theorem 1.1. The following proposition relates integration on a flag manifold to integration on nested Grassmannians (see [51, Theorem 7.1.1 on p. 267] for such a result for flags of elements consisting of two subspaces). Since we will use this fact many times throughout this paper, we include the proof. For a subspace $F \subset \mathbb{R}^n$, we denote by $G_{F,i}$ the Grassmannian of all i-dimensional subspaces contained in F.

Proposition 2.1 *Let $1 \le r \le n-1$ and $\mathbf{r} := (i_1, \dots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. For $G \in L^1(F^n_{\mathbf{r}})$,*

$$\int_{F^n_{\mathbf{r}}} G(\xi_{\mathbf{r}}) d\xi_{\mathbf{r}} = \int_{G_{n,i_r}} \int_{G_{F_r,i_{r-1}}} \dots \int_{G_{F_2,i_1}} G(F_1, \dots, F_{r-1}, F_r)\, dF_1 \dots dF_{r-1} dF_r. \tag{18}$$

For simplicity, we have suppressed the notation to write dF_1 rather than $d\mu_{G_{F_2,i_1}}(F_1)$, which is the Haar probability measure on the Grassmannian of all i_1-dimensional subspaces of the ambient space F_2; similarly for all other indices. This convention will be used throughout.

Proof Fix i_j. Denote by $SO(F_j)$ the subgroup of SO_n acting transitively on $G_{F_j,i_{j-1}}$. For example, if $F_o = \operatorname{span}\{e_1, \dots, e_{i_j}\}$ and $E_o = \operatorname{span}\{e_1, \dots, e_{i_{j-1}}\}$, then elements of $SO(F_o)$ are given by

$$\begin{pmatrix} SO_{i_j} & 0 \\ 0 & I_{n-i_j} \end{pmatrix}.$$

And, the stabilizer of E_o in $SO(F_o)$ is

$$\begin{pmatrix} SO_{i_{j-1}} & 0 & 0 \\ 0 & SO_{i_j - i_{j-1}} & 0 \\ 0 & 0 & I_{n-i_j} \end{pmatrix}.$$

The measure $\mu_{G_{F_j},i_{j-1}}$ is invariant under $SO(F_j)$. Further, for $g \in SO_n$ and a Borel subset $A \subset G_{F_j,i_{j-1}}$, we have $\mu_{G_{gF_j,i_{j-1}}}(gA) = \mu_{G_{F_j,i_{j-1}}}(A)$.

We will show that both integrals are invariant under the action of SO_n. Fix $g \in SO_n$. We start with the integral on the right-hand side of (18):

$$\begin{aligned} &\int_{G_{n,i_r}} \int_{G_{F_r,i_{r-1}}} \dots \int_{G_{F_2,i_1}} G(g^{-1} \cdot (F_1, \dots, F_r))\, dF_1 \dots dF_{r-1} dF_r \\ &\quad = \int_{G_{n,i_r}} \int_{G_{gF_r,i_{r-1}}} \dots \int_{G_{gF_2,i_1}} G(F_1, \dots, F_r)\, d(gF_1) \dots d(gF_{r-1}) d(gF_r) \\ &\quad = \int_{G_{n,i_r}} \int_{G_{gF_r,i_{r-1}}} \dots \int_{G_{gF_3,i_2}} \int_{G_{F_2,i_1}} G(F_1, \dots, F_r)\, dF_1 d(gF_2) \dots d(gF_{r-1}) d(gF_r), \end{aligned}$$

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where we have sent $(F_1,\dots,F_r)\to g\cdot(F_1,\dots,F_r)$ and then used the invariance property

$$\mu_{G_{gF_j,i_{j-1}}}(gA)=\mu_{G_{F_j,i_{j-1}}}(A).$$

Continuing this way for all inner integrals and using the SO_n-invariance of the measure $\mu_{G_{n,i_r}}$ for the outer integral, the above expression reduces to

$$\int_{G_{n,i_r}}\int_{G_{F_r,i_{r-1}}}\cdots\int_{G_{F_2,i_1}}G(F_1,\dots,F_r)\,dF_1\dots dF_{r-1}dF_r.$$

Note that at each step $(F_1,\dots,F_r)$ remains an element of $F^n_{\mathbf{r}}$, this is to say that the inclusion relation is preserved. The invariance of the integral on the left-hand side of (18) is a consequence of the SO_n-invariance of the measure $\mu_{F^n_{\mathbf{r}}}$. The proposition now follows by the uniqueness of the SO_n-invariant probability measure on $F^n_{\mathbf{r}}$ (see, for example, Section 13.3 in [51]). ∎

The following fact allows one to view an integral of a function on a partial flag as an integral over the full flag manifold. In this case, to avoid confusion, the subspaces of flag manifolds are indexed by their dimension.

Proposition 2.2 *Let $1\le r\le n-1$ and $\mathbf{r}:=(i_1,\dots,i_r)$ be an increasing sequence of integers between 1 and $n-1$. For a function G on the partial flag $F^n_{\mathbf{r}}$, denote by $\widetilde{G}$ its trivial extension to the full flag manifold F^n, i.e., $\widetilde{G}(F_1,\dots,F_{n-1}):=G(F_{i_1},\dots,F_{i_r})$. Then*

$$\int_{F^n}\widetilde{G}(\eta)d\eta=\int_{F^n_{\mathbf{r}}}G(\xi_{\mathbf{r}})d\xi_{\mathbf{r}}. \tag{19}$$

Proof We "integrate out" the Grassmannians that do not contain subspaces that G depends on by repeatedly using the identity

$$\int_{G_{F_{j+1},j}}\int_{G_{F_j,j-1}}f(F_{j-1})dF_{j-1}dF_j=\int_{G_{F_{j+1},j-1}}f(F_{j-1})dF_{j-1}.$$

On the right-hand side, we integrate over the set of all $(j-1)$-dimensional subspaces in the ambient $(j+1)$-dimensional space. On the left-hand side, we integrate over the same set of planes (up to a null set) stepwise, we step from one j-dimensional subspace in the ambient $(j+1)$-dimensional space to the next, and in each such subspace, we consider all $(j-1)$-dimensional subspaces. The above identity holds since we are using probability measures on each nested Grassmannian. Applying the latter iteratively, we get

$$\begin{aligned}
&\int_{F^n}\widetilde{G}(\eta)d\eta\\
&=\int_{G_{n,n-1}}\int_{G_{F_{n-1},n-2}}\cdots\int_{G_{F_2,1}}\widetilde{G}(F_1,\dots,F_{n-1})dF_1\dots dF_{n-2}dF_{n-1}\\
&=\int_{G_{n,n-1}}\int_{G_{F_{n-1},n-2}}\cdots\int_{G_{F_2,1}}G(F_{i_1},\dots,F_{i_r})dF_1\dots dF_{n-2}dF_{n-1}\\
&=\int_{G_{n,n-1}}\cdots\int_{G_{F_{i_1+1},i_1}}G(F_{i_1},\dots,F_{i_r})\left(\int_{G_{F_{i_1},i_1-1}}\cdots\int_{G_{F_2,1}}dF_1\dots dF_{i_1-1}\right)dF_{i_1}\dots dF_{n-1}
\end{aligned}$$

$$= \int_{G_{n,n-1}} \cdots \int_{G_{F_{i_1+1},i_1}} G(F_{i_1},\ldots,F_{i_r})dF_{i_1}\ldots dF_{n-1}$$
$$= \int_{G_{n,n-1}} \cdots \left(\int_{G_{F_{i_2},i_2-1}} \cdots \int_{G_{F_{i_1+1},i_1}} G(F_{i_1},\ldots,F_{i_r})dF_{i_1}\ldots dF_{i_2-1} \right) \ldots dF_{n-1}$$
$$= \int_{G_{n,n-1}} \cdots \left(\int_{G_{F_{i_2},i_1}} G(F_{i_1},\ldots,F_{i_r})dF_{i_1} \right) \ldots dF_{n-1}$$
$$= \cdots$$
$$= \int_{G_{n,i_r}} \int_{G_{F_{i_r},i_{r-1}}} \cdots \int_{G_{F_{i_2},i_1}} G(F_{i_1},\ldots,F_{i_r})dF_{i_1}\ldots dF_{i_{r-1}}dF_{i_r}$$
$$= \int_{F^n_{\mathbf{r}}} G(\xi_{\mathbf{r}})d\xi_{\mathbf{r}}. \qquad \blacksquare$$

We now turn to the invariance properties of the functionals $\Phi_{\mathbf{r}}$ and $\Psi_{\mathbf{r}}$. Although self-contained proofs are possible, they require somewhat involved machinery. Since all of the ingredients are available in the literature [12, 16, 20], we have chosen to gather the essentials without proofs. For readers less familiar with the relevant work, we will explain the main points behind the affine invariance of the functionals $\Phi_{[k]}(K)$ and $\Psi_{[k]}(K)$ along the way. There are two important changes of variables: a "global" change of variables on the Grassmannian $G_{n,k}$ or the flag manifold $F^n_{\mathbf{r}}$ and a "local" change of variables on each element $F \in G_{n,k}$ or $\xi_{\mathbf{r}} \in F^n_{\mathbf{r}}$.

Let $g \in SL_n$, $F \in G_{n,k}$, and $A \subset F$ be a full-dimensional Borel set, then $|gA| = |\det(g|_F)||A|$. This determinant of the transformation g restricted to the subspace F, $\det(g|_F)$, is the Jacobian in the following change of variables:

$$\int_{gF} f(g^{-1}t)dt = \int_F f(t)|\det(g|_F)|dt. \tag{20}$$

Denote it as in [16] by $\sigma_k(g,F) := |\det(g|_F)| = \frac{|gA|}{|A|}$.

For the relevant manifolds M considered in this paper, denote by $\sigma_M(g,F)$ the Jacobian determinant in the following change of variables:

$$\int_M f(F)dF = \int_M f(gF)\sigma_M(g,F)dF. \tag{21}$$

Furstenberg and Tzkoni proved in [16] that

$$\sigma_{G_{n,k}}(g,F) = \sigma_k^{-n}(g,F) \tag{22}$$

and

$$\sigma_{F^n_{\mathbf{r}}}(g,\xi_{\mathbf{r}}) = \sigma_{i_1}^{-i_2}(g,F_1)\sigma_{i_2}^{i_1-i_3}(g,F_2)\cdots\sigma_{i_r}^{i_{r-1}-n}(g,F_r), \tag{23}$$

where $\mathbf{r} := (i_1,\ldots,i_r)$. The linear invariance of the dual affine quermassintegrals $\Psi_{[k]}$ now follows immediately. Indeed, for $g \in SL_n$,

$$\begin{aligned}\Psi^{kn}_{[k]}(gL) &= \int_{G_{n,k}} |gL \cap F|^n dF = \int_{G_{n,k}} |gL \cap gF|^n \sigma_{G_{n,k}}(g,F)dF \\ &= \int_{G_{n,k}} (\sigma_k(g,F)|L\cap F|)^n \, \sigma_k^{-n}(g,F)dF = \Psi^{kn}_{[k]}(L),\end{aligned}$$

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where we have used (21), (20) with $f = 1_L$, and (22). Now we turn toward the proof of Theorem 1.1. We start with the case of dual $\mathbf{r}$-flag affine quermassintegrals.

Proposition 2.3 *Let $1 \le r \le n-1$ and $\mathbf{r} := (i_1, \dots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. For every compact body L in $\mathbb{R}^n$ and every $g \in SL_n$,*

$$\Psi_{\mathbf{r}}(gL) = \Psi_{\mathbf{r}}(L). \tag{24}$$

Proof Let us start by expressing $\sigma_{F^n_{\mathbf{r}}}(g, \xi_{\mathbf{r}})$ in terms of sections. For this, note that

$$\sigma_{i_j}(g, F_j) = \frac{|g(L \cap F_j)|}{|L \cap F_j|},$$

where as a subset of F_j we use the section $L \cap F_j$. By (23) with $i_0 = 0$ and $i_{r+1} = n$, we have

$$\sigma_{F^n_{\mathbf{r}}}(g, \xi_{\mathbf{r}}) := \prod_{j=1}^{r} \sigma_{i_j}^{-i_{j+1}+i_{j-1}}(g, F_j) = \prod_{j=1}^{r} \frac{|L \cap F_j|^{i_{j+1}-i_{j-1}}}{|gL \cap gF_j|^{i_{j+1}-i_{j-1}}}.$$

Using the change of variables (21) with the above expression for $\sigma_{F^n_{\mathbf{r}}}$, yields

$$\begin{aligned}
\Psi_{\mathbf{r}}^{n i_r}(gL) &= \int_{F^n_{\mathbf{r}}} \prod_{j=1}^{r} |gL \cap F_j|^{i_{j+1}-i_{j-1}} d\xi_{\mathbf{r}} \\
&= \int_{F^n_{\mathbf{r}}} \prod_{j=1}^{r} |gL \cap gF_j|^{i_{j+1}-i_{j-1}} \sigma_{F^n_{\mathbf{r}}}(g, \xi_{\mathbf{r}}) d\xi_{\mathbf{r}} \\
&= \int_{F^n_{\mathbf{r}}} \prod_{j=1}^{r} |gL \cap gF_j|^{i_{j+1}-i_{j-1}} \prod_{j=1}^{r} \frac{|L \cap F_j|^{i_{j+1}-i_{j-1}}}{|gL \cap gF_j|^{i_{j+1}-i_{j-1}}} d\xi_{\mathbf{r}} \\
&= \int_{F^n_{\mathbf{r}}} \prod_{j=1}^{r} |L \cap F_j|^{i_{j+1}-i_{j-1}} d\xi_{\mathbf{r}} \\
&= \Psi_{\mathbf{r}}^{n i_r}(L).
\end{aligned}$$

■

To recall the linear invariance of the operator $\Phi_{[k]}$, we follow Grinberg [20]. Observe that for $F \in G_{n,k}$ and $g \in SL_n$ upper-triangular with respect to the decomposition $\mathbb{R}^n = F + F^{\perp}$, we have

$$|P_F(g^t L)| = |g P_F L| = |\det(g|_F)| |P_F L| = \sigma_k(g, F) |P_F L|,$$

where g^t stands for the transpose of g. While for $l \in SO_n$, we have $P_F(l^t L) = P_{lF}(L)$. Since any $g \in SL_n$ can be written as a product of a rotation and an upper-triangular matrix, combining the two observations yields the following.

Lemma 2.4 [20] *Let L be a compact body in $\mathbb{R}^n$, $F \in G_{n,k}$, and $g \in SL_n$. Then*

$$|P_F(g^t L)| = |P_{gF} L| \sigma_k(g, F). \tag{25}$$

The linear invariance of the affine quermassintegrals $\Phi_{[k]}$ can now be seen as follows: let $g \in SL_n$,

$$\Phi_{[k]}^{-kn}(g^t L) = \int_{G_{n,k}} |P_F(g^t L)|^{-n} dF = \int_{G_{n,k}} |P_{gF} L|^{-n} \sigma_k^{-n}(g, F) dF = \Phi_{[k]}^{-kn}(L),$$

where we have used (25) and (21) taking into account (22).

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Proposition 2.5 *Let $1 \le r \le n-1$ and $\mathbf{r} := (i_1, \ldots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. Let A be an affine volume preserving map in $\mathbb{R}^n$. Then, for every compact body L in $\mathbb{R}^n$,*

$$\Phi_{\mathbf{r}}(AL) = \Phi_{\mathbf{r}}(L). \tag{26}$$

Proof We will first prove the theorem in the case $A := g \in SL_n$. Using (25) for the projection onto each F_j, (23), and making the change of variables (21), we get

$$\begin{aligned}
\Phi_{\mathbf{r}}^{-ni_r}(g^t L) &= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |P_{F_j}(g^t L)|^{-i_{j+1}+i_{j-1}} d\xi_{\mathbf{r}} \\
&= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} \left(|P_{gF_j} L| \sigma_{i_j}(g, F_j)\right)^{-i_{j+1}+i_{j-1}} d\xi_{\mathbf{r}} \\
&= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |P_{gF_j} L|^{-i_{j+1}+i_{j-1}} \prod_{j=1}^{r} \sigma_{i_j}^{-i_{j+1}+i_{j-1}}(g, F_j) d\xi_{\mathbf{r}} \\
&= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |P_{gF_j} L|^{-i_{j+1}+i_{j-1}} \sigma_{F_{\mathbf{r}}^n}(g, \xi_{\mathbf{r}}) d\xi_{\mathbf{r}} \\
&= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |P_{F_j} L|^{-i_{j+1}+i_{j-1}} d\xi_{\mathbf{r}} \\
&= \Phi_{\mathbf{r}}^{-ni_r}(L).
\end{aligned}$$

The general case follows easily. This proves (26). ∎

The proof of Theorem 1.1 is now complete.

3 Inequalities

We start by proving an extension of the inequality of Busemann–Straus and Grinberg (8) to flag manifolds.

Proposition 3.1 *Let $1 < r \le n-1$ and $\mathbf{r} := (i_1, \ldots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. Then, for every compact body L in $\mathbb{R}^n$,*

$$\Psi_{\mathbf{r}}(L) \le \Psi_{\mathbf{r}}(r_L B_2^n) \tag{27}$$

with equality if and only if L is an origin-symmetric ellipsoid (up to a set of measure zero).

Proof Inequality (8) implies that for every n, every $E \in G_{n,m}$, any $1 \le \ell \le m-1$ and every compact body $L \subseteq \mathbb{R}^n$,

$$\int_{G_{E,\ell}} |(L \cap E) \cap F|^m d\mu_{G_{E,\ell}}(F) \le \frac{\omega_\ell^m}{\omega_m^\ell} |L \cap E|^\ell, \tag{28}$$

with equality iff $L \cap E$ is an ellipsoid (up to a measure 0 set; see [18]). Using (18) and (28), we have that

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$$\begin{aligned}
\Psi_{\mathbf{r}}^{i_r n}(L) &= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |L \cap F_j|^{i_{j+1}-i_{j-1}} d\xi_{\mathbf{r}} \\
&= \int_{G_{n,i_r}} \int_{G_{F_r,i_{r-1}}} \cdots \int_{G_{F_2,i_1}} \prod_{j=1}^{r} |L \cap F_j|^{i_{j+1}-i_{j-1}} dF_1 \dots dF_{r-1} dF_r \\
&= \int_{G_{n,i_r}} \int_{G_{F_r,i_{r-1}}} \cdots \int_{G_{F_3,i_2}} \prod_{j=2}^{r} |L \cap F_j|^{i_{j+1}-i_{j-1}} \times \\
&\quad \times \left(\int_{G_{F_2,i_1}} |L \cap F_1|^{i_2} dF_1 \right) dF_2 \dots dF_{r-1} dF_r \\
&= \int_{G_{n,i_r}} \int_{G_{F_r,i_{r-1}}} \cdots \int_{G_{F_3,i_2}} \prod_{j=2}^{r} |L \cap F_j|^{i_{j+1}-i_{j-1}} \times \\
&\quad \times \left(\int_{G_{F_2,i_1}} |(L \cap F_2) \cap F_1|^{i_2} dF_1 \right) dF_2 \dots dF_{r-1} dF_r \\
&\le \frac{\omega_{i_1}^{i_2}}{\omega_{i_2}^{i_1}} \int_{G_{n,i_r}} \int_{G_{F_r,i_{r-1}}} \cdots \int_{G_{F_3,i_2}} \prod_{j=2}^{r} |L \cap F_j|^{i_{j+1}-i_{j-1}} \times \\
&\quad \times |L \cap F_2|^{i_1} dF_2 \dots dF_{r-1} dF_r \\
&\le \cdots \\
&\le |L|^{i_r} \prod_{j=1}^{r} \frac{\omega_{i_j}^{i_{j+1}}}{\omega_{i_{j+1}}^{i_j}}.
\end{aligned}$$

The last inequality is an equality only when L is an origin-symmetric ellipsoid, up to a set of measure zero (see [18]). Since for the Euclidean ball all inequalities in the previous chain are actually equalities, we can compute the constants and by the linear-invariance property established by Furstenberg and Tzkoni, we conclude the proof. ∎

Our next result is a type of Blaschke–Santaló and reverse Blaschke–Santaló inequality for $\mathbf{r}$-flag affine quermassintegrals. These inequalities concern the volume of the polar body. For a compact set L, we define the *polar body* L° (with respect to the origin) as the convex body

$$L^\circ := \{x \in \mathbb{R}^n : \langle x, y \rangle \le 1, \ \forall y \in L\}.$$

It is straightforward to check the following inclusion: for every compact set L in $\mathbb{R}^n$ and $F \in G_{n,k}$,

$$P_F L^\circ \subseteq (L \cap F)^\circ. \tag{29}$$

If, in addition, L is convex and 0 is in the interior of L,

$$P_F L^\circ = (L \cap F)^\circ. \tag{30}$$

Recall that the Blaschke–Santaló inequality (e.g., [17, 49]) states that for every origin-symmetric convex body L in $\mathbb{R}^n$,

$$|L||L^\circ| \le |B_2^n|^2. \tag{31}$$

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Moreover, (31) holds when L is a convex body and L° is centered, i.e., the centroid of L° is at the origin (see [49]). We do not know the first reference for (31) in case of compact sets; (31) for centered star-shaped sets appears in [33]. For origin-symmetric compact sets with interior points, (31) follows from a more general result in [10] (as a limiting case of Theorem 1.1 with $f = 1_{L'}$, where L' is the closure of the interior of L). For a general proof, see [38]. An approximate reverse form of this inequality is known as the Bourgain–Milman theorem [5]: for every compact, convex set L with $0 \in \operatorname{int}(L)$,

$$|L||L^\circ| \geq c^n |B_2^n|^2, \tag{32}$$

for some absolute constant $c > 0$. For further background and alternate proofs of this inequality, see [19, 26, 40, 41].

The next proposition is the aforementioned Blaschke–Santaló inequality and its (approximate) reversal in the setting of $\mathbf{r}$-flag manifolds:

Proposition 3.2 *Let $1 \leq r \leq n-1$ and $\mathbf{r} := (i_1, \ldots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. Then, for an origin-symmetric compact body L in $\mathbb{R}^n$,*

$$\Phi_{\mathbf{r}}(L^\circ)\Psi_{\mathbf{r}}(L) \leq \Phi_{\mathbf{r}}(B_2^n)\Psi_{\mathbf{r}}(B_2^n). \tag{33}$$

Moreover, if L is a convex body in $\mathbb{R}^n$ with $0 \in \operatorname{int}(L)$, we have that

$$\Phi_{\mathbf{r}}(L^\circ)\Psi_{\mathbf{r}}(L) \geq c\, \Phi_{\mathbf{r}}(B_2^n)\Psi_{\mathbf{r}}(B_2^n), \tag{34}$$

where $c > 0$ is an absolute constant (from the reverse Santaló inequality (32)).

Proof First, note that

$$\Phi_{\mathbf{r}}^{i_r n}(B_2^n)\Psi_{\mathbf{r}}^{i_r n}(B_2^n) = \left(\prod_{j=1}^{r} |B_2^n \cap F_j|^{i_{j+1}-i_{j-1}}\right)^2,$$

where F_j are i_j-dimensional subspaces as defined in Section 1.2 and the expression is independent of $\xi_{\mathbf{r}} := (F_1, \ldots, F_r)$. Replacing $L \cap F_j$ by its convex hull in F_j, using the Blaschke–Santaló inequality (31) and (29), we have

$$\begin{aligned}
\Psi_{\mathbf{r}}^{i_r n}(L) &= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |L \cap F_j|^{i_{j+1}-i_{j-1}} d\xi_{\mathbf{r}} \\
&\leq \int_{F_{\mathbf{r}}^n} \left(\prod_{j=1}^{r} |B_2^n \cap F_j|^{i_{j+1}-i_{j-1}}\right)^2 \prod_{j=1}^{r} \frac{1}{|(L \cap F_j)^\circ|^{i_{j+1}-i_{j-1}}} d\xi_{\mathbf{r}} \\
&\leq \int_{F_{yr}^n} \left(\prod_{j=1}^{r} |B_2^n \cap F_j|^{i_{j+1}-i_{j-1}}\right)^2 \prod_{j=1}^{r} \frac{1}{|P_{F_j} L^\circ|^{i_{j+1}-i_{j-1}}} d\xi_{\mathbf{r}} \\
&= \Phi_{\mathbf{r}}^{i_r n}(B_2^n)\Psi_{\mathbf{r}}^{i_r n}(B_2^n)\Phi_{\mathbf{r}}^{-i_r n}(L^\circ).
\end{aligned}$$

On the other hand, using the reverse Blaschke–Santaló inequality (32), (30), and (9), we get

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$$
\begin{aligned}
\Psi_{\mathbf{r}}^{i_r n}(L) &= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |L \cap F_j|^{i_{j+1}-i_{j-1}} d\xi_{\mathbf{r}} \\
&\geq c^{\sum_{j=1}^{r} i_j(i_{j+1}-i_{j-1})} \int_{F_{\mathbf{r}}^n} \left(\prod_{j=1}^{r} |B_2^n \cap F_j|^{i_{j+1}-i_{j-1}} \right)^2 \prod_{j=1}^{r} \frac{1}{|(L \cap F_j)^\circ|^{i_{j+1}-i_{j-1}}} d\xi_{\mathbf{r}} \\
&= c^{i_r n} \int_{F_{\mathbf{r}}^n} \left(\prod_{j=1}^{r} |B_2^n \cap F_j|^{i_{j+1}-i_{j-1}} \right)^2 \prod_{j=1}^{r} \frac{1}{|P_{F_j} L^\circ|^{i_{j+1}-i_{j-1}}} d\xi_{\mathbf{r}} \\
&= c^{i_r n} \Phi_{\mathbf{r}}^{i_r n}(B_2^n) \Psi_{\mathbf{r}}^{i_r n}(B_2^n) \Phi_{\mathbf{r}}^{-i_r n}(L^\circ).
\end{aligned}
$$

The proof is complete. ∎

We now turn to the proof of (5) in the setting of **r**-flag manifolds. Let us first prove an analogue of (5) for **r**-flag affine quermassintegrals at the expense of a universal constant using known techniques. The case $r = 1$ of the following corollary has been proved in [42] before.

Corollary 3.3 *Let $1 \leq r \leq n-1$ and $\mathbf{r} := (i_1, \ldots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. Then, for every convex body L,*

$$\Phi_{\mathbf{r}}(L) \geq c\, \Phi_{\mathbf{r}}(r_L B_2^n),$$

where $c > 0$ is an absolute constant.

Proof As $\Phi_{\mathbf{r}}(L)$ is translation-invariant, we may assume that L is centered. The Blaschke–Santaló inequality implies $r_L r_{L^\circ} \leq 1$, so with (34) and (27), we obtain

$$\Phi_{\mathbf{r}}(L) \geq c \frac{\Phi_{\mathbf{r}}(B_2^n)\Psi_{\mathbf{r}}(B_2^n)}{\Psi_{\mathbf{r}}(L^\circ)} \geq c \frac{\Phi_{\mathbf{r}}(B_2^n)\Psi_{\mathbf{r}}(B_2^n)}{\Psi_{\mathbf{r}}(r_{L^\circ} B_2^n)} = \frac{c}{r_{L^\circ}} \Phi_{\mathbf{r}}(B_2^n) \geq c\, \Phi_{\mathbf{r}}(r_L B_2^n).$$

The proof is complete. ∎

The constant in the above corollary can be made 1 using (5) and an argument similar to that of Proposition 3.1, as we argue next.

Proposition 3.4 *Let $1 < r \leq n-1$ and $\mathbf{r} := (i_1, \ldots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. Then, for every convex body L in $\mathbb{R}^n$,*

$$\Phi_{\mathbf{r}}(L) \geq \Phi_{\mathbf{r}}(r_L B_2^n)\,, \tag{35}$$

with equality if and only if L is an ellipsoid.

Proof First, note that (5) can be written equivalently as

$$\int_{G_{n,k}} |P_E L|^{-n}\, dE \leq \omega_n^k \omega_k^{-n} |L|^{-k}\,, \tag{36}$$

and

$$\Phi_{\mathbf{r}}^{-i_r n}(r_L B_2^n) = \prod_{j=1}^{r} |r_L B_2^{i_j}|^{i_{j-1}-i_{j+1}} = \left(\prod_{j=1}^{r} \omega_{i_j}^{i_{j-1}-i_{j+1}} \right) \omega_n^{i_r} |L|^{-i_r}\,. \tag{37}$$

We repeatedly use the following simple reformulation of (36) that for every n, every $E \in G_{n,m}$, and any $1 \le \ell \le m-1$,

$$\int_{G_{E,\ell}} |P_F L|^{-m} dF = \int_{G_{E,\ell}} |P_F(P_E L)|^{-m} dF \le \omega_m^{\ell} \omega_\ell^{-m} |P_E L|^{-\ell}. \tag{38}$$

Using (18) and (36)–(38), we obtain

$$\begin{aligned}
&\Phi_{\mathbf{r}}^{-i_r n}(L)\\
&= \int_{F_{\mathbf{r}}^n} \prod_{j=1}^{r} |P_{F_j} L|^{i_{j-1}-i_{j+1}} d\xi_{\mathbf{r}}\\
&= \int_{G_{n,i_r}} \cdots \int_{G_{F_3,i_2}} \prod_{j=2}^{r} |P_{F_j} L|^{i_{j-1}-i_{j+1}} \left(\int_{G_{F_2,i_1}} |P_{F_1}(P_{F_2} L)|^{-i_2} dF_1 \right) dF_2 \dots dF_r\\
&\le \omega_{i_2}^{i_1} \omega_{i_1}^{-i_2} \int_{G_{n,i_r}} \cdots \int_{G_{F_3,i_2}} \prod_{j=2}^{r} |P_{F_j} L|^{i_{j-1}-i_{j+1}} \left(|P_{F_2} L|^{-i_1}\right) dF_2 \dots dF_r\\
&= \omega_{i_2}^{i_1} \omega_{i_1}^{-i_2} \int_{G_{n,i_r}} \cdots \int_{G_{F_4,i_3}} \prod_{j=3}^{r} |P_{F_j} L|^{i_{j-1}-i_{j+1}} \left(\int_{G_{F_3,i_2}} |P_{F_2}(P_{F_3} L)|^{-i_3} dF_2 \right) dF_3 \dots dF_r\\
&\le (\omega_{i_2}^{i_1} \omega_{i_1}^{-i_2})(\omega_{i_3}^{i_2} \omega_{i_2}^{-i_3}) \int_{G_{n,i_r}} \cdots \int_{G_{F_4,i_3}} \prod_{j=3}^{r} |P_{F_j} L|^{i_{j-1}-i_{j+1}} \left(|P_{F_3} L|^{-i_2}\right) dF_3 \dots dF_r\\
&\dots\\
&\le \prod_{j=1}^{r} \omega_{i_{j+1}}^{i_j} \omega_{i_j}^{-i_{j+1}} \quad |L|^{-i_r} = \left(\prod_{j=1}^{r} \omega_{i_j}^{i_{j-1}-i_{j+1}} \right) \omega_n^{i_r} \quad |L|^{-i_r} = \Phi_{\mathbf{r}}^{-i_r n}(r_L B_2^n),
\end{aligned}$$

where for the penultimate equality we have used the convention that $i_0 = 0$ and $i_{r+1} = n$. This proves (35). For the Euclidean ball, all inequalities in the above argument are equalities [38]. By the affine invariance of $\Phi_{\mathbf{r}}$, the equality holds only when L is an ellipsoid. ∎

The next proposition shows that for a convex body L in $\mathbb{R}^n$, all the quantities $\Phi_{\mathbf{r}}(L)$ lie between the volume-radius r_L and the mean width $W(L) = 2\int_{S^{n-1}} h_L(\theta) d\sigma(\theta)$; here, h_L is the support function of L, i.e., $h_L(\theta) = \sup_{x \in L} \langle x, \theta \rangle$. We also set

$$W_L := \inf_{T \in SL_n} W(TL). \tag{39}$$

It will be convenient to use Urysohn's inequality ($k = 1$ in (4)) in the following form. For any convex body K in $\mathbb{R}^n$,

$$\left(\frac{|K|}{|B_2^n|} \right)^{1/n} \le \frac{W(K)}{2}.$$

Proposition 3.5 *Let $1 \le r \le n-1$ and $\mathbf{r} := (i_1, \dots, i_r)$ be an increasing sequence of integers between 1 and $n-1$. Then, for any convex body L in $\mathbb{R}^n$,*

$$r_L \le \frac{\Phi_{\mathbf{r}}(L)}{\Phi_{\mathbf{r}}(B_2^n)} \le \frac{W_L}{2}. \tag{40}$$

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Proof The left-most inequality follows from (35). Next, for $s > 0$, Hölder's inequality and the fact that $h_L(x) = h_{P_F L}(x)$ for $x \in F$, give

$$\left(\int_{G_{n,k}} W(P_F L)^{-s} dF\right)^{-1/s} \leq \int_{G_{n,k}} W(P_F L)\, dF = W(L). \tag{41}$$

Toward proving the right-hand side, we write

$$\Phi_{\mathbf{r}}^{-ni_r}(L) = \int_{G_{n,i_r}} \cdots \int_{G_{F_3,i_2}} \int_{G_{F_2,i_1}} \prod_{j=1}^{r} |P_{F_j} L|^{i_{j-1}-i_{j+1}} dF_1 dF_2 \ldots dF_r. \tag{42}$$

Working with the inner-most integral (recall $i_0 = 0$), we invoke Urysohn's inequality, followed by (41) to obtain

$$\int_{G_{F_2,i_1}} \frac{1}{|P_{F_1} L|^{i_2}} dF_1 \geq \int_{G_{F_2,i_1}} \frac{2^{i_1 i_2}}{\omega_{i_1}^{i_2} W(P_{F_1} L)^{i_1 i_2}} dF_1 \geq \frac{2^{i_1 i_2}}{\omega_{i_1}^{i_2} W(P_{F_2} L)^{i_1 i_2}}.$$

Similarly, by Urysohn's inequality,

$$\frac{1}{|P_{F_2} L|^{i_3 - i_1}} \geq \frac{2^{i_2(i_3 - i_1)}}{\omega_{i_2}^{i_3 - i_1} W(P_{F_2} L)^{i_2(i_3 - i_1)}}.$$

Inserting the last two inequalities into (42) and applying the same argument iteratively, we have

$$\begin{aligned}
\Phi_{\mathbf{r}}^{-ni_r}(L) &\geq \int_{G_{n,i_r}} \cdots \int_{G_{F_4,i_3}} \prod_{j=3}^{r} |P_{F_j} L|^{i_{j-1}-i_{j+1}} \int_{G_{F_3,i_2}} \frac{2^{i_2 i_3}}{\omega_{i_2}^{i_3-i_1} \omega_{i_1}^{i_2} W(P_{F_2} L)^{i_2 i_3}} dF_2 dF_3 \ldots dF_r \\
&\geq \int_{G_{n,i_r}} \cdots \int_{G_{F_4,i_3}} \prod_{j=3}^{r} |P_{F_j} L|^{i_{j-1}-i_{j+1}} \frac{2^{i_2 i_3}}{\omega_{i_2}^{i_3-i_1} \omega_{i_1}^{i_2} W(P_{F_3} L)^{i_2 i_3}} dF_3 \ldots dF_r \\
&\geq \ldots \\
&\geq 2^{ni_r} \Phi_{\mathbf{r}}^{-ni_r}(B_2^n) \int_{G_{n,i_r}} \frac{1}{W(P_{F_r} L)^{ni_r}} dF_r \\
&\geq 2^{ni_r} \Phi_{\mathbf{r}}^{-ni_r}(B_2^n) W(L)^{-ni_r}.
\end{aligned}$$

In the above argument, we may replace L by TL, where $T \in SL_n$. Since the left-hand side of this inequality remains the same for all T by Theorem 1.1, we may take the infimum over all T on the right-hand side. This completes the proof. ∎

We conclude this subsection with a discussion of inequalities of isomorphic nature. For convex bodies L in $\mathbb{R}^n$, we define the Banach–Mazur distance to the Euclidean ball B_2^n by

$$d_{BM}(L) := \inf\left\{ab : a > 0, b > 0, \frac{1}{b} B_2^n \subseteq T(L - L) \subseteq a B_2^n, T \in GL_n\right\}.$$

For origin-symmetric convex bodies, this coincides with the standard notion of Banach–Mazur distance (for more information, see, e.g., [52]).

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Proposition 3.6 *Let $1 \le r \le n-1$ and $\mathbf{r} := (i_1, \ldots, i_r)$ be an increasing sequence of integers between* 1 *and* $n-1$*. Then, for any convex body L in $\mathbb{R}^n$,*

$$\Phi_{\mathbf{r}}(L) \le c_1 \min\left\{\sqrt{\frac{n}{i_r}}, \log\left(1 + d_{BM}(L)\right)\right\} \Phi_{\mathbf{r}}(r_L B_2^n), \tag{43}$$

and, if L is also symmetric, then

$$\Psi_{\mathbf{r}}(L) \ge \frac{c_2}{\min\left\{\sqrt{\frac{n}{i_r}}, \log\left(1 + d_{BM}(L)\right)\right\}} \Psi_{\mathbf{r}}(r_L B_2^n), \tag{44}$$

where $c_1, c_2 > 0$ are absolute constants.

The proof relies on several different tools. We draw on ideas from Dafnis and the second-named author [11] to exploit the affine invariance of $\Phi_{\mathbf{r}}(L)$, $\Psi_{\mathbf{r}}(L)$ by using appropriately chosen affine images of L. To this end, recall the following fundamental theorem, which combines work of Figiel–Tomczak-Jaegermann [13], Lewis [28], Pisier [45], and Rogers–Shephard [47] (see Theorem 1.11.5 on page 52 in [6]).

Theorem 3.7 *Let L be a centered convex body in $\mathbb{R}^n$. Then there exists a linear map $T \in SL_n$ such that*

$$W(TL) \le c \log\{1 + d_{BM}(L)\} \sqrt{n} |L|^{1/n},$$

for some absolute constant $c > 0$.

We will also use recent results on isotropic convex bodies. For background, the reader may consult [6], but we will recall all facts that we need here. To each convex body $M \subseteq \mathbb{R}^n$ with unit volume, one can associate an ellipsoid $Z_2(M)$, called the L_2-*centroid body* of M, which is defined by its support function as

$$h_{Z_2(M)}(\theta) := \left(\int_M |\langle x, \theta\rangle|^2 dx\right)^{\frac{1}{2}}.$$

The *isotropic constant* of M is defined by $L_M := r_{Z_2(M)}$. We say that M is *isotropic* if it is centered and $Z_2(M) = L_M B_2^n$. Fix an isotropic convex body M and a k-dimensional subspace F. Ball [3] proved that

$$|M \cap F^\perp|^{\frac{1}{k}} \ge \frac{c}{L_M}; \tag{45}$$

a corresponding inequality for projections,

$$|P_F M| \le \left(c\frac{n}{k} L_M\right)^k, \tag{46}$$

follows immediately from (45) and the Rogers–Shephard inequality [47]:

$$|P_F M||M \cap F^\perp| \le \binom{n}{k}.$$

Next, we recall a variant of $\Psi_{[k]}(M)$ studied by Dafnis and the second-named author [11]. For every $1 \le k \le n-1$ and a compact body M in $\mathbb{R}^n$ with $|M| = 1$, we define

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(47) $$\widetilde{\Phi}_{[k]}(M) := \left(\int_{G_{n,k}} |M \cap F^{\perp}|^n d\mu_{G_{n,k}}(F) \right)^{\frac{1}{nk}}.$$

In [11], it is shown that for every centered convex body M in $\mathbb{R}^n$ of unit volume,

$$\frac{c_1}{L_M} \le \widetilde{\Phi}_{[k]}(M) \le \widetilde{\Phi}_{[k]}(D_n) \simeq 1,$$

where D_n is the Euclidean ball of volume one.

We also invoke Klartag's [24] fundamental result on perturbations of isotropic convex bodies having a well-bounded isotropic constant.

Theorem 3.8 *Let M be a convex body in $\mathbb{R}^n$. For every $\varepsilon \in (0,1)$, there exists a centered convex body $M_{Kl} \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ such that*

(48) $$\frac{1}{1+\varepsilon} M_{Kl} \subseteq M + x \subseteq (1+\varepsilon) M_{Kl}$$

and

(49) $$L_{M_{Kl}} \le \frac{c}{\sqrt{\varepsilon}}.$$

We are now ready to complete the proof.

Proof of Proposition 3.6 By homogeneity of the operators $\Phi_{\mathbf{r}}$ and $\Psi_{\mathbf{r}}$, we can assume that L has unit volume.

First, we will prove the bound (43) for $\mathbf{r}$-flag affine quermassintegrals. By translation invariance of projections, we may further assume that L is centered. Bounding $\Phi_{\mathbf{r}}(L)$ by $W(L)$ according to (40), using affine invariance of $\Phi_{\mathbf{r}}$ and reverse Urysohn inequality from Theorem 3.7, we get

(50) $$\Phi_{\mathbf{r}}(L) \le c \log(1 + d_{BM}(L)) \Phi_{\mathbf{r}}(D_n).$$

For the Euclidean ball D_n of unit volume, we have $|P_F D_n|^{\frac{1}{k}} = |D_n \cap F|^{\frac{1}{k}} \simeq \sqrt{\frac{n}{k}}$ for every $F \in G_{n,k}$, so

$$\Phi_{\mathbf{r}}(D_n) \simeq \left(\prod_{j=1}^{r} \binom{n}{i_j}^{i_j(i_{j+1}-i_{j-1})} \right)^{\frac{1}{2i_r n}}.$$

The AM/GM inequality implies

$$\left(\prod_{j=1}^{r} \binom{n}{i_j}^{i_j(i_{j+1}-i_{j-1})} \right)^{\frac{1}{2i_r n}} \le \sqrt{\frac{n}{i_r n} \sum_{j=1}^{r} \frac{i_j(i_{j+1} - j_{j-1})}{i_j}} \le \sqrt{\frac{n}{i_r}}.$$

Thus,

(51) $$\Phi_{\mathbf{r}}(D_n) = \Psi_{\mathbf{r}}(D_n) \simeq \left(\prod_{j=1}^{r} \binom{n}{i_j}^{i_j(i_{j+1}-i_{j-1})} \right)^{\frac{1}{2i_r n}} \le \sqrt{\frac{n}{i_r}}.$$

Let $K_1 \subset \mathbb{R}^n$ be a centered convex body and $x \in \mathbb{R}^n$ from the conclusion of Theorem 3.8 corresponding to $\varepsilon = \frac{1}{2}$. Then (48) implies $1 = |L|^{1/n} \ge \frac{2}{3}|K_1|^{1/n}$, while (49) implies

$L_{K_1} \simeq 1$. Let $K_2 := \frac{K_1}{|K_1|^{\frac{1}{n}}}$. Then $L_{K_2} = L_{K_1} \simeq 1$ and our choice of ε and the latter volume bound give

$$\Phi_{\mathbf{r}}(L) = \Phi_{\mathbf{r}}(L+x) \leq \frac{3}{2}\Phi_{\mathbf{r}}(K_1) \leq \frac{9}{4}\Phi_{\mathbf{r}}(K_2). \tag{52}$$

Affine invariance of $\Phi_{\mathbf{r}}$ (Theorem 1.1) allows us to assume that K_2 is isotropic. Using (46), (51), $L_{K_2} \simeq 1$, and (51) one more time, we obtain

$$\begin{aligned}
\Phi_{\mathbf{r}}(K_2) &= \left(\int_{F^n_{\mathbf{r}}} \prod_{j=1}^{r} |P_{F_j}K_2|^{-i_{j+1}+i_{j-1}} d\xi_{\mathbf{r}}\right)^{-\frac{1}{i_r n}} \\
&\leq \left(\prod_{j=1}^{r} \binom{n}{i_j}^{i_j(i_{j+1}-i_{j-1})}\right)^{\frac{1}{i_r n}} (cL_{K_2})^{\frac{1}{i_r n}\sum_{j=1}^{r} i_j(i_{j+1}-i_{j-1})} \\
&\leq c\, L_{K_2}\Phi_{\mathbf{r}}(D_n)^2 \\
&\leq c\sqrt{\frac{n}{i_r}}\,\Phi_{\mathbf{r}}(D_n).
\end{aligned}$$

By (52), we have $\Phi_{\mathbf{r}}(L) \leq c'\sqrt{\frac{n}{i_r}}\,\Phi_{\mathbf{r}}(D_n)$, which together with (50) implies (43).

Applying (34), (43) for L°, and the Blaschke–Santaló inequality $r_L r_{L^\circ} \leq 1$, we get

$$\begin{aligned}
\Psi_{\mathbf{r}}(L) &\geq c\frac{\Phi_{\mathbf{r}}(B_2^n)\Psi_{\mathbf{r}}(B_2^n)}{\Phi_{\mathbf{r}}(L^\circ)} \\
&\geq \frac{c}{r_{L^\circ}}\frac{1}{\min\left\{\log\left(1+d_{BM}(L^\circ)\right), \sqrt{\frac{n}{i_r}}\right\}}\frac{\Phi_{\mathbf{r}}(B_2^n)\Psi_{\mathbf{r}}(B_2^n)}{\Phi_{\mathbf{r}}(B_2^n)} \\
&\geq \frac{c}{\min\left\{\log\left(1+d_{BM}(L)\right), \sqrt{\frac{n}{i_r}}\right\}}\Psi_{\mathbf{r}}(r_L B_2^n),
\end{aligned}$$

where we have also used the identity $d_{BM}(L^\circ) = d_{BM}(L)$ for origin-symmetric convex bodies. This proves (44). ∎

4 Flag manifolds and permutations

In this section, we discuss more general quantities involving permutations. We investigate the extent to which SL_n-invariance properties established by Furstenberg and Tzkoni [16] carry over from ellipsoids to compact bodies. In particular, we provide an example of a convex body for which SL_n-invariance fails. Nevertheless, we show that for convex bodies, such quantities cannot be too degenerate in the sense that they admit uniform upper and lower bounds, independent of the body. The key ingredient is the notion of M-ellipsoids, introduced by V. D. Milman [40].

The next definition is motivated by the work of Furstenberg and Tzkoni [16] for ellipsoids.

Definition 4.1 Let Π_n be the set of permutations of $\{1, 2, \ldots, n\}$ and $\omega \in \Pi_n$. For compact bodies L in $\mathbb{R}^n$, we define the *ω-flag quermassintegral* and dual *ω-flag*

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quermassintegrals as follows: if $\omega(n)\neq n$, then

$$\Phi_\omega(L) := \left(\int_{F^n}\prod_{j=1}^{n-1}|P_{F_j}L|^{-\omega(j)+\omega(j+1)-1}d\xi\right)^{-\frac{1}{n(n-\omega(n))}} \tag{53}$$

and

$$\Psi_\omega(L) := \left(\int_{F^n}\prod_{j=1}^{n-1}|L\cap F_j|^{\omega(j)-\omega(j+1)+1}d\xi\right)^{\frac{1}{n(n-\omega(n))}}. \tag{54}$$

When $\omega(n)=n$, we set

$$\Phi_\omega(L) := \int_{F^n}\prod_{j=1}^{n-1}|P_{F_j}L|^{-\omega(j)+\omega(j+1)-1}d\xi \tag{55}$$

and

$$\Psi_\omega(L) := \int_{F^n}\prod_{j=1}^{n-1}|L\cap F_j|^{\omega(j)-\omega(j+1)+1}d\xi. \tag{56}$$

Note that

$$\sum_{j=1}^{n-1}(\omega(j)-\omega(j+1)+1) = n-\omega(n)+\omega(1)-1 \tag{57}$$

and

$$\sum_{j=1}^{n-1}j(\omega(j)-\omega(j+1)+1) = n(n-\omega(n)). \tag{58}$$

Identity (58) guarantees that $\Psi_\omega(L)$ and $\Phi_\omega(L)$ are 1-homogeneous when $\omega(n)\neq n$ and 0-homogeneous when $\omega(n)=n$.

The following fact for dual ω-flag quermassintegrals is from [16]. For ω-flag quermassintegrals, it follows, for example, by duality.

Theorem 4.2 *Let $\mathcal{E}$ be an ellipsoid in $\mathbb{R}^n$ and $\omega\in\Pi_n$. Then*

$$\Psi_\omega(\mathcal{E}) = \Psi_\omega(r_{\mathcal{E}}B_2^n) \text{ and } \Phi_\omega(\mathcal{E}) = \Phi_\omega(r_{\mathcal{E}}B_2^n).$$

An equivalent formulation of the latter result is that for every ellipsoid $\mathcal{E}$,

$$\Psi_\omega(\mathcal{E}) = c_\omega|\mathcal{E}|^{\frac{1}{n}},\ \omega(n)\neq n \text{ and } \Psi_\omega(\mathcal{E}) = c_\omega,\ \omega(n)=n,$$

where c_ω is a constant that depends only on ω. An analogous statement holds for $\Phi_\omega(\mathcal{E})$ (see the proof of Proposition 3.2).

The operators Ψ_ω and Φ_ω are generalizations of $\Psi_{\mathbf{r}}$ and $\Phi_{\mathbf{r}}$. Indeed, let $1\le r\le n-1$, $1\le i_1<i_2<\cdots<i_r\le n-1$, and $\mathbf{r}:=(i_1,\dots,i_r)$. Define ω by $\omega(1)=n-i_1+1$, and $\omega(t+1)=\omega(t)+1$, for $t\neq i_j$, $1\le j\le r$, and $\omega(i_j+1)=\omega(i_j)+1-i_{j+1}+i_{j-1}$ for $1\le j\le r$. Then $\omega\in\Pi_n$ with $\omega(i_j)-\omega(i_j+1)+1=i_{j+1}-i_{j-1}$ for $1\le j\le r$ and $\omega(t)-\omega(t+1)+1=0,\ t\neq i_j$, for $1\le j\le r$. Since $\omega(n)=n-i_r$, for a compact body L in $\mathbb{R}^n$,

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we have

$$\Psi_{\omega}^{ni_r}(L) = \int_{F^n} \prod_{j=1}^{n-1} |L \cap F_j|^{\omega(j)-\omega(j+1)+1} d\xi = \int_{F^n} \prod_{j=1}^{r} |L \cap F_{i_j}|^{i_{j+1}-i_{j-1}} d\xi = \Psi_{\mathbf{r}}^{ni_r}(L),$$

where, in the last equality, we have used (19). Correspondingly, we also have $\Phi_{\omega}(L) = \Phi_{\mathbf{r}}(L)$. In particular, for this permutation ω, $\Psi_{\omega}(L)$ is SL_n-invariant and $\Phi_{\omega}(L)$ is affine invariant. As a particular case of the preceding discussion, let $r = 1$, $i_1 = k$, $1 \leq k \leq n$, and let $\omega(1) = n - k + 1$ and $\omega(t+1) = \omega(t) + 1$ for $t \neq k$ and $\omega(k+1) = \omega(k) - n + 1$. Then $\Phi_{\omega}(L) = \Phi_{[k]}(L)$.

Given that $\Psi_{\mathbf{r}}(L)$ and $\Phi_{\mathbf{r}}(L)$ enjoy invariance properties and arise as permutations, it is natural to investigate the extent to which the invariance from Theorem 4.2 carries over to compact bodies. We do not have a complete answer. However, there are cases outside of those considered above where the invariance holds and also counter-examples where it fails as the next two examples show. It will be convenient to recall that a star-body L is a compact body with $0 \in \operatorname{int}(L)$ such that $\alpha x \in L$ whenever $x \in L$ and $\alpha \in [0,1]$, with continuous radial function $\rho_L(\theta) = \sup\{r \geq 0 : r\theta \in L\}$ $(\theta \in S^{n-1})$.

Example 1 Let $n \geq 3$. Define ω by $\omega(1) = 2$, $\omega(2) = 1$ and $\omega(t) = t$ for all $3 \leq t \leq n$. We will show that for every origin-symmetric star-body L in $\mathbb{R}^n$,

$$\Psi_{\omega}(L) = \frac{4}{\pi}.$$

In particular, $\Psi_{\omega}(L)$ is SL_n-invariant. Note that our choice of the permutation ω satisfies

$$\omega(1) - \omega(2) + 1 = 2, \quad \omega(2) - \omega(3) + 1 = -1, \quad \omega(j) - \omega(j+1) + 1 = 0 \text{ for } 3 \leq j \leq n-1,$$

or equivalently

$$\omega(2) = \omega(1) - 1, \quad \omega(3) = \omega(1) + 1, \quad \omega(j+1) = \omega(1) + (j-1) \text{ for } 3 \leq j \leq n-1.$$

Moreover, ω is the unique permutation satisfying the latter equations, as $1 \leq \omega(j) \leq n$ for all $1 \leq j \leq n$. For a k-dimensional subspace F_k of $\mathbb{R}^n$, denote by S_{F_k} the unit sphere in F_k. Now, using (19) and recalling the notation for ρ_L above, we compute

$$\begin{aligned}
\Psi_{\omega}(L) &= \int_{F^n} \prod_{j=1}^{n-1} |L \cap F_j|^{\omega(j)-\omega(j+1)+1} d\xi \\
&= \int_{F^n} |L \cap F_1|^2 |L \cap F_2|^{-1} d\xi \\
&= \int_{G_{n,2}} |L \cap F_2|^{-1} \int_{G_{F_2,1}} |(L \cap F_2) \cap F_1|^2 dF_1 dF_2 \\
&= \int_{G_{n,2}} |L \cap F_2|^{-1} \int_{S_{F_2}} (2\rho_{L\cap F_2}(\theta))^2 d\sigma(\theta) dF_2 \\
&= \int_{G_{n,2}} |L \cap F_2|^{-1} \frac{4}{|S_{F_2}|} \int_{S_{F_2}} \rho_{L\cap F_2}^2(\theta) d\theta \, dF_2
\end{aligned}$$

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$$= \frac{4}{\pi} \int_{G_{n,2}} |L \cap F_2|^{-1} |L \cap F_2| dF_2$$
$$= \frac{4}{\pi}.$$

When $n = 3$, for permutations ω with $\omega(3) = 3$, $\Psi_\omega(L)$ are absolute constants. Moreover, the discussion following Theorem 4.2 shows that for three of the remaining four permutations ω in Π_3, $\Psi_\omega(L) = \Psi_{\mathbf{r}}(L)$. Altogether, for $n = 3$, for five out of six permutations ω, $\Psi_\omega(L)$ are SL_n-invariant. The next example shows that for the remaining permutation, the invariance does not carry over for all convex bodies.

Example 2 Let $\omega \in \Pi_3$ with $\omega(1) = 1$, $\omega(2) = 3$, and $\omega(3) = 2$. We claim that for a centered cube $Q := [-1,1]^3$ and the diagonal matrix $D = \operatorname{diag}(1, 2, 1/2)$, $\Phi_\omega(DQ) > \Phi_\omega(Q)$. Since $D \in SL_3$, this shows that the operator Φ_ω is not invariant under volume-preserving transformations.

To show this, we first note that for any convex body $L \subset \mathbb{R}^3$

$$\Phi_\omega^{-3}(L) = \int_{G_{3,2}} |P_{F_2} L|^{-2} \int_{G_{F_2,1}} |P_{F_1} L| \, dF_1 dF_2 = \int_{S^2} \frac{W(P_{\phi^\perp} L)}{h^2_{\Pi L}(\phi)} d\sigma(\phi),$$

where $h_{\Pi L}(\phi) = |P_{\phi^\perp} L|$. For $\theta \in S^2$, we have $h_Q(\theta) = \sum_{i=1}^3 |\theta_i|$ and, for $g \in GL_3$, $h_{gL}(\theta) = h_L(g^t\theta)$. We will also use the following facts about projection bodies (see, e.g., [17]). The projection body of a cube is again a cube, $\Pi Q = 2Q$, and for $g \in GL_3$,

$$\Pi(gL) = |\det g| \, g^{-t} \, \Pi L.$$

Let $A = [a_1 \, a_2 \, a_3] \in SL_3$ with columns a_i. Fix $\phi \in S^2$. Let $U \in O_3$ be given in column form by $U = [u \; v \; \phi]$. Since U is orthogonal, $U^t\phi = e_3$ and $U^t\phi^\perp = \operatorname{span}\{e_1, e_2\} = \mathbb{R}^2$. Then

$$W(P_{\phi^\perp} AQ) = 2 \int_{S_{\phi^\perp}} h_{AQ}(\theta) d\sigma(\theta) = 2 \int_{S^1} h_{AQ}(U\theta) d\sigma(\theta) = 2 \int_{S^1} h_Q(A^t U\theta) d\sigma(\theta).$$

Thus, denoting by P the orthogonal projection onto $\mathbb{R}^2$, we have

$$W(P_{\phi^\perp} AQ) = 2 \sum_{i=1}^3 \int_{S^1} |\langle \theta, U^t A e_i \rangle| d\sigma(\theta) = 2 \sum_{i=1}^3 \int_{S^1} |\langle \theta, P U^t A e_i \rangle| d\sigma(\theta) = \frac{4}{\pi} \sum_{i=1}^3 \| P U^t A e_i \|_2.$$

One can verify the last equality in this setting by a direct computation as follows. We realize integration over S^1 as integration over $[0, 2\pi]$. Hence, we write $\theta = (\cos\alpha, \sin\alpha)$, $v \in \mathbb{R}^2$ as $v = \|v\|_2 (\cos\beta, \sin\beta)$ and $\int_{S^1} |\langle \theta, v \rangle| d\sigma(\theta) = \frac{\|v\|_2}{2\pi} \int_0^{2\pi} |\cos\alpha\cos\beta + \sin\alpha\sin\beta| d\alpha = \frac{\|v\|_2}{2\pi} \int_0^{2\pi} |\cos(\alpha - \beta)| d\alpha = \frac{\|v\|_2}{2\pi} \int_0^{2\pi} |\cos\alpha| d\alpha = \frac{2}{\pi} \|v\|_2$. We have that $Ae_i = a_i$, $U^t a_i = (\langle u, a_i \rangle, \langle v, a_i \rangle, \langle \phi, a_i \rangle)^t$ and

$$\|PU^t A e_i\|_2^2 = \|U^t A e_i\|_2^2 - \|(I - P) U^t A e_i\|_2^2 = \|a_i\|_2^2 - \langle \phi, a_i \rangle^2.$$

Therefore,

$$W(P_{\phi^\perp} AQ) = \frac{4}{\pi} \sum_{i=1}^3 \sqrt{\|a_i\|_2^2 - \langle \phi, a_i \rangle^2}.$$

Moreover, by the aforementioned properties of a centered cube, $h_{\Pi(AQ)}(\phi) = h_{\Pi Q}(A^{-1}\phi) = 2\sum_{i=1}^{3}|\langle A^{-1}\phi, e_i\rangle|$. Thus,

$$\Phi_\omega^{-3}(AQ) = \frac{1}{\pi}\int_{S^2} \frac{\sum_{i=1}^{3}\sqrt{\|a_i\|_2^2 - \langle\phi, a_i\rangle^2}}{\left(\sum_{j=1}^{3}|\langle A^{-1}\phi, e_j\rangle|\right)^2} d\sigma(\phi).$$

Set $A := \operatorname{diag}(d_1, d_2, d_3)$ with $\prod_{i=1}^{3} d_i = 1$ and $d_i > 0$. Then the quantity

$$\mathcal{A}(d_1, d_2, d_3) := \int_{S^2} \frac{\sum_{i=1}^{3} d_i\sqrt{1-\phi_i^2}}{\left(\sum_{j=1}^{3}\frac{|\phi_j|}{d_j}\right)^2} d\sigma(\phi)$$

is not constant. Indeed, using MATLAB, for example, one can verify that $\mathcal{A}(1,2,1/2) < \mathcal{A}(1,1,1)$.

In the case of convex bodies, the quantities $\Psi_\omega(K), \Phi_\omega(K)$ are uniformly bounded by a constant that depends on ω only. We will use the following well-known consequence of the celebrated "existence of M-ellipsoids" by V. D. Milman [40].

Theorem 4.3 *Let K be an origin-symmetric convex body in $\mathbb{R}^n$. Then there exists an ellipsoid $\mathcal{E}$ such that $|\mathcal{E}|^{1/n} \le e^c|K|^{1/n}$ and for every $F \in G_{n,k}$,*

$$|P_F\mathcal{E}| \le |P_FK| \le e^{cn}|P_F\mathcal{E}| \tag{59}$$

and

$$|\mathcal{E}\cap F| \le |K\cap F| \le e^{cn}|\mathcal{E}\cap F|, \tag{60}$$

where $c > 0$ is an absolute constant.

Corollary 4.4 *Let $\omega \in \Pi_n$ such that $\omega(n) \ne n$. Let $\delta_\omega(j) := \omega(j) - \omega(j+1) + 1$. Set*

$$I_\omega := \{j \le n : \delta_\omega(j) \ge 0\} \text{ and } \Delta(\omega) := \frac{\min\{\sum_{j\in I_\omega}\delta_\omega(j), \sum_{j\in I_\omega^c}|\delta_\omega(j)|\}}{n-\omega(n)} + 1.$$

We have that

$$e^{-c\Delta(\omega)}c_\omega|K|^{\frac{1}{n}} \le \Psi_\omega(K) \le e^{c\Delta(\omega)}c_\omega|K|^{\frac{1}{n}} \tag{61}$$

and

$$e^{-c\Delta(\omega)}c_\omega|K|^{\frac{1}{n}} \le \Phi_\omega(K) \le e^{c\Delta(\omega)}c_\omega|K|^{\frac{1}{n}}, \tag{62}$$

where $c > 0$ is an absolute constant.

Proof Set $\Delta_+(\omega) := \frac{\sum_{j\in I_\omega}\delta_\omega(j)}{n-\omega(n)}$ and $\Delta_-(\omega) := \frac{\sum_{j\in I_\omega^c}|\delta_\omega(j)|}{n-\omega(n)}$. In the ensuing computation, we use the bounds on $|K\cap F|$ in (60): for $j \in I_\omega$, we use the upper bound, while for $j \notin I_\omega$, we use the lower bound. Thus, we obtain

$$\begin{aligned}\Psi_\omega(K) &= \left(\int_{F_n}\prod_{j=1}^{n-1}|K\cap F_j|^{\omega(j)-\omega(j+1)+1}d\xi\right)^{\frac{1}{n(n-\omega(n))}}\\ &\le \left(\int_{F_n} e^{cn\sum_{j\in I_\omega}(\omega(j)-\omega(j+1)+1)}\prod_{j=1}^{n-1}|\mathcal{E}\cap F_j|^{\omega(j)-\omega(j+1)+1}d\xi\right)^{\frac{1}{n(n-\omega(n))}}\end{aligned}$$

$$
\begin{aligned}
&= e^{c\Delta_+(\omega)}\left(\int_{F_n}\prod_{j=1}^{n-1}|\mathcal{E}\cap F_j|^{\omega(j)-\omega(j+1)+1}d\xi\right)^{\frac{1}{n(n-\omega(n))}}\\
&= e^{c\Delta_+(\omega)}c_\omega|\mathcal{E}|^{\frac{1}{n}}\\
&\le e^{c(\Delta_+(\omega)+1)}c_\omega|K|^{\frac{1}{n}}.
\end{aligned}
$$

One can verify that a similar inequality with the quantity $\Delta_-(\omega)$ holds as well, which leads to the right-hand side in (61). The proof of the other inequalities is identical and hence is omitted. ■

Remark A similar proposition (with the same proof) holds for the case $\omega(n) = n$. Moreover, using Pisier's regular M-position (see [46]), one can get more precise estimates.

5 Functional forms

In this section, we derive functional forms of some of the previous geometric inequalities. Section 5.1 concerns the dual setting with sharp inequalities for restrictions of functions to flags of subspaces; these do not depend on their counterparts for sets. In Section 5.2, we define a new notion of $\mathbf{r}$-flag affine quermassintegrals for functions and establish a double-sided approximate isoperimetric inequality.

5.1 Functional forms of dual r-flag affine quermassintegrals

Let f be a bounded integrable function on $\mathbb{R}^n$. We denote by $I(f)$ the functional form of the dual $\mathbf{r}$-flag affine quermassintegral

$$
I(f) := \int_{F_{\mathbf{r}}^n}\prod_{j=1}^{r}\|f|_{F_j}\|_1^{i_{j+1}-i_{j-1}}d\xi_{\mathbf{r}}.
$$

Theorem 5.1 *For every $g \in SL_n$, $I(g\cdot f) = I(f)$, where $g\cdot f(x) = f(g^{-1}x)$.*

Proof Starting with the left-hand side, $I(g\cdot f)$, we do a global change of variables (21) on the flag manifold:

$$
I(g\cdot f) = \int_{F_{\mathbf{r}}^n}\prod_{j=1}^{r}\|g\cdot f|_{F_j}\|_1^{i_{j+1}-i_{j-1}}d\xi_{\mathbf{r}} = \int_{F_{\mathbf{r}}^n}\prod_{j=1}^{r}\|g\cdot f|_{g\cdot F_j}\|_1^{i_{j+1}-i_{j-1}}\sigma_{F_{\mathbf{r}}^n}(g,\xi)d\xi_{\mathbf{r}},
$$

where by (23) $\sigma_{F_{\mathbf{r}}^n}(g,\xi) = \sigma_{i_1}^{-i_2}(g,F_1)\sigma_{i_2}^{i_1-i_3}(g,F_2)\cdots\sigma_{i_r}^{i_{r-1}-n}(g,F_r)$. Now we do r local changes of variables (20) on each nested subspace F_j in the product. For each $1\le j\le r$, we thus have

$$
\|g\cdot f|_{g\cdot F_j}\|_1 = \|f|_{F_j}\|_1\,\sigma_{i_j}(g,F_j).
$$

For the product under the integral, we obtain

$$
\prod_{j=1}^{r}\|g\cdot f|_{g\cdot F_j}\|_1^{i_{j+1}-i_{j-1}} = \prod_{j=1}^{r}\left(\|f|_{F_j}\|_1\,\sigma_{i_j}(g,F_j)\right)^{i_{j+1}-i_{j-1}} = \prod_{j=1}^{r}\|f|_{F_j}\|_1^{i_{j+1}-i_{j-1}}\sigma_{F_{\mathbf{r}}^n}^{-1}(g,\xi).
$$

■

The latter result admits several generalizations as in [12] for functional forms of dual quermassintegrals. Rather than taking $L_1(F_j)$ norms, one can take $L_{p_j}(F_j)$ norms and replace the powers $i_{j+1}-i_{j-1}$ by α_j. As long as $\frac{\alpha_j}{p_j}=i_{j+1}-i_{j-1}$ and the integrals exist, the conclusion of Theorem 5.1 will hold. Theorem 5.1 also generalizes to a product of m functions. One can replace $\|f|_{F_j}\|_1^{i_{j+1}-i_{j-1}}$ by $\prod_{i=1}^m \|f_i|_{E_j}\|_{p_{i,j}}^{\alpha_{i,j}}$. For the analogue of Theorem 5.1 to hold in this case, we have to require $\sum_{i=1}^m \frac{\alpha_{i,j}}{p_{i,j}}=i_{j+1}-i_{j-1}$. Another way to generalize functional forms of dual quermassintegrals is to replace

$$\|f|_{F_j}\|_1^{i_{j+1}-i_{j-1}} \quad \text{by} \quad \frac{\|f|_{F_j}\|_{p_j}^{\alpha_j}}{\|g|_{F_j}\|_{q_j}^{\beta_j}} \quad \text{with} \quad \frac{\alpha_j}{p_j}-\frac{\beta_j}{q_j}=i_{j+1}-i_{j-1},$$

to ensure they remain invariant under volume-preserving transformations. Letting $q_j\to\infty$ modifies the integrand to $\frac{\|f|_{F_j}\|_{p_j}^{\alpha_j}}{\|f|_{F_j}\|_{\infty}^{\beta_j}}$ and the condition on the powers and norms to $\frac{\alpha_j}{p_j}=i_{j+1}-i_{j-1}$. Note that in this case the invariance holds for arbitrary powers β_j. As a particular case, this proves invariance under volume-preserving transformations of the integrand appearing in the next theorem. One can also take the quotient of products of functions, replacing

$$\|f|_{F_j}\|_1^{i_{j+1}-i_{j-1}} \quad \text{by} \quad \frac{\prod_{i=1}^m \|f_i|_{F_j}\|_{p_{i,j}}^{\alpha_{i,j}}}{\prod_{l=1}^{m'} \|g_l|_{F_j}\|_{q_{l,j}}^{\beta_{l,j}}} \quad \text{with} \quad \sum_{i=1}^m \frac{\alpha_{i,j}}{p_{i,j}}-\sum_{l=1}^{m'} \frac{\beta_{l,j}}{q_{l,j}}=i_{j+1}-i_{j-1}.$$

Here again, we can let $q_{l,j}\to\infty$, obtaining the corresponding generalization with no restrictions on $\beta_{l,j}$.

Theorem 5.2 *Let f be a non-negative bounded integrable function on $\mathbb{R}^n$, then*

$$\int_{F_{\mathbf{r}}^n}\prod_{j=1}^r \frac{\|f|_{F_j}\|_1^{i_{j+1}-i_{j-1}}}{\|f|_{F_j}\|_\infty^{i_{j+1}-i_j}}\,d\xi_{\mathbf{r}} \le \prod_{j=1}^r \frac{\omega_{i_j}^{i_{j+1}}}{\omega_{i_{j+1}}^{i_j}}\|f\|_1^{i_r}.$$

Proof The result follows by iteration of an inequality on $G_{n,k}$ for one function from our previous work [12, Theorem 1.2]:

$$\int_{G_{n,k}} \frac{\|f|_E\|_1^n}{\|f|_E\|_\infty^{n-k}}\,dE \le \frac{\omega_k^n}{\omega_n^k}\|f\|_1^k. \tag{63}$$

Applying the latter inequality repeatedly, we get

$$\begin{aligned}
&\int_{F_{\mathbf{r}}^n}\prod_{j=1}^r \frac{\|f|_{F_j}\|_1^{i_{j+1}-i_{j-1}}}{\|f|_{F_j}\|_\infty^{i_{j+1}-i_j}}\,d\xi_{\mathbf{r}}\\
&\quad=\int_{G_{n,i_r}}\cdots\int_{G_{F_3,i_2}}\prod_{j=2}^r \frac{\|f|_{F_j}\|_1^{i_{j+1}-i_{j-1}}}{\|f|_{F_j}\|_\infty^{i_{j+1}-i_j}}\int_{G_{F_2,i_1}}\frac{\|f|_{F_1}\|_1^{i_2}}{\|f|_{F_1}\|_\infty^{i_2-i_1}}\,dF_1dF_2\dots dF_r\\
&\quad\le \frac{\omega_{i_1}^{i_2}}{\omega_{i_2}^{i_1}}\int_{G_{n,i_r}}\cdots\int_{G_{F_3,i_2}}\prod_{j=2}^r \frac{\|f|_{F_j}\|_1^{i_{j+1}-i_{j-1}}}{\|f|_{F_j}\|_\infty^{i_{j+1}-i_j}}\|f|_{F_2}\|_1^{i_1}\,dF_2\dots dF_r
\end{aligned}$$

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$$= \frac{\omega_{i_1}^{i_2}}{\omega_{i_2}^{i_1}} \int_{G_{n,i_r}} \cdots \int_{G_{F_4,i_3}} \prod_{j=3}^{r} \frac{\|f|_{F_j}\|_1^{i_{j+1}-i_{j-1}}}{\|f|_{F_j}\|_\infty^{i_{j+1}-i_j}} \int_{G_{F_3,i_2}} \frac{\|f|_{F_2}\|_1^{i_3}}{\|f|_{F_2}\|_\infty^{i_3-i_2}} dF_2 dF_3 \dots dF_r$$
$$\le \frac{\omega_{i_1}^{i_2}}{\omega_{i_2}^{i_1}} \frac{\omega_{i_2}^{i_3}}{\omega_{i_3}^{i_2}} \int_{G_{n,i_r}} \cdots \int_{G_{F_4,i_3}} \prod_{j=3}^{r} \frac{\|f|_{F_j}\|_1^{i_{j+1}-i_{j-1}}}{\|f|_{F_j}\|_\infty^{i_{j+1}-i_j}} \|f|_{F_3}\|_1^{i_2} dF_3 \dots dF_r$$
$$= \cdots$$
$$\le \prod_{j=1}^{r} \frac{\omega_{i_j}^{i_{j+1}}}{\omega_{i_{j+1}}^{i_j}} \|f\|_1^{i_r}. \qquad \blacksquare$$

In [12], more general versions of (63) are proved with several functions and different powers. These also carry over to extremal inequalities on flag manifolds by mimicking the previous proof. As a sample, we mention just one statement. Let $1 \le q \le i_1$ and let $f_1, \dots, f_q$ be non-negative bounded integrable functions on $\mathbb{R}^n$, then

$$\int_{F_{\mathbf{r}}^n} \prod_{k=1}^{q} \prod_{j=2}^{r} \frac{\|f_k|_{F_j}\|_1^{\frac{i_{j+1}-i_j}{i_j}}}{\|f_k|_{F_j}\|_\infty^{\frac{i_{j+1}-i_j}{i_j}}} \frac{\|f_k|_{F_1}\|_1^{\frac{i_2}{i_1}}}{\|f_k|_{F_1}\|_\infty^{\frac{i_2-i_1}{i_1}}} d\xi_{\mathbf{r}} \le \prod_{j=1}^{r} \frac{\omega_{i_j}^{i_{j+1}}}{\omega_{i_{j+1}}^{i_j}} \prod_{k=1}^{q} \|f_k\|_1.$$

5.2 Functional forms of the r-flag affine quermassintegrals

In this subsection, we extend the notion of **r**-flag affine quermassintegrals to functions. In particular, this will lead to functional versions of affine quermassintegrals. This is motivated by recent work of Bobkov, Colesanti, and Fragalá [4] and V. Milman and Rotem [39]. The latter authors proposed and studied a notion of quermassintegrals for log-concave or even quasi-concave functions, which we now recall.

Definition 5.3 Suppose that $f : \mathbb{R}^n \to [0, \infty)$ is upper-semicontinuous and quasi-concave. For $1 \le k \le n$, let

$$V_k(f) := \int_0^\infty V_k(\{f \ge t\}) dt.$$

The above definition is consistent with the notion of projection of a function onto a subspace as introduced by Klartag and V. Milman in [25]. Namely, let $f : \mathbb{R}^n \to [0, \infty]$ be a non-negative function and $F \in G_{n,k}$. Define the orthogonal projection of f onto F as the function $P_F f : F \to [0, \infty]$ given by

$$(P_F f)(z) := \sup_{y \in F^\perp} f(z + y).$$

Note that if K is compact and $f := \mathbf{1}_K$, then $P_F f := \mathbf{1}_{P_F(K)}$. Moreover, from the definition, one has

$$\{z \in F : (P_F f)(z) > t\} = P_F(\{x \in \mathbb{R}^n : f(x) > t\}).$$

Assume from now on that $f : \mathbb{R}^n \to [0, \infty)$ and

(64) $$\{x \in \mathbb{R}^n : f(x) \ge t\} \text{ is a compact body for each } t > 0.$$

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For $1 \le k \le n-1$, we define the affine quermassintegral of f by

$$\Phi_{[k]}(f) := \int_0^\infty \Phi_{[k]}(\{f \ge t\})dt = \int_0^\infty \left(\int_{G_{n,k}} |\{P_F f \ge t\}|^{-n} dF \right)^{-\frac{1}{nk}} dt. \tag{65}$$

Note that for $f = \mathbf{1}_K$, we have $\Phi_{[k]}(\mathbf{1}_K) = \int_0^1 \Phi_{[k]}(K)dt = \Phi_{[k]}(K)$; hence, the notions coincide for sets. For $1 \le r \le n-1$ and $1 \le i_1 < i_2 < \cdots < i_r \le n-1$, $\mathbf{r} := (i_1, \cdots, i_r)$, we define the $\mathbf{r}$-flag affine quermassintegrals of f by

$$\Phi_{\mathbf{r}}(f) := \int_0^\infty \Phi_{\mathbf{r}}(\{f \ge t\})dt. \tag{66}$$

For comparison, we recall that for every $f : \mathbb{R}^n \to [0, \infty]$,

$$\int_{\mathbb{R}^n} f(x)dx = \int_0^\infty |\{f \ge t\}|dt.$$

For $\lambda \in \mathbb{R}\backslash\{0\}$ and f as above, we write

$$f_{(\lambda)} : \mathbb{R}^n \to [0, \infty], \text{ as } f_{(\lambda)}(x) := f\left(\frac{x}{\lambda}\right),$$

and if $T \in GL_n$,

$$f \circ T : \mathbb{R}^n \to [0, \infty], \text{ as } f \circ T(x) := f(T^{-1}x).$$

Note that if $f := \mathbf{1}_K$, then

$$f_{(\lambda)}(x) = \mathbf{1}_{\lambda K}(x) \text{ and } f \circ T(x) = \mathbf{1}_{TK}(x).$$

For $f : \mathbb{R}^n \to [0, \infty]$, $\lambda > 0$, and $T \in GL_n$, we have

$$\{f \circ T \ge t\} = T(\{f \ge t\}) \text{ and } \{f_{(\lambda)} \ge t\} = \lambda\{f \ge t\}.$$

Thus, by the 1-homogeneity of the $\mathbf{r}$-flag affine quermassintegrals for sets, and the affine invariance of these quantities, we obtain the following theorem.

Theorem 5.4 *Let $f : \mathbb{R}^n \to [0, \infty]$ satisfy (64). Let $\lambda > 0$ and T be an affine volume-preserving map. Then*

$$\Phi_{\mathbf{r}}(f_{(\lambda)}) = \lambda \Phi_{\mathbf{r}}(f) \text{ and } \Phi_{\mathbf{r}}(f \circ T) = \Phi_{\mathbf{r}}(f).$$

We recall the symmetric decreasing rearrangement of an integrable function f. For a set $A \subseteq \mathbb{R}^n$ with finite volume, the decreasing rearrangement A^* is defined as $A^* := r_A B_2^n$, where r_A is the volume-radius of A. The symmetric decreasing rearrangement f^* of f is defined as the radial function f^* such that, for $t > 0$,

$$\{f \ge t\}^* = \{f^* \ge t\}.$$

In particular, $r_{\{f \ge t\}} B_2^n = \{f^* \ge t\}$. Using (66) and (35), we have, for non-negative quasi-concave functions f on $\mathbb{R}^n$,

$$\Phi_{\mathbf{r}}(f) = \int_0^\infty \Phi_{\mathbf{r}}(\{f \ge t\})dt \ge \int_0^\infty \Phi_{\mathbf{r}}(r_{\{f \ge t\}} B_2^n)dt = \int_0^\infty \Phi_{\mathbf{r}}(\{f^* \ge t\})dt = \Phi_{\mathbf{r}}(f^*).$$

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Lastly, let f be a non-negative quasi-concave function on $\mathbb{R}^n$ satisfying (64). We define

$$d_{BM}(f) := \sup_{t>0} d_{BM}(\{f \geq t\}).$$

The results of Section 3 lead to the following double-sided inequality for $\Phi_{\mathbf{r}}(f)$:

Theorem 5.5 *Let f be a non-negative quasi-concave function on $\mathbb{R}^n$ satisfying (64). Then*

$$\Phi_{\mathbf{r}}(f^*) \leq \Phi_{\mathbf{r}}(f) \leq c' \min\left\{\sqrt{\frac{n}{i_r}}, \log(1 + d_{BM}(f))\right\} \Phi_{\mathbf{r}}(f^*).$$

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