L-IDEALS OF $M(G)$ DETERMINED BY CONTINUITY OF TRANSLATION

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1. Introduction

G denotes a locally compact abelian group and $M(G)$ the convolution algebra of regular bounded Borel measures on G. An ideal I of $M(G)$ closed in the usual (total variation) norm topology is called an *L-ideal* if $\mu \in I$, $\nu \ll \mu$ (ν absolutely continuous with respect to μ) implies that $\nu \in I$. Here we are concerned with the L-ideals $L^1(G)$, $L^{\frac{1}{2}}(G)$, and $M_0(G)$ where, as usual, $L^1(G)$ denotes the set of measures absolutely continuous with respect to Haar measure, $L^{\frac{1}{2}}(G)$ denotes the radical of $L^1(G)$ in $M(G)$ and $M_0(G)$ denotes the set of measures whose Fourier-Stieltjes transforms vanish at infinity.

It has long been known that $L^1(G)$ is determined as a subset of $M(G)$ by the norm continuity of translation. This is perhaps most easily seen by noting that $\mu \in L^1(G)$ if and only if $|\mu * \delta(x)(K) - \mu(K)| \to 0$ as $x \to 0$ for every compact subset K of G. Goldberg and Simon proved in (3) that the continuity of translation under the supremum norm of Fourier-Stieltjes transforms characterises $M_0(G)$, and they conjectured that $L^{\frac{1}{2}}(G)$ is determined by the analogous property with respect to the Gelfand (spectral) norm. We verify that conjecture, give a simple proof of the $M_0(G)$ result and show that in each case norm convergence can be replaced by an appropriate pointwise convergence. The case of $L^{\frac{1}{2}}(G)$ is the more interesting and will be discussed in Section 3 after the more straightforward results have been proved in Section 2.

First, however, we need some notation to make our assertions precise. Elements of the dual space, $M(G)^*$, can be regarded as generalised functions, i.e. as members $f = (f_\mu)$ of the product $\prod_{\mu \in M(G)} L^\infty(\mu)$ which satisfy the appropriate consistency conditions. Observe that the $\sigma(M(G)^*, M(G))$-topology is realised in this representation as the product of the $\sigma(L^\infty(\mu), L^1(\mu))$-topologies.

Any bounded measurable function on G can be regarded as an element of $M(G)^*$ by the definition $f_\mu = f$ (μ a.e.) ($\forall \mu \in M(G)$). In this way we regard the dual group $G^\wedge$ as a set of generalised functions, while we adopt the usual representation of elements of the maximal ideal space, Δ, of $M(G)$ as generalised characters (cf. e.g. (**1**)). For $f \in \Delta$, we write $\mu^\wedge(f) = \int f_\mu d\mu$; this, of course, defines the Gelfand transform of μ and, by restricting to the dual group $G^\wedge$, the Fourier-Stieltjes transform.

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Given a set F of generalised functions we define

$$T(F) = \{\mu \in M(G)\colon\ x \rightsquigarrow (\delta(x) * \mu)\hat{}(f) \text{ is continuous for } f \in F\}$$

$$T_u(F) = \{\mu \in M(G)\colon\ x \rightsquigarrow (\delta(x) * \mu)\hat{}(f) \text{ is equicontinuous over } f \text{ in } F\}.$$

Our main result is then

Theorem

(i) $L^{\natural}(G) = T(\Delta) = T_u(\Delta)$;

(ii) $M_0(G) = T(G\hat{}) = T_u(G\hat{})$.

Observe that, since $(\delta(x) * \mu)\hat{}(f) = \delta(x)\hat{}(f)\mu\hat{}(f)$ for $f \in \Delta$, it is enough to consider continuity of translation at 0. In particular, for the cases covered by the theorem, μ belongs to T_u if and only if $\| \delta(x) * \mu - \mu \| \to 0$ as $x \to 0$, where $\| \ \|$ denotes successively the Gelfand norm and the Fourier-Stieltjes norm.

A corresponding result characterises $L^1(G)$ as $T(U) = T_u(U)$, where U is the unit ball of $M(G)^*$. This is obtained by noting that the characteristic functions of compact subsets of G can be regarded as members of U, and applying the result mentioned in the second paragraph. Observe that this can also be established by noting that U is the closure in $M(G)^*$ of the trigonometric polynomials with supremum norm not greater than 1, and applying the methods we use in this paper. This is, of course, an absurdly inefficient way to establish the result—but we did exactly that in a preliminary version. We thank Dr J. W. Baker who was the first to bring this to our attention.

2. The easy cases

Proposition 1.

$$M_0(G) \subseteq T_u(G\hat{}),\ L^{\natural}(G) \subseteq T_u(\Delta).$$

Proof. For the first inclusion, fix $\mu \in M_0(G)$ $(\mu \neq 0)$, $x_\alpha \to 0$, and $\varepsilon > 0$. Choose the compact K in G such that $|\hat{\mu}(\gamma)| < \frac{\varepsilon}{2}$ off K and then α_0 such that $|\gamma(x_\alpha) - 1| < \frac{\varepsilon}{2} \|\mu\|^{-1}$ uniformly on K for $\alpha \geqq \alpha_0$. Then

$$\sup_G |(\delta(x_\alpha) * \mu)\hat{}(\gamma) - \mu\hat{}(\gamma)| = \sup_G |\gamma(x_\alpha) - 1| \ |\mu\hat{}(\gamma)| < \varepsilon$$

for $\alpha \geqq \alpha_0$.

For the second inclusion, note that if $\mu \in L^{\natural}(G)$ then $\mu \in M_0(G)$ and the spectral norm of $\delta(x_\alpha) * \mu - \mu$ coincides with the Fourier-Stieltjes norm.

Proposition 2.

$$T(G\hat{}) \subseteq M_0(G).$$

Proof. Suppose that $\mu \in T(G^\wedge)\backslash M_0(G)$. Then there exists $\varepsilon>0$ such that the set

$$K(\varepsilon) = \{\gamma \in G^\wedge : |\mu^\wedge(\gamma)| \geqq \varepsilon\}$$

is not compact in $G^\wedge$. The closure of (the canonical image of) $K(\varepsilon)$ in the Bohr compactification, $bG^\wedge$, of $G^\wedge$ is compact. Any subset of $G^\wedge$ whose canonical image in $bG^\wedge$ is compact is already compact in $G^\wedge$, (cf. (**2**)), so the closure of $K(\varepsilon)$ must intersect $bG^\wedge\backslash G^\wedge$. This guarantees the existence of $\phi \in bG^\wedge\backslash G^\wedge$ and a net $\{\gamma_\alpha\}$ of elements of $K(\varepsilon)$ such that $\gamma_\alpha \to \phi$ in $bG^\wedge$. Passing to a subnet, if necessary, we may also assume that $\gamma_\alpha \to f$ in Δ. It is obvious that

$$\delta(x)^\wedge(f) = \phi(x) \ (\forall x \in G).$$

Now choose $x_\beta \to 0$ in G such that $\phi(x_\beta) \not\to 1$. Then

$$(\delta(x_\beta) * \mu)^\wedge(f) - \mu^\wedge(f) = (\phi(x_\beta) - 1)\mu^\wedge(f) \not\to 0,$$

which is a contradiction.

Now we have $T(G^\wedge) \subseteq M_0(G) \subseteq T_u(G^\wedge)$, hence equality throughout and a generalisation of the Goldberg-Simon result.

3. The radical ideal

Before tackling $T(\Delta(G))$ we must first establish some properties of $L^{\ddagger}(G)$. The most basic, which we give in the next lemma, was proved by Taylor (**5**) but we prefer to give a proof in the framework of generalised characters.

Lemma 1. *Let N be an L-ideal of $M(G)$ and $N^{\ddagger}$ the radical of N in $M(G)$. Then $N^{\ddagger}$ is the L-ideal $\{\mu \in M(G): \mu^n \in N^\perp, n = 1, 2, \ldots\}^\perp$.*

Proof. $N^{\ddagger}$ is clearly a closed ideal so we check the L-space property by showing $\gamma \in G^\wedge, \mu \in N^{\ddagger} \Rightarrow \gamma.\mu \in N^{\ddagger}$. Let Θ denote the hull of N. Now suppose $\gamma \in G^\wedge$, $\mu \in N^{\ddagger}$ and $(\gamma.\mu)^\wedge(f) \neq 0$. Then $\gamma.f \in \Delta\backslash\Theta$. Hence $\nu^\wedge(\gamma.f) \neq 0$ for some $\nu \in N$. But $\gamma.\nu \in N$ by hypothesis so $f \in \Delta\backslash\Theta$.

Now suppose $\mu \not\perp \nu$ where $\nu^n \in N^\perp$ for all n. We choose $\omega \neq 0$ absolutely continuous with respect to both μ, ν. Then $\omega^n \perp N$ for all n, and so

$$\inf_{\lambda \in N} \| \omega^n - \lambda \|^{1/n} = \| \omega^n \|^{1/n} \neq 0.$$

Hence the spectral norm of the image of ω in $M(G)/N$ is non-zero and thus $\omega \notin N^{\ddagger}$ which forces $\mu \notin N^{\ddagger}$.

Conversely suppose $\mu \notin N^{\ddagger}$ and choose $\nu \ll \mu$, $\nu \geqq 0$, such that $\nu \perp N^{\ddagger}$. There exists $f \in \Theta$ such that $\nu^\wedge(f) \neq 0$. Clearly $f_\nu \not\equiv 0$, so for all $\varepsilon>0$, $\nu^\wedge(|f|^\varepsilon) \neq 0$. Let g be a limit point of $(|f|^\varepsilon)$ as $\varepsilon \to 0$.

Then g is an idempotent, $g_\lambda \equiv 0$ $(\forall \lambda \in N)$, $g.\nu$ is a positive non-zero measure and $(g.\nu)^n = g.\nu^n$ for all n. Hence $(g.\nu)^n \perp N$ for all n, and the proof is complete.

Lemma 2. *Let A be an L-subalgebra of $M(G)$ such that $A \not\subset L^{\ddagger}(G)$ then there exists a probability measure μ in A such that $\mu^2 \ll \mu$ and $\mu \perp L^{\ddagger}(G)$.*

Proof. Apply Lemma 1 with $N = L^{\natural}(G)$ to derive

$$L^{\natural}(G) = \{\mu \in M(G) \colon \ \mu^n \perp L^{\natural}(G) \quad n = 1, 2, \ldots\}^{\perp}.$$

Hence there exists $\lambda \in A$, $\lambda \geqq 0$, $\| \lambda \| = 1$, such that $\lambda^n \perp L^{\natural}(G)$ for all n. Set $\mu = \sum_{n=1}^{\infty} 2^{-n}\lambda^n$.

Lemma 3. *Let $\phi\colon G \to H$ be a continuous surjective homomorphism with compact kernel K. Let $\phi^*\colon M(G) \to M(H)$ be the induced homomorphism of measure algebras, and let m_K be the Haar measure of the group K. Then ϕ^* induces an isomorphism between $L^{\natural}(G) * m_K$ and $L^{\natural}(H)$.*

Proof. Note that ϕ^* preserves both algebraic and L-structure, in particular $\phi^*(L^1(G)) \subseteq L^1(H)$. Also ϕ^* induces an isomorphism between $M(G)*m_K$ and $M(H)$. Thus $L^1(G)*m_K$ is isomorphic to $L^1(H)$. There are various ways to complete the proof. For example the assumption $\phi^*(L^{\natural}(G)) \not\subset L^{\natural}(H)$ would lead by the previous lemma to the existence of some $\mu * m_K \in L^{\natural}(G)$ all of whose powers were singular to $L^1(G)$, thus contradicting Lemma 1. Also

$$(\phi^*)^{-1}(L^{\natural}(H) \cap (M(G)*m_K))$$

is clearly the radical in $M(G) * m_K$ of $L^{\natural}(G)*m_K$ and is *a fortiori* contained in $L^{\natural}(G)$.

In view of this result we will eventually prove that $\phi^*(T(\Delta(G))) = T(\Delta(H))$, at the moment we require only:

Lemma 4. *With the notation of Lemma* 3,

$$\phi^*(T(\Delta(G))) \subseteq T(\Delta(H)).$$

Proof. In the contrary case we may suppose that there exist $\mu \in T(\Delta(G))$, $f \in \Delta(H)$, $\varepsilon > 0$ and a net $y_\alpha \to 0$ in H, such that for all α,

$$|\,(\delta(y_\alpha) * \phi^*(\mu) - \phi^*(\mu))^\wedge(f)\,| \geqq \varepsilon. \tag{1}$$

Choose z_α arbitrarily such that $\phi(z_\alpha) = y_\alpha$ and some compact neighbourhood V of 0 in G. Then there exists α_0 such that $z_\alpha \in V + \ker \phi$, for all $\alpha \geqq \alpha_0$. Since $\ker \phi$ is compact there is some subnet (z_β) such that $z_\beta \to z$.

Then $y_\beta = \phi(z_\beta) \to \phi(z)$, so, of course, $z \in \ker \phi$. Now write $x_\beta = z_\beta - z$, to obtain $\phi(x_\beta) = y_\beta$ and $x_\beta \to 0$ in G. Observe next that, since $f \in \Delta(H)$, $f \circ \phi^*$ is a well-defined member of $\Delta(G)$. Because μ belongs to $T(\Delta(G))$, this gives

$$|\,(\delta(x_\beta) * \mu)^\wedge(f \circ \phi^*) - \mu^\wedge(f \circ \phi^*)\,| \to 0,$$

equivalently

$$|\,\delta(\phi(x_\beta)) * \phi^*(\mu)^\wedge(f) - \phi^*(\mu)^\wedge(f)\,| \to 0,$$

and this contradicts (1).

Lemma 5. *It suffices to prove $L^{\natural}(G) = T(\Delta(G))$ for metrisable groups.*

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Proof. Evidently $T(\Delta(G)) \subset T(\mathrm{cl}\, G^\wedge)$, so by Proposition 2 we have $T(\Delta(G)) \subset M_0(G)$ in the general case. Now given $\mu \in T(\Delta(G)$, there is a σ-compact subgroup Λ of $G^\wedge$ such that $\hat{\mu} \equiv 0$ outside of Δ. Then $G/\Lambda^\perp$ is metrisable but $\Lambda^\perp$, the annihilator of Λ need not be compact. Therefore take some compact $\mathscr{G}_\delta$ subgroup, K, of $\Lambda^\perp$ so that $\Lambda^\perp/K$ is metrisable (cf. (4), Section 64) and $H = G/K$ is metrisable. Writing ϕ for the quotient map as in Lemma 1, we have $\phi^*(\mu) \in T(\Delta(H)) = L^\ddagger(H)$. Thus, by Lemma 3, $\mu * m_K$ belongs to $m_K * L^\ddagger(G) \subset L^\ddagger(G)$. By the choice of K, $\mu = \mu * m_K$, and the lemma is proved.

Lemma 6. *$T(\Delta(G))$ is an L-ideal.*

Proof. $T(\Delta(G))$ is clearly a linear subspace, and is seen to be (variation) norm closed from the formula

$$|(\delta(x) * \mu)^\wedge(f) - \mu^\wedge(f)| \leqq 2\| \mu_n - \mu \| + |(\delta(x) * \mu_n)^\wedge(f) - \mu_n^\wedge(f)|,$$

which is valid for μ, $\mu_n \in M(G)$, $x \in G$, $f \in \Delta(G)$.

To complete the proof that $T(\Delta(G))$ is an L-space it is sufficient to check that $\mu \in T(\Delta(G))$ implies $\gamma . \mu \in T(\Delta(G))$. This follows immediately from the fact that $\Delta(G)$ is a semigroup containing $G^\wedge$.

It remains only to check that $T(\Delta(G))$ is an ideal. In fact, suppose that $\mu \in T(\Delta(G))$, $\nu \in M(G)$, $f \in \Delta(G)$, and note that

$$|(\delta(x) * \nu * \mu)^\wedge(f) - (\nu * \mu)^\wedge(f)| \leqq \| \nu \| \; |(\delta(x) * \mu)^\wedge(f) - \mu^\wedge(f)|.$$

The result now follows.

We consider now one of the few cases in which there is a universal extension theorem for generalised characters.

Lemma 7. *If H is an open subgroup of G then every element of $\Delta(H)$ extends to $\Delta(G)$.*

Proof. Given $f \in \Delta(H)$, we let γ be the character of H_d (the group H with the discrete topology) induced by f; in other words we define

$$\gamma(x) = \delta(x)^\wedge(f) \; (\forall x \in H).$$

Let γ' be any extension of γ to the group G_d. Now define f' in $\Delta\,(G)$ as follows: given $\mu \in M(G)$ then μ is of the form

$$\sum_{k=1}^{\infty} \delta(x_k) * \mu_k, \tag{2}$$

where each μ_k belongs to $M(H)$ and the x_k belong to distinct cosets of H. Now write

$$\mu^\wedge(f') = \sum_{k=1}^{\infty} \gamma'(x_k)\mu_k^\wedge(f). \tag{3}$$

(Observe that (3) is the only candidate for the definition of f', so that, once its validity has been checked, we will have shown that $\Delta(G)$ is determined by $\Delta(H)$

modulo the extension $H_d^\wedge$ to $G_d^\wedge$.) Any other representation of μ of the type given in (2) takes the form $\sum_{k=1}^{\infty} \delta(x_k') * \mu_k'$, where $x_k' \in x_k + H$, and since the closed cosets are mutually disjoint we have $\delta(x_k) * \mu_k = \delta(x_k') * \mu_k'$, for all k. Hence the first consistency check requires simply that $\gamma'(x_k)\mu_k^\wedge(f) = \gamma'(x_k')(\mu_k')^\wedge(f)$, and this is seen to be true by applying f to the relation $\delta(x_k - x_k') * \mu_k = \mu_k'$ which is expressed in $M(H)$. Therefore (3) is a good definition but it remains to check that f' is a complex homomorphism of $M(G)$. In view of (2) the relations which occur in checking $f'(\mu + \nu) = f'(\mu) + f'(\nu)$, $f'(\mu * \nu) = f'(\mu)f'(\nu)$ can be broken down into countably many relations each of which involves only measures supported by some fixed coset. These " elementary " relations must then take the form

$$\sum_{k=1}^{\infty} \delta(y_k) * \lambda_k = \sum_{j=1}^{\infty} \delta(z_j) * \sigma_j,$$

where λ_k, σ_j belong to $M(H)$; y_k, z_j belong to $x + H$, say, for all k, j. Multiplying both sides of (4) by $\delta(-x)$ we obtain a relation in $M(H)$ and therefore f' respects (4). This completes the proof.

Lemma 8. *If H is an open subgroup of G and $T\,(\Delta(H)) = L^{\frac{1}{2}}(H)$ then $T(\Delta(G)) = L^{\frac{1}{2}}(G)$.*

Proof. If $T(\Delta(G)) \neq L^{\frac{1}{2}}(G)$, then using the fact that both these spaces are translation invariant we can find μ supported by H such that $\mu \in T(\Delta(G))$ but $\mu \perp L^{\frac{1}{2}}(G)$. If now $x_\alpha \to x$ in H and $f \in \Delta(H)$ we simply extend f to f' in $\Delta(G)$ by the previous lemma to see that $(\delta(x_\alpha) * \mu)^\wedge(f') = (\delta(x_\alpha) * \mu)^\wedge(f)$ tends to $\mu^\wedge(f') = \mu^\wedge(f)$. Thus $\mu \in T(\Delta(H))$. On the other hand $L^{\frac{1}{2}}(G) \cap M(H) = L^{\frac{1}{2}}(H)$ so we have a contradiction.

The problem is now reduced to the study of non-discrete groups of the form $\mathbf{R}^q \times K$ where K is compact, and can be reduced still further by means of the open subgroup property. In fact either m_K-almost all elements of K have infinite order or for some positive integer n the kernel of the homomorphism $\boldsymbol{n}$: $K \to K$ defined by $\boldsymbol{n}(x) = nx$ has positive m_K measure. In the latter case, normalising the Haar measure of K as 1, we have $m_K(\text{Ker } \boldsymbol{n})$ is the reciprocal of the index of Ker $\boldsymbol{n}$ in K. This shows that Ker $\boldsymbol{n}$ has finite index and hence is open. Now choose n to be minimal with respect to this property—then almost all elements of Ker $\boldsymbol{n}$ have order n. In this way we may define $\Psi(K)$, the essential exponent of K, to be ∞ or the least positive integer n such that K has an open subgroup almost all of whose elements have order n. Moreover, as far as proving the theorem is concerned we may as well restrict attention to the groups $G = \mathbf{R}^q \times K$ described above, where, in addition, K is of *reduced form* in the sense that all elements of K have order $\leqq \Psi(K)$ and almost all elements of K have order $\Psi(K)$.

The strategy is to employ induction on the essential exponent. In particular the base of the induction is when $\Psi(K) = 1$, hence $G = \mathbf{R}^q$. In order to accom-

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modate this case in the statement of the next lemma we find it a technical convenience to introduce the quantity $\Phi(K)$ defined by

$$\Phi(K) = \Psi(K), \text{ for } \Psi(K) \neq 1; \ \Phi(K) = \infty, \text{ otherwise.}$$

Let us recall here that a subset X of G is independent if

$$\sum_{i=1}^{k} m_i x_i = 0, \quad m_i \in \mathbf{Z}, \quad x_i \in X,$$

implies $m_i x_i = 0$ for all i, and that $gp(X)$ denotes the subgroup of G generated by X.

Lemma 9. *Let G be a non-discrete group of the form $\mathbf{R}^q \times K$, with K compact and metrisable. Suppose that $\mu \in M(G)$ satisfies $\mu \perp L^1(G)$ and is such that, for all y in G and for all positive integers p such that $\mathbf{p}(K)$ is infinite,*

$$\delta(y) * m_{\mathbf{p}(K)} * \mu \perp \mu.$$

Then there exists a sequence (x_n) in G with the following properties:

(i) $x_n \to 0$ *as* $n \to \infty$,

(ii) $X = \{x_n\colon n = 1, 2, \ldots\}$ *is an independent set,*

(iii) *the order of x_n equals $\Phi(K)$ for all n,*

(iv) *for all non-zero y in $gp(X)$, $\delta(y) * \mu \perp \mu$.*

Proof. It is straightforward to check that we may assume, without loss of generality, that μ is a probability measure and that K is in reduced form. This we do and consider three cases.

(*a*) Suppose that $\Psi(K) = \infty$. We define the sequence (x_n) $n = 1, 2, \ldots$ by induction, starting by introducing the auxiliary element $x_0 = 0$. Fix a sequence (V_n) of symmetric neighbourhoods of zero in $\{0\} \times K$ such that diam $(V_n) \to 0$. Suppose that $x_0, x_1, \ldots, x_n$ have been chosen, for some $n \geqq 0$, such that $x_j \in V_j$ and x_j has infinite order for $j = 1, \ldots, n$; the set $X_n = \{x_1, \ldots, x_n\}$ is independent and for all non-zero $y \in gp(X_n)$, $\delta(y) * \mu \perp \mu$. (The conditions are, of course vacuously true when $n = 0$.) It will be possible to continue the induction provided the (measurable) set

$$N(p, y, n) = \{x \in V_{n+1}\colon \| \delta(px+y) * \mu - \mu \| < 2\}$$

is m_K null for all non-zero integers p and all elements y of the (countable) group $gp(X_n)$. In the contrary case we have the existence of a positive integer p and a (possibly zero) $y \in gp(X_n)$ such that $m_K(N(p, y, n)) > 0$. Define the probability measure λ $(= \lambda(p, y, n))$ to be the normalised restriction of m_K to $N(p, y, n)$. Observe that $\mathbf{p}^*(\lambda)$ is again a probability measure. Now for any $f \in C_0(G)$, $\|f\|_\infty \leqq 1$, we have

$$|(\mathbf{p}^*(\lambda) * \delta(y) * \mu)^\wedge(f) - \mu^\wedge(f)| = |\textstyle\int ((\delta(y) * \mu)^\wedge({}_x f) - \mu^\wedge(f)) d\mathbf{p}^*(\lambda)(x)|$$
$$\leqq \int |(\delta(y) * \mu)^\wedge({}_{px} f) - \mu^\wedge(f)| \, d\lambda(x).$$

where ${}_x f(y) = f(x+y)$ $(x, y \in G)$.

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Since the last quantity is strictly less than two, we see that μ is not singular to $\delta(y) * p^*(\lambda) * \mu$ which is in turn absolutely continuous with respect to $\delta(y) * m_{p(K)} * \mu$. Therefore $\mu \not\perp \delta(y) * m_{p(K)} * \mu$. But $p(K)$ is infinite because $\Psi(K)$ is infinite, so the assumption of the lemma is contradicted and the inductive construction proceeds without hindrance.

(*b*) Suppose that $\Phi(K)$ is finite. Here $\Psi(K) = \Phi(K) = k$, for some positive integer k. Since K is in reduced form $ak(K) = \{0\}$ for all integers a and $b(K)$ is infinite for $0<b<k$. Hence $p(K)$ is infinite unless p is a multiple of k. In the analogous construction to that given in (*a*) we choose each x_n, $n \geqq 1$, to have order k and hence ignore the case where p is a multiple of k in the discussion of $N(p, y, n)$. This means that we obtain the required contradiction as before. (Observe that in neither of these cases did we make explicit use of the assumption that μ is singular to $L^1(G)$, but in the next case this will not be a consequence of the other assumption which is vacuously true.)

(*c*) Suppose $\Phi(K) = \infty$, $\Psi(K) = 1$. Since K is in reduced form this implies that $G = \mathbf{R}^q$. Accordingly we choose the neighbourhoods V_n in $\mathbf{R}^q$ and add the hypothesis that they are compact. The induction proceeds as in case (*a*). $N(p, y, n)$ has the same formal definition but λ is now the normalised restriction of $m_{\mathbf{R}^q}$ to $N(p, y, n)$. One obtains as before that μ is not singular to

$$\delta(y) * p^*(\lambda) * \mu$$

but on this occasion $p^*(\lambda) \in L^1(G)$, hence μ is not singular to $L^1(G)$. This is the required contradiction which justifies stage $n+1$ of the induction in case (*c*), and the proof of the lemma is now complete.

The next lemma is the last (conceptually the first!) step required to establish the machinery of an inductive attack on the theorem. G is assumed to be of the form $\mathbf{R}^q \times K$.

Lemma 10. *Suppose that μ is a positive measure in $M(G)$ which has the property $\mu^2 \ll \mu$ and satisfies the conclusion of Lemma 9 for some sequence (x_n). Then μ does not belong to $T(\Delta(G))$.*

Proof. We simply prove the existence of a generalised character f such that

$$(\delta(x_n) * \mu)\hat{}(f) \not\to \mu\hat{}(f) \text{ as } n \to \infty.$$

To do this first construct a generalised character g on the L-subalgebra A generated by μ and $\{\delta(y)\colon y \in gp\{x_n\colon n = 1, 2, \ldots\}\}$ as follows. Using the independence of (x_n) choose $\gamma \in G_d\hat{}$ such that $\gamma(x_n) \not\to 1$. Every ν in A is absolutely continuous with respect to some measure ω of the form

$$\omega = \sum_{i=1}^{\infty} \delta(y_i) + \sum_{i=1}^{\infty} \delta(z_i) * \mu, \; y_i, z_i \in gp\{x_n\}.$$

Define $g_\omega(y_i) = \gamma(y_i)$, $g_\omega(t) = \gamma(z_i)$ $((\delta(z_i) * \mu)$ a.e.). Since $\delta(z_i) * \mu \perp \delta(z_j) * \mu$ unless $z_i = z_j$ this is a consistent definition and it is easy to see that g is, indeed, a generalised character of A.

By construction $|g_\omega| \equiv 1$ (ω a.e.) for all ω in A, hence, by Theorem 1.2 of (1), g belongs to the Silov boundary of A. This, in turn, guarantees the existence of some f in $\Delta(G)$ whose restriction to A is g, and the lemma is proved.

We can now complete the proof of the theorem.

Proposition 3. *Let $G = \mathbf{R}^q \times K$, where K is compact and metrisable. Then $T(\Delta(G)) \subseteq L^{\ddagger}(G)$.*

Proof. We assume that K is in reduced form and proceed by induction on $\Psi(K)$. Note by Lemmas 2 and 6 that if the conclusion fails for any group G then there exists a probability measure μ in $T(\Delta(G))$ such that $\mu^2 \ll \mu$ and $\mu \perp L^{\ddagger}(G)$. In the case $G = \mathbf{R}^q$ the last statement ensures that Lemma 9 applies; we obtain a contradiction via Lemma 10. Accordingly we have proved the result when $\Psi(K) = 1$. Next suppose that n is a positive integer and the result is known for $\Psi(K) \leqq n$. Consider G such that $\Psi(K) = n+1$. If the result fails for G, there exists $\mu \geqq 0$, $\|\mu\| = 1$, with $\mu^2 \ll \mu$, and $\mu \in T(\Delta(G)) \cap L^{\ddagger}(G)^{\perp}$. Let p be a positive integer such that $p(K)$ is infinite. As we saw in the proof of Lemma 9, $p(K) = b(K)$ for some $0<b<n+1$. Consider now the canonical epimorphism $\phi\colon G \to G/p(K) = H$. Note first that $H \approx \mathbf{R}^q \times (K/p(K))$, and $\Psi(K/p(K)) \leqq n$ (since $b(K/p(K)) = \{0\}$). By the inductive hypothesis we have $T(\Delta(H)) \subseteq L^{\ddagger}(H)$. By Lemma 4, $\phi^*(\mu)$ belongs to $T(\Delta(H))$, hence to $L^{\ddagger}(H)$. Now Lemma 3 shows that $\mu * m_{p(K)} \in L^{\ddagger}(G)$ and hence $\delta(y) * \mu * m_{p(K)} \in L^{\ddagger}(G)$ for all y in G. But μ is singular to $L^{\ddagger}(G)$ so that the conditions of Lemma 9 apply. Once more we arrive at a contradiction using Lemma 10.

The result is now established whenever $\Psi(K) < \infty$. Consider therefore G with $\Psi(K) = \infty$ for which the result fails. There exists a positive μ with $\mu^2 \ll \mu$, $\mu \in T(\Delta(G))$, $\mu \perp L^{\ddagger}(G)$. For any positive integer p consider the canonical epimorphism $\phi\colon G \to G/p(K) = H$. $p(K/p(K)) = \{0\}$ so $\Psi(K/p(K)) < \infty$ and we have $T(\Delta(H)) \subseteq L^{\ddagger}(H)$—this leads to a contradiction as before.

We have now completed the proof of the theorem.

The same methods give us a slightly stronger result, i.e. $T(\partial M(G)) = L^{\ddagger}(G)$ where $\partial M(G)$ is the Silov boundary of $M(G)$. To see this, first note that in the proof of Lemma 10 it is possible to choose the extension f of g in the Silov boundary of $M(G)$. For the analogous result to Lemma 4, let $\phi\colon G \to H$ be a continuous surjective homomorphism with compact kernel K. Since $\phi^*\colon m_K * M(G) \to M(H)$ is an isomorphism, $f \in \partial M(H)$ implies $f \circ \phi^*$ when restricted to $m_K * M(G)$ is in the Silov boundary of that algebra. This being so, the complex homomorphism has an extension to a member of the Silov boundary of $M(G)$. However $m_K * M(G)$ is an ideal and so such an extension is necessarily unique. Thus $f \circ \phi^* \in \partial M(G)$. The rest of the proof of Lemma 4 for $\partial M(G)$ is as for $\Delta(G)$. Finally for $\gamma \in G^\wedge$, $\mu \to \gamma . \mu$ is a Banach algebra automorphism of M(G), so that $\gamma f \in \partial M(G)$ if and only if $f \in \partial M(G)$. This supplies the information necessary for the proof of the result corresponding to Lemma 6. The adaptation of the remaining lemmas is straightforward.

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