MORE ON GALOIS COHOMOLOGY, DEFINABILITY, AND DIFFERENTIAL ALGEBRAIC GROUPS

OMAR LEÓN SÁNCHEZ, DAVID MERETZKY, AND ANAND PILLAY

Abstract. As a continuation of the work of the third author in [5], we make further observations on the features of Galois cohomology in the general model theoretic context. We make explicit the connection between forms of definable groups and first cohomology sets with coefficients in a suitable automorphism group. We then use a method of twisting cohomology (inspired by Serre's algebraic twisting) to describe arbitrary fibres in cohomology sequences—yielding a useful "finiteness" result on cohomology sets.

Applied to the special case of differential fields and Kolchin's constrained cohomology, we complete results from [3] by proving that the first constrained cohomology set of a differential algebraic group over a bounded, differentially large, field is countable.

§1. Introduction. As in the third author's paper [5], which develops some basic features of definable cohomology, in this note we are mainly concerned with translations from the Galois cohomological language to the model-theoretic language of definable sets and atomic extensions. Nonetheless, the general statements here do yield new results in the differential cohomology context; that is, when specialized to Kolchin's constrained cohomology, see Example 1.3(ii), Remark 4.5(iii), and Section 5.

We work in the (general) model-theoretic context of [5, Section 2]; namely:

ASSUMPTION 1.1. We fix a (first-order) structure M and an arbitrary subset A such that:

(i) M is atomic over A, meaning that for every (finite) tuple a from M the type $tp(a/A)$ is isolated,

(ii) for any (finite) tuples a and b from M, if $tp(a/A) = tp(b/A)$, then there is $\sigma \in \mathrm{Aut}(M/A)$ such that $\sigma(a) = b$, and

(iii) the theory of M eliminates imaginaries (alternatively one could work in M^{eq}).

We remark that if (i) and (ii) hold for real tuples then they hold for all of M^{eq}.

Notation and conventions. Throughout we let $\mathcal{G}$ denote the automorphism group $\mathrm{Aut}(M/A)$. We also let G be an arbitrary (not necessarily abelian) group definable in M with parameters from A. Note that $\mathcal{G}$ acts naturally on G by group automorphisms. We make $\mathcal{G}$ into a topological group by taking

$$\{\mathrm{Fix}(a) \ : \ a \text{ is a finite tuple from } M\}$$

as open neighbourhoods at the identity. We further equip G with its discrete topology, thus making the natural action of $\mathcal{G}$ on G continuous.

We briefly recall the basic notions (and some results) on definable cohomology from [5]. By a (1-)cocycle of $\mathcal{G}$ with values in G (with respect to the natural action of $\mathcal{G}$ on G) we mean a crossed-homomorphism from $\mathcal{G}$ to G; namely, a map

$$\Phi : \mathcal{G} \to G$$

such that

$$\Phi(\sigma_1 \cdot \sigma_2) = \Phi(\sigma_1) \cdot \sigma_1(\Phi(\sigma_2)).$$

A cocycle Φ is said to be definable if there is a (finite) tuple a from M and an A-definable (partial) function $h(x, y)$ such that for any $\sigma \in \mathcal{G}$ we have

$$\Phi(\sigma) = h(a, \sigma(a)).$$

One readily checks that definable cocycles are continuous. The set of definable cocycles (from $\mathcal{G}$ to G) is denoted by $Z^1_{\text{def}}(\mathcal{G}, G)$. Two cocycles Φ and Ψ are said to be cohomologous if there is $b \in G$ such that

$$\Psi(\sigma) = b^{-1} \cdot \Phi(\sigma) \cdot \sigma(b), \quad \text{for all } \sigma \in \mathcal{G}.$$

The relation of two cocycles being cohomologous is an equivalence relation on $Z^1_{\text{def}}(\mathcal{G}, G)$. The equivalence classes are called cohomology classes. The equivalence class of the trivial cocycle (the one that maps each $\sigma \in \mathcal{G}$ to the identity of G) is denoted by $B^1_{\text{def}}(\mathcal{G}, G)$ and its elements are called definable (1-)coboundaries from $\mathcal{G}$ to G. The set of all equivalence classes is called the first definable cohomology set of $\mathcal{G}$ with coefficients in G (over A), and is denoted by $H^1_{\text{def}}(\mathcal{G}, G)$.

REMARK 1.2.
(1) When $M = \text{dcl}(A)$ the cohomology $H^1_{\text{def}}(\mathcal{G}, G)$ is trivial.
(2) In general, the first cohomology set $H^1_{\text{def}}(\mathcal{G}, G)$ has the structure of a *pointed set* with distinguished element being $B^1_{\text{def}}(\mathcal{G}, G)$. When G is abelian, the set of definable cocycles $Z^1_{\text{def}}(\mathcal{G}, G)$ has a natural structure of an abelian group (by point-wise multiplication) with $B^1_{\text{def}}(\mathcal{G}, G)$ a subgroup; and thus, in this case, $H^1_{\text{def}}(\mathcal{G}, G)$ inherits a group structure given by the factor group $Z^1_{\text{def}}(\mathcal{G}, G)/B^1_{\text{def}}(\mathcal{G}, G)$.

EXAMPLE 1.3.
(i) *Algebraic Galois cohomology.* In the case when $A = k$ is a perfect field and $M = k^{\text{alg}}$ is the field-theoretic algebraic closure of A, the group G is simply the k^{alg}-rational points of an algebraic group over k and, by [5, Lemma 2.6], we are in the classical situation of (algebraic) Galois cohomology. More precisely, the definable cohomology $H^1_{\text{def}}(\mathcal{G}, G)$ agrees with the Galois cohomology over k with coefficients in G usually denoted $H^1(k, G)$. Most of our statements are inspired from well-known results in this context (see for instance [8] or [11]).
(ii) *Differential constrained cohomology.* In the case when A is a differential field of characteristic zero in finitely many commuting derivations (k, Δ) and M is a differential closure $(k^{\text{diff}}, \Delta)$, by [6], the definable group G is simply the k^{diff}-rational points of a differential algebraic group over k. Also, by

[5, Section 3], the definable cohomology $H^1_{\mathrm{def}}(\mathcal{G}, G)$ coincides with Kolchin's constrained cohomology $H^1_\Delta(k, G)$ from [2, Chapter VII]. It is worth noting that, while the results of the current paper are well known in classical Galois cohomology, they are as a matter of fact novel in the context of differential constrained cohomology (they do not appear in [2] or elsewhere to the authors' knowledge).

We recall that an A-definable (right) principal homogeneous space for G is an A-definable set X together with an A-definable regular (also called strictly transitive) right G-action. Two such spaces are said to be A-definably isomorphic if there is an A-definable bijection preserving the G-actions. The collection of A-definable principal homogeneous spaces, up to A-definable isomorphism, is a pointed set (with G as distinguished element) denoted by $P_{\mathrm{def}}(G)$. The main result of [5] establishes that as pointed sets $P_{\mathrm{def}}(G)$ and $H^1_{\mathrm{def}}(\mathcal{G}, G)$ are naturally isomorphic.

Let $\mathrm{Aut}_{\mathrm{def}}(G)$ denote the (abstract) group of M-definable group automorphisms of G. In Section 4 of [5], $H^1_{\mathrm{def}}(\mathcal{G}, \mathrm{Aut}_{\mathrm{def}}(G))$ is defined and an argument is sketched showing that this pointed set classifies A-forms of G (definition following Remark 2.1) up to A-definable isomorphism. Section 2 of this paper is devoted to a full proof of this result, which is then used in Section 4.

Then, in Section 3, we are explicit on the construction of exact sequences in definable cohomology from (normal) short exact sequences for G. We further discuss the method of twisting definable cohomology by inner automorphisms, in the spirit of Serre's algebraic twisting (cf. [11, Chapter I, Section 5.3] and [12]), and utilise it to describe arbitrary fibres of the maps in the (original) cohomology sequence. Recall that our cohomologies are just pointed sets, not necessarily groups, thus the kernel of a morphism does not give us information about the other fibres.

In Section 4, we put together the aforementioned results to show that if N is a normal A-definable subgroup of G and $\mu \in H^1_{\mathrm{def}}(\mathcal{G}, G)$, then there is a *surjection* from $H^1_{\mathrm{def}}(\mathcal{G}, N_\mu)$, where N_μ is the A-form of N corresponding to the cohomology class in $H^1_{\mathrm{def}}(\mathcal{G}, \mathrm{Aut}_{\mathrm{def}}(N))$ induced by conjugation of N by μ, to the fibre

$$\mathfrak{P}(\mu) := (\pi^1)^{-1}(\pi^1(\mu)),$$

where $\pi^1 : H^1_{\mathrm{def}}(\mathcal{G}, G) \to H^1_{\mathrm{def}}(\mathcal{G}, G/N)$ is the induced morphism (of pointed sets) from the canonical projection $\pi : G \to G/N$. As an application/corollary, we prove that if $H_{\mathrm{def}}(\mathcal{G}, G/N)$ is finite and $H^1_{\mathrm{def}}(\mathcal{G}, N_\mu)$ is also finite, for all $\mu \in H^1_{\mathrm{def}}(\mathcal{G}, G)$, then $H^1_{\mathrm{def}}(\mathcal{G}, G)$ is finite as well.

The corollary above, in the special case of Example 1.3(ii), is used in [3, Lemma 2.6]. A sketch of the argument is pointed out there, in the differential case, with details left to the reader. One of the points of the current paper is to give a detailed account in the more general model theoretic environment. In fact, as we point out in Section 5, one obtains also that if $H_{\mathrm{def}}(\mathcal{G}, G/N)$ has cardinality $\leq \kappa$ and $H^1_{\mathrm{def}}(\mathcal{G}, N_\mu)$ has cardinality $\leq \kappa$, for all $\mu \in H^1_{\mathrm{def}}(\mathcal{G}, G)$, then $H^1_{\mathrm{def}}(\mathcal{G}, G)$ has cardinality $\leq \kappa$ too. In Section 5, we apply this back in the differential context to prove the *countability* of $H^1_\Delta(k, G)$ whenever k is a bounded, differentially large field, and G is an arbitrary differential algebraic group over k.

§2. Definable cohomology and A-forms of G. We carry on the notation and assumptions from the previous section; in particular, those on M, A, $\mathcal{G}$, and G. We let $\mathrm{Aut}_{\mathrm{def}}(G)$ denote the set of definable (with parameters in M) group automorphisms of G. Note that this is a group—with respect to composition. Also note that the automorphism group $\mathcal{G}$ acts naturally on $\mathrm{Aut}_{\mathrm{def}}(G)$; namely, if $\sigma \in \mathcal{G}$ and $\phi \in \mathrm{Aut}_{\mathrm{def}}(G)$, then $\sigma(\phi)$ is obtained by applying σ to the graph of ϕ (a definable set), which yields another definable automorphism of G. Furthermore, one readily checks that for $\phi_1, \phi_2 \in \mathrm{Aut}_{\mathrm{def}}(G)$ we have

$$\sigma(\phi_1 \cdot \phi_2) = \sigma(\phi_1) \cdot \sigma(\phi_2). \tag{2.1}$$

In other words, the action of $\mathcal{G}$ in $\mathrm{Aut}_{\mathrm{def}}(G)$ is by group automorphisms. Furthermore, equipping $\mathrm{Aut}_{\mathrm{def}}(G)$ with its discrete topology, elimination of imaginaries implies that this action is continuous.

Using the action of $\mathcal{G}$ on $\mathrm{Aut}_{\mathrm{def}}(G)$, we define a (*1-*)*cocycle* from $\mathcal{G}$ to $\mathrm{Aut}_{\mathrm{def}}(G)$ to be a map

$$\Phi : \mathcal{G} \to \mathrm{Aut}_{\mathrm{def}}(G)$$

such that

$$\Phi(\sigma_1 \cdot \sigma_2) = \Phi(\sigma_1) \cdot \sigma_1(\Phi(\sigma_2)).$$

The trivial group homomorphism from $\mathcal{G}$ to $\mathrm{Aut}_{\mathrm{def}}(G)$ is a cocycle, called the *trivial* cocycle. Two cocycles Φ and Ψ from $\mathcal{G}$ to $\mathrm{Aut}_{\mathrm{def}}(G)$ are said to be cohomologous if there exist $\phi \in \mathrm{Aut}_{\mathrm{def}}(G)$ such that

$$\Psi(\sigma) = \phi^{-1} \cdot \Phi(\sigma) \cdot \sigma(\phi) \quad \text{for all } \sigma \in \mathcal{G}.$$

Being cohomologous is an equivalence relation: transitivity follows by (2.1). We say that the cocycle Φ is *definable* if if there exists a tuple a from M and an A-definable (partial) function $h(x, y, z)$ such that

$$\Phi(\sigma)(-) = h(a, \sigma(a), -) \quad \text{for all } \sigma \in \mathcal{G}.$$

We note that the trivial cocycle is definable, and if Φ is a cocycle that is cohomologous to a definable cocycle then Φ is itself definable. Furthermore, one readily verifies that definable cocycles are continuous. The set of definable cocycles is denoted by $Z^1_{\mathrm{def}}(\mathcal{G}, \mathrm{Aut}_{\mathrm{def}}(G))$. The set of cohomology classes of $Z^1_{\mathrm{def}}(\mathcal{G}, \mathrm{Aut}_{\mathrm{def}}(G))$ (under the equivalence relation of being cohomologous) is the first definable cohomology of $\mathcal{G}$ with coefficients in $\mathrm{Aut}_{\mathrm{def}}(G)$ and is denoted by $H^1_{\mathrm{def}}(\mathcal{G}, \mathrm{Aut}_{\mathrm{def}}(G))$.

REMARK 2.1. As in the case of the definable cohomology $H^1_{\mathrm{def}}(\mathcal{G}, G)$, the cohomology $H^1_{\mathrm{def}}(\mathcal{G}, \mathrm{Aut}_{\mathrm{def}}(G))$ has the structure of a pointed set, with the cohomology class of the trivial cocycle (also called the set of *coboundaries* from $\mathcal{G}$ to $\mathrm{Aut}_{\mathrm{def}}(G)$) being the distinguished element. Further, in the case when the group $\mathrm{Aut}_{\mathrm{def}}(G)$ is abelian, $H^1_{\mathrm{def}}(\mathcal{G}, \mathrm{Aut}_{\mathrm{def}}(G))$ has the structure of an abelian group. This follows from the fact that, in this case, the natural (commutative) group structure on the set of maps from $\mathcal{G}$ to $\mathrm{Aut}_{\mathrm{def}}(G)$ given by

$$\Psi_1 * \Psi_1(\sigma) = \Psi_1(\sigma) \cdot \Psi_2(\sigma)$$

restricts to a (commutative) group structure on $Z^1_{\mathrm{def}}(\mathcal{G}, \mathrm{Aut}_{\mathrm{def}}(G))$.

By an *A-form of G* we mean an A-definable group H which is definably (over M) isomorphic to G. Let H be an A-form of G. Note that, for any definable (group) isomorphism $f : G \to H$ and $\sigma \in \mathcal{G}$, the map $\sigma(f) : G \to H$ is again a definable isomorphism. Thus, we have a well-defined map $\Phi_{H,f} : \mathcal{G} \to \mathrm{Aut}_{\mathrm{def}}(G)$ given by

$$\Phi_{H,f}(\sigma) = f^{-1} \circ \sigma(f).$$

Furthermore, $\Phi_{H,f}$ is a definable cocycle. It is clearly definable and

$$\begin{aligned}
\Phi_{H,f}(\sigma_1 \cdot \sigma_2) &= f^{-1} \circ \sigma_1(\sigma_2(f)) \\
&= f^{-1} \circ \sigma_1(f \circ f^{-1} \circ \sigma_2(f)) \\
&= f^{-1} \circ \sigma_1(f) \circ \sigma_1(f^{-1} \circ \sigma_2(f)) \\
&= \Phi_{H,f}(\sigma_1) \cdot \sigma_1(\Phi_{H,f}(\sigma_2)).
\end{aligned}$$

Now, if $g : G \to H$ is another definable isomorphism, then $\Phi_{H,f}$ and $\Phi_{H,g}$ are cohomologous. Indeed,

$$\Phi_{H,g}(\sigma) = (f^{-1}g)^{-1}\Phi_{H,f}(\sigma)\sigma(f^{-1}g).$$

More generally, if N is another A-form of G which is (group) definably isomorphic to H over A, say witnessed by $\eta : N \to H$, and $p : G \to N$ is a definable isomorphism, then $\Phi_{H,f}$ and $\Phi_{N,p}$ are cohomologous. Indeed

$$\Phi_{N,p}(\sigma) = (f^{-1}\eta\, p)^{-1}\Phi_{H,f}(\sigma)\sigma(f^{-1}\eta\, p).$$

Let $\mathfrak{F}_{\mathrm{def}}(G)$ denote the set of A-forms of G up to A-definable isomorphism. The above discussion yields a well-defined map

$$\mathfrak{F}_{\mathrm{def}}(G) \to H^1_{\mathrm{def}}(\mathcal{G}, \mathrm{Aut}_{\mathrm{def}}(G))$$

given by mapping (H, f) to the cohomology class of $\Phi_{H,f}$, where H an A-form of G and $f : G \to H$ a definable (over M) isomorphism. We now show that the displayed map is an isomorphism of pointed sets. The distinguished element in $\mathfrak{F}_{\mathrm{def}}(G)$ being G. Clearly, $\Phi_{G,\mathrm{Id}}$ is the trivial cocycle from $\mathcal{G}$ to $\mathrm{Aut}_{\mathrm{def}}(G)$, hence (G, Id) maps to the cohomology class of the trivial cocycle (i.e., the set of coboundaries). The remaining of this section is devoted to showing that this map is a bijection.

REMARK 2.2. It follows from our assumptions (see Assumptions 1.1) that a definable set X is A-definable if and only X is fixed setwise by any element of $\mathcal{G}$. We will use this to show injectivity of the above map.

Let us show injectivity of $\mathfrak{F}_{\mathrm{def}}(G) \to H^1_{\mathrm{def}}(\mathcal{G}, \mathrm{Aut}_{\mathrm{def}}(G))$. Take (H, f) and (N, p), where H and N are A-forms of G and $f : G \to H$ and $p : G \to N$ are definable isomorphisms. Assume $\Phi_{H,f}$ is cohomologous to $\Phi_{N,p}$, then there exists $\phi \in \mathrm{Aut}_{\mathrm{def}}(G)$ such that for all $\sigma \in \mathcal{G}$

$$\Phi_{N,p}(\sigma) = \phi^{-1}\Phi_{H,f}(\sigma)\,\sigma(\phi).$$

In other words,

$$p^{-1}\sigma(p) = \phi^{-1}f^{-1}\sigma(f)\sigma(\phi),$$

rearranging we get

$$f \phi p^{-1} = \sigma(f \phi p^{-1}).$$

It follows, see Remark 2.2, that the isomorphism $f \phi p^{-1} : N \to H$ is A-definable. Thus N and H are identified in $\mathfrak{F}_{\mathrm{def}}(G)$, showing the map is indeed injective.

We now show surjectivity. Let $\Phi \in Z^1_{\mathrm{def}}(\mathcal{G}, \mathrm{Aut}_{\mathrm{def}}(G))$. Then there is tuple a from M and an A-definable (partial) function $h(x, y, z)$ such that $\Phi(\sigma)(-) = h(a, \sigma(a), -)$ for all $\sigma \in \mathcal{G}$.

LEMMA 2.3. *Let a and h be as above. For any b, c, d tuples from M realising $tp(a/A)$ we have that $h(b, c, -) \in \mathrm{Aut}_{\mathrm{def}}(G)$. Furthermore,*

$$h(b, c, -) = h(b, d, h(d, c, -)).$$

It follows that $h(b, b, -) = \mathrm{Id}_G$ and $h(b, c, -)^{-1} = h(c, b, -)$.

PROOF. Let $\sigma, \tau, \eta \in \mathcal{G}$ be such that

$$\sigma(a) = b, \; \sigma\tau\eta(a) = c \; \text{ and } \; \sigma\tau(a) = d.$$

Then

$$\begin{aligned}
h(b, c, -) &= h(\sigma(a), \sigma\tau\eta(a), -) \\
&= \sigma(\Phi(\tau\eta))(-) \\
&= \sigma(\Phi(\tau)\tau(\Phi(\eta)))(-) \\
&= h(\sigma(a), \sigma\tau(a), h(\sigma\tau(a), \sigma\tau\eta(a), -)) \\
&= h(b, d, h(d, c, -)).
\end{aligned}$$

All of the remaining properties follow from this, e.g., $h(b, c, -) \in \mathrm{Aut}_{\mathrm{def}}(G)$ as $h(b, c, -) = h(b, a, h(a, c, -))$. ⊣

Now let Y be the set of realisations of $tp(a/A)$ in M. Then Y is an A-definable set. Set $Z = Y \times G$. Define a relation R on Z given by

$$(b, d) \, R \, (c, e) \iff h(b, c, e) = d.$$

It follows, using Lemma 2.3, that R is an equivalence relation (which is of course A-definable). Let $H = Z/R$. Then, by elimination of imaginaries, we identify the quotient H with an A-definable set. We now equip H with a A-definable group structure. Define, for $(b, d), (c, e) \in Z$,

$$(b, d) * (c, e) = (b, d \cdot h(b, c, e)) \in Z.$$

Using Lemma 2.3, one readily checks that this binary operation $*$ on Z is R-invariant and thus induces a A-definable binary operation on H that we also denote by $*$. We show that this is in fact a group structure on H. We prove this by showing that there is a definable (over M) bijection f between H and G such that the induced group structure on H is precisely $*$. Note that this further shows that H is a A-form of G.

Let $f : Z \to G$ be defined by $f(b, d) = h(a, b, d)$. Since $h(a, b, -) \in \mathrm{Aut}_{\mathrm{def}}(G)$, the function f is clearly surjective. Furthermore, again using Lemma 2.3, one checks that

$$f(a, b, d) = f(a, c, e) \iff h(b, c, e) = d \iff (b, d) \, R \, (c, e).$$

Thus f induces a definable (over M) bijection from $H = Z/R$ to G, which we also denote by f. Now, using Lemma 2.3, the induced group structure on H via this bijection is

$$\begin{aligned}
f^{-1}(f(b,d) \cdot f(c,e)) &= f^{-1}(h(a,b,d) \cdot h(a,c,e)) \\
&= f^{-1}(h(a,b,d) \cdot h(a,b,h(b,c,e))) \\
&= f^{-1}(h(a,b,d \cdot h(b,c,e))) \\
&= f^{-1}(f(b,d \cdot h(b,c,e))) \\
&= (b,d \cdot h(b,c,e)) \\
&= (b,d) * (c,e).
\end{aligned}$$

We have thus shown that $(H, *)$ is a A-form of G (witnessed by $f^{-1} : G \to H$). We claim that (H, f^{-1}) maps to Φ (namely $\Phi = \Phi_{H,f^{-1}}$). Indeed, for any $\sigma \in \mathcal{G}$, using once again Lemma 2.3, we have

$$\begin{aligned}
\Phi(\sigma)\sigma(f)(b,d) = h(a, \sigma(a), \sigma(f)(b,d)) &= h(a, \sigma(a), h(\sigma(a), b, d)) \\
&= h(a,b,d) = f(b,d),
\end{aligned}$$

and so

$$\Phi(\sigma) = f \cdot \sigma(f^{-1}),$$

this shows that $\Phi = \Phi_{H,f^{-1}}$, as desired.

We have thus shown:

PROPOSITION 2.4. *There is a natural isomorphism (as pointed sets) between* $\mathfrak{F}_{\mathrm{def}}(G)$ *and* $H^1_{\mathrm{def}}(\mathcal{G}, \mathrm{Aut}_{\mathrm{def}}(G))$.

§3. Exact sequences and twisting in definable cohomology. We carry forward the conventions and assumptions on M, A, $\mathcal{G}$, and G from previous sections. Let X be an A-definable set in the structure M. Then $\mathcal{G}$ acts naturally on X. The 0th definable cohomology from $\mathcal{G}$ to X, denoted $H^0_{\mathrm{def}}(\mathcal{G}, X)$, is the set of $\mathcal{G}$-invariant points of X. By our assumptions (see Assumptions 1.1), this coincides with the points of X whose entries are all in $\mathrm{dcl}(A)$. In the case when X has in addition the structure of an A-definable pointed set (i.e., has a distinguished point which is $\mathcal{G}$-invariant and thus in $\mathrm{dcl}(A)$), then $H^0_{\mathrm{def}}(\mathcal{G}, X)$ inherits the structure of a pointed set sharing its distinguished element with X. Furthermore, when $X = G$ (an A-definable group), the 0th definable cohomology $H^0_{\mathrm{def}}(\mathcal{G}, G)$ is clearly a subgroup of G.

Now consider an A-definable map $f : X \to X'$ of A-definable sets. Since f preserves $\mathcal{G}$-invariants, it induces a map in cohomology

$$f^0 : H^0_{\mathrm{def}}(\mathcal{G}, X) \to H^0_{\mathrm{def}}(\mathcal{G}, X').$$

In the case that X and X' have in addition the structure of A-definable pointed sets (the distinguished elements being $\mathcal{G}$-invariant) and f is a homomorphism of pointed sets (i.e., maps the distinguished element of X to that of X'), the above map f^0 is a homomorphism of pointed sets. Furthermore, when $X = G$ and $X' = G'$ are

A-definable groups and f is a group homomorphism, the map

$$f^0 : H^0_{\text{def}}(\mathcal{G}, G) \to H^0_{\text{def}}(\mathcal{G}, G')$$

is a group homomorphism.

Recall, from the introduction, that $Z^1_{\text{def}}(\mathcal{G}, G)$ and $H^1_{\text{def}}(\mathcal{G}, G)$ denote the sets of definable cocycles and cohomology from $\mathcal{G}$ to G, respectively. Let $f : G \to G'$ be an A-definable group homomorphism (between the A-definable groups G and G'). For any definable 1-cocycle $\Phi \in Z^1_{\text{def}}(\mathcal{G}, G)$, the composition $f \circ \Phi : \mathcal{G} \to G'$ is again a definable 1-cocycle. Furthermore, if Ψ is cohomologous to Φ, then $f \circ \Phi$ is cohomologous to $f \circ \Psi$ as elements of $Z^1_{\text{def}}(\mathcal{G}, G')$. Indeed, suppose $b \in G$ is such that $\Phi(\sigma) = b^{-1}\Psi(\sigma)\sigma(b)$ for all $\sigma \in \mathcal{G}$, then one easily verifies

$$(f \circ \Psi)(\sigma) = (f(b))^{-1} (f \circ \Psi)(\sigma) \, \sigma(f(b)).$$

Thus, the assignment $\Psi \mapsto (f \circ \Phi)$ defines a map from $Z^1_{\text{def}}(\mathcal{G}, G)$ to $Z^1_{\text{def}}(\mathcal{G}, G')$ that induces a map in cohomology

$$f^1 : H^1_{\text{def}}(\mathcal{G}, G) \to H^1_{\text{def}}(\mathcal{G}, G').$$

Moreover, f^1 is a morphism of pointed sets (meaning it preserves distinguished elements), and when G and G' are abelian then it is a group homomorphism.

3.1. Exact sequence in cohomology. We now fix an arbitrary (not necessarily normal) A-definable subgroup N of G. This yields the natural short exact sequence of A-definable pointed sets

$$1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} G/N \to 1. \tag{3.1}$$

Note that, since N is not necessarily normal in G, the left-coset space G/N is not necessarily a group. It is however an A-definable (left) homogeneous space for G (i.e., G/N is an A-definable set equipped with an A-definable transitive left G-action—in this case the natural G-action on the left). While the inclusion map ι is a group homomorphism, the projection map π is generally just a morphism of A-definable homogenous G-spaces. Furthermore, G/N has a natural structure of an A-definable pointed set, with distinguished element being the coset N, and so π is also a homomorphism of A-definable pointed sets. When N is normal in G, the above sequence is a normal short exact sequence of A-definable groups.

Let $a \in H^0_{\text{def}}(\mathcal{G}, G/N)$, here we think of G/N as a pointed set and so the 0th cohomology (which the same as the $\mathcal{G}$-invariants of G/N) is also a pointed set as explained above. Then, a is of the form $\pi(b)$ for some $b \in G$. For any $\sigma \in \mathcal{G}$ we have

$$\pi(\sigma(b)) = \sigma(\pi(b)) = \sigma(a) = a = \pi(b)$$

and so $b^{-1} \cdot \sigma(b) \in N$. Thus the assignment $\sigma \mapsto b^{-1} \cdot \sigma(b)$ defines a map from $\mathcal{G}$ to N. Moreover, this map clearly belongs to $Z^1_{\text{def}}(\mathcal{G}, N)$; and if b' is another element of G with $\pi(b') = a$, then $b' = bc$ for some $c \in N$ and hence

$$(bc)^{-1} \cdot \sigma(bc) = c^{-1}(b^{-1} \cdot \sigma(b))\sigma(c).$$

This shows that the assignment $\sigma \mapsto (b')^{-1} \cdot \sigma(b')$ is cohomologous to the previous one. Therefore, the cohomology class of this assignment is uniquely determined by

a, and this class is denoted by $\delta(a) \in H^1_{\mathrm{def}}(\mathcal{G}, N)$. We have now a morphism

$$\delta : H^0_{\mathrm{def}}(\mathcal{G}, G/N) \to H^1_{\mathrm{def}}(\mathcal{G}, N)$$

of pointed sets (and of groups when G is abelian) called the connecting homomorphism.

We are now in the position to state (and prove) how exactness transfers when passing to the sequence in cohomology.

THEOREM 3.1. *Suppose N is an A-definable subgroup of G* (*not necessarily normal*). *Then, the sequence in cohomology induced from* (3.1)

$$1 \to H^0_{\mathrm{def}}(\mathcal{G}, N) \xrightarrow{\iota^0} H^0_{\mathrm{def}}(\mathcal{G}, G) \xrightarrow{\pi^0} H^0_{\mathrm{def}}(\mathcal{G}, G/N) \xrightarrow{\delta} H^1_{\mathrm{def}}(\mathcal{G}, N) \xrightarrow{\iota^1} H^1_{\mathrm{def}}(\mathcal{G}, G)$$

is exact (*as a sequence of pointed sets*). *Furthermore, if N is normal in G, then the sequence remains exact when extended by*

$$H^1_{\mathrm{def}}(\mathcal{G}, G) \xrightarrow{\pi^1} H^1_{\mathrm{def}}(\mathcal{G}, G/N).$$

PROOF. It is easy to check that the sequence is exact at $H^0_{\mathrm{def}}(\mathcal{G}, N)$ and at $H^0_{\mathrm{def}}(\mathcal{G}, G)$. We now prove it is exact at $H^0_{\mathrm{def}}(\mathcal{G}, G/N)$. Let $a \in H^0_{\mathrm{def}}(\mathcal{G}, G/N)$ and $b \in G$ such that $\pi(b) = a$. Then, $\delta(a) = 1$ if and only if there is $c \in N$ such that

$$b^{-1}\sigma(b) = c^{-1}\sigma(c), \quad \text{for all } \sigma \in \mathcal{G};$$

that is, $bc^{-1} \in H^0_{\mathrm{def}}(\mathcal{G}, G)$. Thus, $\delta(a) = 1$ if and only there is $d \in H^0_{\mathrm{def}}(\mathcal{G}, G)$ such that $\pi^0(d) = a$.

We now check the sequence in exact at $H^1_{\mathrm{def}}(\mathcal{G}, N)$. Let $\eta \in H^1_{\mathrm{def}}(\mathcal{G}, N)$ and Φ a cocycle in $Z^1_{\mathrm{def}}(\mathcal{G}, N)$ with cohomology class η. Then, $\iota^1(\eta) = 1$ if and only there is $b \in G$ such that $\Phi(\sigma) = b^{-1}\sigma(b)$ for all $\sigma \in \mathcal{G}$. The latter implies

$$\pi(b)^{-1}\sigma(\pi(b)) = \pi(b^{-1}\sigma(b)) = \pi(\Phi(\sigma)) = 1$$

and so $\pi(b) \in H^0_{\mathrm{def}}(\mathcal{G}, G/N)$. Clearly, $\delta(\pi(b)) = \eta$, and exactness at $H^1_{\mathrm{def}}(\mathcal{G}, N)$ follows.

Finally, assuming N is normal in G, we check exactness at $H^1_{\mathrm{def}}(\mathcal{G}, G)$. Let $\Phi \in Z^1_{\mathrm{def}}(\mathcal{G}, N)$. Then, since $im(\iota) = ker(\pi)$, we get that $\pi \circ \iota \circ \Phi$ is the trivial cocycle from $\mathcal{G}$ to G/N. It follows that $im(\iota^1) \subseteq ker(\pi^1)$. Now, let $\Psi \in Z^1_{\mathrm{def}}(\mathcal{G}, G)$ be such that $\pi \circ \Psi$ is a coboundary from $\mathcal{G}$ to G/N. This means that there is $d \in G/N$ such that

$$\pi(\Psi(\sigma)) = d^{-1} \cdot \sigma(d), \quad \text{for all } \sigma \in \mathcal{G}.$$

Let $a \in G$ such that $\pi(a) = d$. Then we can rearrange the above to

$$\pi(a \cdot \Psi(\sigma) \cdot \sigma(a)^{-1}) = 1.$$

Since ι is injective and $im(\iota) = ker(\pi)$, there is a unique $c \in N$ such that $\iota(c) = a \cdot \Psi(\sigma) \cdot \sigma(a)^{-1}$. If we set $\Lambda : \mathcal{G} \to N$ to be $\Lambda(\sigma) = c$, where c is the unique c just found, then one can readily check that Λ is a definable cocycle from $\mathcal{G}$ to N. Moreover, $\iota \circ \Lambda$ is cohomologous to Ψ as witnessed by $a \in G$. This proves $ker(\pi^1) \subseteq im(\iota^1)$, as desired. ⊣

3.2. Twisting definable cohomology. Recall that when G is abelian, the cohomology sequence in Theorem 3.1 is an exact sequence of groups and group homomorphism, and so fibres are just translates of the kernels. However, in the general situation (when G is not necessarily abelian), the exact sequence is just a sequence of pointed sets and thus carry less information than in the abelian case. Indeed, the kernel of a morphism of pointed sets does not generally yield any information about the other fibres.

As in algebraic Galois cohomology [8, Section 1.3], this can be overcome using a method introduced by Serre [12, Section 1.5] called *twisting*. We now discuss how a special case (which suffices for our purposes) of this method transfers to our general definable model-theoretic context. Let $\Phi \in Z^1_{\mathrm{def}}(\mathcal{G}, G)$ and define a new (twisted) action of $\mathcal{G}$ on G by

$$\sigma * g = C_{\Phi(\sigma)}(\sigma(g)) \ \ (\ = \Phi(\sigma)\sigma(g)\Phi(\sigma)^{-1} \ )$$

where, for any $h \in G$, $C_h : G \to G$ denotes conjugation by h; namely, $C_h(g) = hgh^{-1}$ for all $g \in G$. We call this the induced action of $\mathcal{G}$ on G twisted via conjugation by Φ. When clarity is needed we denote this action by saying that $\mathcal{G}_\Phi$ acts on G. One readily checks that this is indeed a group action and in fact $\mathcal{G}_\Phi$ acts on G by group automorphisms. Furthermore, since $\Phi : \mathcal{G} \to G$ is continuous, this action is continuous.

With respect to this (non-natural) action of $\mathcal{G}_\Phi$ on G, we can define the notion of definable (1-)cocycles, denoted $Z^1_{\mathrm{def}}(\mathcal{G}_\Phi, G)$, as those maps $\Psi : \mathcal{G} \to G$ such that

$$\Psi(\sigma_1\sigma_2) = \Psi(\sigma_1) \cdot \sigma_1 * (\Psi(\sigma_2)).$$

As before, such cocycle is said to be definable if there is tuple a from M and an A-definable function $h(x, y)$ such that

$$\Psi(\sigma) = h(a, \sigma(a)), \quad \text{for all } \sigma \in \mathcal{G}.$$

We remark that definable cocycles from $\mathcal{G}_\Phi$ to G are continuous (see [5] for more discussion of continuity and definability of cocycles). Two cocycles Ψ and Λ in $Z^1_{\mathrm{def}}(\mathcal{G}_\Phi, G)$ are cohomologous if there is $b \in G$ such that

$$\Psi(\sigma) = b^{-1} \cdot \Lambda(\sigma) \cdot (\sigma * b), \quad \text{for all } \sigma \in \mathcal{G}.$$

This again yields an equivalence relation. The set of equivalence classes is the first definable cohomology set $H^1_{\mathrm{def}}(\mathcal{G}_\Phi, G)$ twisted by Φ. This has a natural structure of a pointed set.

REMARK 3.2. Note that when G is abelian the twisted action of $\mathcal{G}_\Phi$ on G is just the natural action of $\mathcal{G}$ on G and so the cohomologies $H^1_{\mathrm{def}}(\mathcal{G}_\Phi, G)$ and $H^1_{\mathrm{def}}(\mathcal{G}, G)$ coincide.

As with the 0th definable cohomology set $H^0_{\mathrm{def}}(\mathcal{G}, G)$, the 0th definable twisted cohomology $H^0_{\mathrm{def}}(\mathcal{G}_\Phi, G)$ is defined as the set of $\mathcal{G}_\Phi$-invariant points of G, which is a subgroup of G (as $\mathcal{G}_\Phi$ acts on G by group automorphisms).

Now let N be an A-definable normal subgroup of G. We can then restrict the action of $\mathcal{G}_\Phi$ on G to N (since N is normal in G). This yields a continuous (twisted) action by group automorphisms of $\mathcal{G}_\Phi$ on N. In a similar way as above, we can construct the sets of twisted definable cocycles and cohomology $Z^1_{\mathrm{def}}(\mathcal{G}_\Phi, N)$ and $H^1_{\mathrm{def}}(\mathcal{G}_\Phi, N)$,

respectively. Furthermore, letting $\pi : G \to G/N$ be the natural projection, we can twist the natural action of $\mathcal{G}$ on G/N by the cocycle $\pi \circ \Phi \in Z^1_{\mathrm{def}}(\mathcal{G}, G/N)$. Then one can construct the twisted definable cocycles and cohomology sets $Z^1_{\mathrm{def}}(\mathcal{G}_{\pi\Phi}, G/N)$ and $H^1_{\mathrm{def}}(\mathcal{G}_{\pi\Phi}, G/N)$. For convenience, and it should not cause confusion, we denote $\mathcal{G}_{\pi\Phi}$ (acting on G/N) simply by $\mathcal{G}_\Phi$ and the set of definable cocycles and cohomology by $Z^1_{\mathrm{def}}(\mathcal{G}_\Phi, G/N)$ and $H^1_{\mathrm{def}}(\mathcal{G}_\Phi, G/N)$.

Consider the natural short exact sequence of A-definable groups (recall that we are assuming that N is normal in G)

$$1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} G/N \to 1.$$

With the respect to the above twisted actions of $\mathcal{G}_\Phi$ on N, G, and G/N, one readily checks that for $\sigma \in \mathcal{G}$, $n \in N$, and $g \in G$ we have

$$\iota(\sigma * n) = \sigma * \iota(n) \quad \text{and} \quad \pi(\sigma * g) = \sigma * \pi(g).$$

It follows from these equalities, exactly as we did in the previous sub-section, that we can induce morphisms of pointed sets in cohomology

$$\iota^i_\Phi : H^i_{\mathrm{def}}(\mathcal{G}_\Phi, N) \to H^i_{\mathrm{def}}(\mathcal{G}_\Phi, G) \quad \text{and} \quad \pi^i_\Phi : H^i_{\mathrm{def}}(\mathcal{G}_\Phi, G) \to H^i_{\mathrm{def}}(\mathcal{G}_\Phi, G/N)$$

for $i = 0, 1$. Furthermore, we get a connecting morphism (or pointed sets)

$$\delta_\Phi : H^0(\mathcal{G}_\Phi, G/N) \to H^1_{\mathrm{def}}(\mathcal{G}_\Phi, N),$$

and obtain, by adapting the argument in the proof of Theorem 3.1, the (twisted) exact sequence

$$1 \to H^0_{\mathrm{def}}(\mathcal{G}_\Phi, N) \xrightarrow{\iota^0_\Phi} H^0_{\mathrm{def}}(\mathcal{G}_\Phi, G) \xrightarrow{\pi^0_\Phi} H^0_{\mathrm{def}}(\mathcal{G}_\Phi, G/N)$$
$$\xrightarrow{\delta} H^1_{\mathrm{def}_\Phi}(\mathcal{G}_\Phi, N) \xrightarrow{\iota^1_\Phi} H^1_{\mathrm{def}}(\mathcal{G}_\Phi, G) \xrightarrow{\pi^1_\Phi} H^1_{\mathrm{def}}(\mathcal{G}_\Phi, G/N)$$

of pointed sets.

While one is not generally able to multiply cocycles in $Z^1_{\mathrm{def}}(\mathcal{G}, G)$, unless G is abelian say, by going to the twisted context this is possible. We make this precise in the following result which also shows how the cohomologies $H^1_{\mathrm{def}}(\mathcal{G}, G)$ and $H^1_{\mathrm{def}}(\mathcal{G}_\Phi, G)$ are related (in fact in bijection).

PROPOSITION 3.3. Let $f_\Phi : Z^1_{\mathrm{def}}(\mathcal{G}_\Phi, G) \to Z^1_{\mathrm{def}}(\mathcal{G}, G)$ be the map defined by $f_\Phi(\Psi) = \Psi \cdot \Phi$; namely, for $\sigma \in G$, $f_\Phi(\Psi)(\sigma) = \Psi(\sigma) \cdot \Phi(\sigma)$. Then, f_Φ is a bijection. Furthermore, it induces a bijection in cohomology $F_\Phi : H^1_{\mathrm{def}}(\mathcal{G}_\Phi, G) \to H^1_{\mathrm{def}}(\mathcal{G}, G)$ which maps the distinguished element to the cohomology class of Φ.

PROOF. We first prove that indeed $f_\Phi(\Psi) \in Z^1_{\mathrm{def}}(\mathcal{G}, G)$ for every $\Psi \in Z^1_{\mathrm{def}}(\mathcal{G}_\Phi, G)$.

Indeed, it is clearly definable (since Φ and Ψ are definable) and for $\sigma_1, \sigma_2 \in \mathcal{G}$ we have

$$\begin{aligned}
f_\Phi(\Psi)(\sigma_1\sigma_2) &= \Psi(\sigma_1\sigma_2) \cdot \Phi(\sigma_1\sigma_2) \\
&= \Psi(\sigma_1) \cdot \sigma_1 * (\Psi(\sigma_2)) \cdot \Phi(\sigma_1) \cdot \sigma_1(\Phi(\sigma_2)) \\
&= \Psi(\sigma_1) \cdot \Phi(\sigma_1) \cdot \sigma_1(\Psi(\sigma_2)) \cdot \Phi(\sigma_1)^{-1} \cdot \Phi(\sigma_1) \cdot \sigma_1(\Phi(\sigma_2)) \\
&= f_\Phi(\Psi)(\sigma_1) \cdot \sigma_1(f_\Phi(\Psi)(\sigma_2)).
\end{aligned}$$

One easily checks that f_Φ has inverse $f_\Phi^{-1} : Z_{\text{def}}^1(\mathcal{G}, G) \to Z_{\text{def}}^1(\mathcal{G}_\Phi, G)$ given by

$$f_\Phi^{-1}(\Lambda)(\sigma) = \Lambda(\sigma) \cdot (\Phi(\sigma))^{-1}.$$

Now, to pass to cohomology, let $\Psi, \Lambda \in Z_{\text{def}}^1(\mathcal{G}_\Phi, G)$ be cohomologous witnessed by $b \in G$; namely,

$$\Lambda(\sigma) = b^{-1} \cdot \Psi(\sigma) \cdot \sigma * b, \quad \text{for all } \sigma \in \mathcal{G}.$$

Multiplying on the right by $\Phi(\sigma)$ we get that

$$\begin{aligned}
f_\Phi(\Lambda)(\sigma) &= \Lambda(\sigma) \cdot \Phi(\sigma) \\
&= b^{-1} \cdot \Psi(\sigma) \cdot \sigma * b \cdot \Phi(\sigma) \\
&= b^{-1} \cdot \Psi(\sigma) \cdot \Phi(\sigma) \cdot \sigma(b) \cdot \Phi(\sigma)^{-1} \cdot \Phi(\sigma) \\
&= b^{-1} \cdot f_\Phi(\Psi)(\sigma) \cdot \sigma(b).
\end{aligned}$$

Thus, $f_\Phi(\Lambda)$ and $f_\Phi(\Psi)$ are cohomologous as elements in $Z_{\text{def}}^1(\mathcal{G}, G)$, and so f_Φ induces a well-defined map $F_\Phi : H_{\text{def}}^1(\mathcal{G}_\Phi, G) \to H_{\text{def}}^1(\mathcal{G}, G)$. Let 1 be the trivial cocycle in $Z_{\text{def}}^1(\mathcal{G}_\Phi, G)$, then, by definition, F_Φ maps the distinguished element of $H_{\text{def}}^1(\mathcal{G}_\Phi, G)$ to the cohomology class of

$$f_\Phi(1) = \Phi,$$

as desired. Finally, one can similarly check that the inverse function f_Φ^{-1} induces a well-defined map in cohomology

$$F_\Phi^{-1} : H_{\text{def}}^1(\mathcal{G}, G) \to H_{\text{def}}^1(\mathcal{G}_\Phi, G)$$

which an easy computation shows that it is in fact the inverse for F_Φ.   ⊣

Let $\Psi \in Z_{\text{def}}^1(\mathcal{G}_\Phi, G)$ and $\sigma \in \mathcal{G}$. As $\pi : G \to G/N$ is a homomorphism, we have

$$\begin{aligned}
f_{\pi\Phi}(\pi \circ \Psi)(\sigma) &= \pi \circ \Psi(\sigma) \cdot \pi \circ \Phi(\sigma) \\
&= \pi(\Psi(\sigma) \cdot \Phi(\sigma)) \\
&= \pi \circ f_\Phi(\Psi)(\sigma),
\end{aligned}$$

which shows that the following diagram commutes:

$$\begin{array}{ccc}
Z_{\text{def}}^1(\mathcal{G}_\Phi, G) & \xrightarrow{\pi \circ (-)} & Z_{\text{def}}^1(\mathcal{G}_\Phi, G/N) \\
{\scriptstyle f_\Phi} \downarrow & & \downarrow {\scriptstyle f_{\pi\Phi}} \\
Z_{\text{def}}^1(\mathcal{G}, G) & \xrightarrow{\pi \circ (-)} & Z_{\text{def}}^1(\mathcal{G}, G/N).
\end{array}$$

This induces a commutative diagram in cohomology

$$\begin{array}{ccc}
H_{\text{def}}^1(\mathcal{G}_\Phi, G) & \xrightarrow{\pi^1} & H_{\text{def}}^1(\mathcal{G}_\Phi, G/N) \\
{\scriptstyle F_\Phi} \downarrow & & \downarrow {\scriptstyle F_{\pi\Phi}} \\
H_{\text{def}}^1(\mathcal{G}, G) & \xrightarrow{\pi^1} & H_{\text{def}}^1(\mathcal{G}, G/N)
\end{array}$$

which, together with Proposition 3.3 applied to f_Φ and $f_{\pi\Phi}$, shows that the bijection

$$F_\Phi : H^1_{\mathrm{def}}(\mathcal{G}_\Phi, G) \to H^1_{\mathrm{def}}(\mathcal{G}, G)$$

restricts to a bijection between the kernel of π^1_Φ and the fibre $(\pi^1)^{-1}(\pi^1(\mu))$, where μ is the cohomology class of Φ. Thus, we have proved the following.

COROLLARY 3.4. *With the above notation and μ denoting the cohomology class of Φ, there is a natural bijection between $\ker(\pi^1_\Phi)$ and the fibre*

$$\mathfrak{P}(\mu) := (\pi^1)^{-1}(\pi^1(\mu)).$$

This demonstrates how, using the method of twisting, one can describe the fibres of π^1 in the original (non-twisted) cohomology sequence.

REMARK 3.5. We note that if Ψ is a cocycle in $Z^1_{\mathrm{def}}(\mathcal{G}, G)$ which is cohomologous to Φ, witnessed by $b \in G$ (i.e., $\Phi(\sigma) = b^{-1}\Psi(\sigma)\sigma(b)$), then the inner automorphism of G given by conjugation $C_b : G \to G$ commutes with the twisted actions of G_Φ and G_Ψ on G. More precisely,

$$C_b(\sigma *_\Phi g) = \sigma *_\Psi C_b(g), \quad \text{for all } \sigma \in \mathcal{G}, g \in G.$$

Let $\Xi \in Z^1_{\mathrm{def}}(\mathcal{G}_\Phi, G)$, we claim $C_b \circ \Xi \in Z^1_{\mathrm{def}}(\mathcal{G}_\Psi, G)$. Let $\sigma, \tau \in \mathcal{G}$,

$$\begin{aligned}
C_b \circ \Xi(\sigma\tau) &= C_b(\Xi(\sigma) \cdot \sigma *_\Phi \Xi(\tau)) \\
&= C_b(\Xi(\sigma)) \cdot C_b(\sigma *_\Phi \Xi(\tau)) \\
&= C_b(\Xi(\sigma)) \cdot \sigma *_\Psi C_b(\Xi(\tau)).
\end{aligned}$$

$C_b \circ \Xi$ is clearly a definable cocycle. It is straightforward to see that there are natural isomorphisms of pointed sets

$$C_b \circ - : Z^1_{\mathrm{def}}(\mathcal{G}_\Phi, G) \to Z^1_{\mathrm{def}}(\mathcal{G}_\Psi, G) \quad \text{and} \quad H^i_{\mathrm{def}}(\mathcal{G}_\Phi, G) \to H^i_{\mathrm{def}}(\mathcal{G}_\Psi, G)$$

for $i = 0, 1$. Thus, for any $\mu \in H^1_{\mathrm{def}}(\mathcal{G}, G)$, we may and will denote by $Z^1_{\mathrm{def}}(\mathcal{G}_\mu, G)$ and $H^i_{\mathrm{def}}(\mathcal{G}_\mu, G)$ the set of definable cocycles $Z^1_{\mathrm{def}}(\mathcal{G}_\Phi, G)$ and cohomology sets $H^i_{\mathrm{def}}(\mathcal{G}_\Phi, G)$ for $i = 0, 1$, respectively, with Φ any definable cocycle having cohomology class μ. Similar observations apply to $Z^1_{\mathrm{def}}(\mathcal{G}_\mu, N)$ and $H^i_{\mathrm{def}}(\mathcal{G}_\mu, N)$, and to $Z^1_{\mathrm{def}}(\mathcal{G}_\mu, G/N)$ and $H^i_{\mathrm{def}}(\mathcal{G}_\mu, G/N)$, where N is an A-definable normal subgroup of G.

§4. Further remarks on fibres and a finiteness result. We fix, throughout this section, an A-definable normal subgroup N of G. Let $\mu \in H^1_{\mathrm{def}}(\mathcal{G}, G)$. In this section we deploy the results from the previous two sections to establish an isomorphism of pointed sets between the twisted cohomology $H^1_{\mathrm{def}}(\mathcal{G}_\mu, N)$ (see Remark 3.5) and the non-twisted $H^1_{\mathrm{def}}(\mathcal{G}, N_\mu)$, where N_μ is the natural A-form of N corresponding to μ (see details below). We then conclude with a finiteness result in definable cohomology.

But first we note an easy consequence of Corollary 3.4. Consider, once again, the short exact sequence of A-definable groups

$$1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} G/N \to 1.$$

For $\mu \in H^1_{\text{def}}(\mathcal{G}, G)$, we let

$$\mathfrak{P}(\mu) := (\pi^1)^{-1}(\pi^1(\mu))$$

denote the fibre of $\pi^1 : H^1_{\text{def}}(\mathcal{G}, G) \to H^1_{\text{def}}(\mathcal{G}, G/N)$ above $\pi^1(\mu)$. Then, $\mathfrak{P}(\mu)$ has the structure of a pointed set by specifying μ as its distinguished element. In the next statement we use the notation discussed in Remark 3.5.

LEMMA 4.1. *For each $\mu \in H^1_{\text{def}}(\mathcal{G}, G)$, there is a natural surjective morphism of pointed sets from $H^1_{\text{def}}(\mathcal{G}_\mu, N)$ to $\mathfrak{P}(\mu)$.*

PROOF. With $\Phi \in Z^1_{\text{def}}(\mathcal{G}, G)$ having cohomology class μ, it follows from Section 3.2 that $ker(\pi^1_\Phi)$ is contained in the image of $H^1_{\text{def}}(\mathcal{G}_\Phi, N)$ under ι^1_Φ. By Corollary 3.4, this kernel is naturally in bijection with $\mathfrak{P}(\mu)$. The result now follows by noting that ι^1_Φ maps the distinguished element of $H^1_{\text{def}}(\mathcal{G}_\Phi, N)$ to that of $H^1_{\text{def}}(\mathcal{G}_\Phi, G)$ and, in turn, the map F_Φ from Proposition 3.3 maps the latter distinguished element to μ. ⊣

For $\mu \in H^1_{\text{def}}(\mathcal{G}, G)$, let $\Phi \in Z^1_{\text{def}}(\mathcal{G}, G)$ be a cocycle with cohomology class μ. Using the notation of Section 2, consider the map $\Lambda_\Phi : \mathcal{G} \to \text{Aut}_{\text{def}}(N)$ given by

$$\Lambda_\Phi(\sigma)(n) = C_{\Phi(\sigma)}(n)$$

for all $n \in N$. Recall that $C_h : N \to N$ denotes conjugation by $h \in G$; namely, $C_h(n) = h \cdot n \cdot h^{-1}$ for all $n \in N$. Since N is a normal in G, the image of Λ_Φ is indeed in $\text{Aut}_{\text{def}}(N)$. Furthermore, Λ_Φ is a definable cocycle from $\mathcal{G}$ to $\text{Aut}_{\text{def}}(N)$. It is clearly definable and for $\sigma_1, \sigma_2 \in \mathcal{G}$ and $n \in N$ we have

$$\begin{aligned}
\Lambda_\Phi(\sigma_1\sigma_2)(n) &= C_{\Phi(\sigma_1\sigma_2)}(n) \\
&= \Phi(\sigma_1\sigma_2) \cdot n \cdot (\Phi(\sigma_1\sigma_2))^{-1} \\
&= \Phi(\sigma_1) \cdot \sigma_1(\Phi(\sigma_2)) \cdot n \cdot (\sigma_1(\Phi(\sigma_2)))^{-1} \cdot (\Phi(\sigma_1))^{-1} \\
&= \Phi(\sigma_1) \cdot \sigma_1(C_{\Phi(\sigma_2)})(n) \cdot (\Phi(\sigma_1))^{-1} \\
&= C_{\Phi(\sigma_1)}(\sigma_1(C_{\Phi(\sigma_2)})(n)) \\
&= \Lambda_\Phi(\sigma_1) \cdot \sigma_1(\Lambda_\Phi(\sigma_2))(n).
\end{aligned}$$

Furthermore, if $\Psi \in Z^1_{\text{def}}(\mathcal{G}, G)$ is cohomologous to Φ, we have that Λ_Ψ is cohomologous to Λ_Φ in $Z^1_{\text{def}}(\mathcal{G}, \text{Aut}_{\text{def}}(N))$. Indeed, if $b \in G$ witnesses the fact that Ψ and Φ are cohomologous, then conjugation $C_b \in \text{Aut}_{\text{def}}(N)$ witnesses that Λ_Ψ is cohomologous to Λ_Φ.

We thus have a well-defined map from $H^1_{\text{def}}(\mathcal{G}, G)$ to $H^1_{\text{def}}(\mathcal{G}, \text{Aut}_{\text{def}}(N))$ mapping μ to the cohomology class of Λ_Φ. By Proposition 2.4, this yields a map from $H^1_{\text{def}}(\mathcal{G}, G)$ to $\mathfrak{F}_{\text{def}}(N)$ (the collection of A-forms of N up to A-definable isomorphism). For $\mu \in H^1_{\text{def}}(\mathcal{G}, G)$ we denote the associated A-form of N by N_μ. We note that when G is abelian Λ_Φ is the trivial cocycle and so N_μ is definably isomorphic to N over A.

We now state and prove the main result of this section.

THEOREM 4.2. *Let $\mu \in H^1_{\text{def}}(\mathcal{G}, G)$. Set N_μ to be the A-form of N associated with μ, as discussed above. Then, there is natural isomorphism of pointed sets between the*

twisted cohomology $H^1_{\mathrm{def}}(\mathcal{G}_\mu, N)$ *and non-twisted cohomology* $H^1_{\mathrm{def}}(\mathcal{G}, N_\mu)$. *Note that the latter has coefficients in the A-form* N_μ.

Let $f : N_\mu \to N$ be the definable isomorphism (over M) constructed in Section 2 (right before Proposition 2.4). We recall the construction of f to obtain an explicit formula. Let $\Phi \in Z^1_{\mathrm{def}}(\mathcal{G}, G)$ be a definable cocycle with cohomology class μ. Let a be a tuple from M and $h(x, y)$ an A-definable function witnessing that Φ is definable; that is, $\Phi(\sigma) = h(a, \sigma(a))$ for all $\sigma \in \mathcal{G}$. Recall that above we defined $\Lambda_\Phi : \mathcal{G} \to \mathrm{Aut}_{\mathrm{def}}(N)$ by $\Lambda_\Phi(\sigma)(-) = C_{\Phi(\sigma)}(-)$ and showed it is a definable cocycle from $\mathcal{G}$ to $\mathrm{Aut}_{\mathrm{def}}(N)$. It follows that

$$\Lambda_\Phi(\sigma)(-) = h(a, \sigma(a)) \cdot (-) \cdot h(a, \sigma(a))^{-1}.$$

In other words, if we set $\tilde{h}$ to be the A-definable (partial) function

$$\tilde{h}(x, y, z) = h(x, y) \cdot z \cdot h(x, y)^{-1}$$

we get

$$\Lambda_\Phi(\sigma)(-) = \tilde{h}(a, \sigma(a), -), \quad \text{for all } \sigma \in \mathcal{G},$$

that is, a and $\tilde{h}(x, y, z)$ witness that the cocycle Λ_Φ is definable. Now let Y be the realisations of $tp(a/A)$ in M, then

$$N_\mu = (Y \times N)/R,$$

where R is the equivalence relation

$$(b, d)\, R\, (c, e) \quad \Longleftrightarrow \quad \tilde{h}(b, c, e) = d \quad \Longleftrightarrow \quad h(b, c) \cdot e \cdot h(b, c)^{-1} = d.$$

The group structure on N_μ is the one induced by

$$(b, d) * (c, e) = (b, d \cdot \tilde{h}(b, c, e)) = (b, d \cdot h(b, c) \cdot e \cdot h(b, c)^{-1})$$

for $(b, c), (c, e) \in Y \times N$. Finally, the function $f : N_\mu \to N$ is given by

$$f(m) = \tilde{h}(a, \tau(a), n) = \Lambda_\Phi(\tau)(n) = C_{\Phi(\tau)}(n),$$

where m is the R-equivalence class of $(\tau(a), n) \in Y \times N$. Recall that Y is the $\mathrm{Aut}(M/A)$-orbit of a, so every element in Y is of the form $\tau(a)$ for some $\tau \in \mathrm{Aut}(M/A)$.

The upshot is that now Theorem 4.2 is a consequence of the following lemma.

LEMMA 4.3. *Let* $f : N_\mu \to N$ *be as above. Set*

$$t_f : Z^1_{\mathrm{def}}(\mathcal{G}, N_\mu) \to Z^1_{\mathrm{def}}(\mathcal{G}_\mu, N)$$

to be the map defined by $t_f(\Psi) = f \circ \Psi$. *Then,* t_f *is a bijection. Furthermore, it induces an isomorphism of pointed sets* $T_f : H^1_{\mathrm{def}}(\mathcal{G}, N_\mu) \to H^1_{\mathrm{def}}(\mathcal{G}_\mu, N)$.

PROOF. We first show that f commutes with the actions of $\mathcal{G}$ on N_μ and $\mathcal{G}_\mu$ on N; that is, we prove that for $\sigma \in \mathcal{G}$ and $m \in N_\mu$ we have

$$f(\sigma(m)) = \sigma * f(m). \tag{4.1}$$

Indeed, let m be the R-equivalence class of $(\tau(a), n) \in Y \times N$, by construction of f we have (see discussion above)

$$f(m) = C_{\Phi(\tau)}(n) \quad \text{and} \quad f(\sigma(m)) = C_{\Phi(\sigma\tau)}(\sigma(n)).$$

The latter follows as $\sigma(m)$ is the R-equivalence class of $(\sigma\tau(a), \sigma(n))$. Using this, we get

$$
\begin{aligned}
\sigma * (f(m)) &= \sigma * (C_{\Phi(\tau)}(n)) \\
&= C_{\Phi(\sigma)}\left(\sigma\left(C_{\Phi(\tau)}(n)\right)\right) \\
&= C_{\Phi(\sigma)} C_{\sigma(\Phi(\tau))}\sigma(n) \\
&= C_{\Phi(\sigma\tau)}\sigma(n) \\
&= f(\sigma(m)),
\end{aligned}
$$

as desired.

Let $\Psi \in Z_{\mathrm{def}}^1(\mathcal{G}, N_\mu)$. It follows from (4.1), as we have done in previous sections, that $f \circ \Psi$ is a cocycle from $\mathcal{G}_\mu$ to N. As f is a definable function, $t_f(\Psi) = f \circ \Psi$ is in fact in $Z_{\mathrm{def}}^1(\mathcal{G}_\mu, N)$. One can analogously consider $f^{-1} : N \to N_\mu$ and easily argue that the map

$$t_{f^{-1}} : Z_{\mathrm{def}}^1(\mathcal{G}_\mu, N) \to Z_{\mathrm{def}}^1(\mathcal{G}, N_\mu)$$

given by $t_{f^{-1}}(\Lambda) = f^{-1} \circ \Lambda$ is the inverse of t_f. Note here that one can use (4.1) to argue that f^{-1} commutes with the relevant actions.

It also follows from (4.1), as we have explicitly done in previous sections, that t_f induces a (well defined) map $T_f : H_{\mathrm{def}}^1(\mathcal{G}, N_\mu) \to H_{\mathrm{def}}^1(\mathcal{G}_\mu, N)$. Clearly, since $f : N_\mu \to N$ is a group homomorphism, the map t_f maps the trivial cocycle of $Z_{\mathrm{def}}^1(\mathcal{G}, N_\mu)$ to that of $Z_{\mathrm{def}}^1(\mathcal{G}_\mu, N)$. Thus, T_f maps distinguished element to distinguished element. Furthermore, one readily checks that the induced map $T_{f^{-1}} : H_{\mathrm{def}}^1(\mathcal{G}_\mu, N) \to H_{\mathrm{def}}^1(\mathcal{G}, N_\mu)$ is the inverse of T_f, and hence T_f is indeed an isomorphism of pointed sets. $\dashv$

Here is the promised finiteness result.

COROLLARY 4.4. *If $H_{\mathrm{def}}^1(\mathcal{G}, G/N)$ is finite and $H_{\mathrm{def}}^1(\mathcal{G}, N_\mu)$ is also finite, for all $\mu \in H_{\mathrm{def}}^1(\mathcal{G}, G)$, then $H_{\mathrm{def}}^1(\mathcal{G}, G)$ is finite as well.*

PROOF. Since $H_{\mathrm{def}}^1(\mathcal{G}, G/N)$ is finite, there are $\mu_1, \ldots, \mu_r \in H_{\mathrm{def}}^1(\mathcal{G}, G)$ such that

$$H_{\mathrm{def}}^1(\mathcal{G}, G/N) = \{\pi^1(\mu_1), \ldots, \pi^1(\mu_r)\}.$$

Note that the π^1-fibre in $H_{\mathrm{def}}^1(\mathcal{G}, G)$ over $\pi^1(\mu_i)$ coincides with the (pointed) set $\mathfrak{P}(\mu_i)$ as defined above. By Lemma 4.1 and Theorem 4.2, this set is a homomorphic image of the cohomology $H_{\mathrm{def}}^1(\mathcal{G}, N_{\mu_i})$, where N_{μ_i} is the A-form of N associated with μ_i. By our assumption, the latter cohomology set is finite, and so the π^1-fibre in $H_{\mathrm{def}}^1(\mathcal{G}, G)$ over $\pi^1(\mu_i)$ is finite for all $i = 1, \ldots, r$. It follows that $H_{\mathrm{def}}^1(\mathcal{G}, G)$ is finite. $\dashv$

REMARK 4.5.

(i) In the case when G is abelian the conclusion of the above corollary holds under the assumption that $H_{\mathrm{def}}^1(\mathcal{G}, G/N)$ and $H_{\mathrm{def}}^1(\mathcal{G}, N)$ are finite. This

follows immediately from Theorem 3.1 as in this case the cohomology
sequence is an exact sequence of groups (and group homomorphisms).

(ii) In the context of classical (algebraic) Galois cohomology (see Example
1.3(i)), the corollary is well known and appears in Section 6.4 of [8] and
as Corollary 3 of Proposition 39, Chapter I of [11]. It finds key application
in Section 4 of Chapter III of [11]. Our arguments are based on (but are
somewhat simpler and more explicit than) those appearing in [8] (cf. Sections
1.3, 2.2, and 6.4 of this reference).

(iii) In the context of Kolchin's (differential) constrained cohomology (see
Example 1.3(ii)), the above corollary (which is not treated in Kolchin's book
[2] or elsewhere, to the author's knowledge) is used in [3, Lemma 2.6]. There,
a sketch of the argument is pointed out (for the differential case) leaving
details to the reader.

§5. Differential Galois cohomology. In this section we complete the results on
differential constrained cohomology of differential algebraic groups over bounded,
differentially large, fields from [3]. We also clarify a proof from [1] about the algebraic
Galois cohomology of algebraic groups over bounded fields.

For model-theoretic notions, differential algebraic notions and their inter-
relations, see the introductions to [3] and [7], and the various references there
(such as Poizat's [9]). But we recall some key definitions: The field K (which we are
assuming to be of characteristic 0) is *bounded* or equivalently has Serre's property
(F), if K has only finitely many extensions of degree n, for each n. A field K is said to
be *large* if whenever W is a K-irreducible variety with a smooth K-point, then V has
a Zariski-dense set of K-points. A differential field (K, ∂) is said to be *differentially
large* if K is large as a field, and for any differential field extension (L, ∂), if K is
existentially closed in L (as fields), then (K, ∂) is existentially closed in (L, ∂). We
will also be using notation from Example 1.3.

We will make use of a slight generalization or version of Corollary 4.4. So here
we are in the same general context as the previous section: M is a structure, A a
subset of the universe of M as in Assumption 1.1, G is a group definable in M
over A, N a normal subgroup of G definable in M over A, π denotes the surjective
homomorphism $G \to G/N$, and $\mathcal{G} = Aut(M/A)$.

COROLLARY 5.1.   *Let κ be any infinite cardinal. Suppose that $|H^1_{\mathrm{def}}(\mathcal{G}, G/N)| \leq \kappa$
and $|H^1_{\mathrm{def}}(\mathcal{G}, N_\mu)| \leq \kappa$ for all $\mu \in H^1_{\mathrm{def}}(\mathcal{G}, G)$. Then $|H^1_{\mathrm{def}}(\mathcal{G}, G)| \leq \kappa$.*

PROOF.   This has an identical proof as Corollary 4.4: choose μ_i for $i < \lambda \leq \kappa$
such that $H^1_{\mathrm{def}}(\mathcal{G}, G/N) = \{\pi^1(\mu_i) : i < \lambda\}$. For each i, the π^1-fibre over $\pi^1(\mu_i)$ is
the image of $H^1_{\mathrm{def}}(\mathcal{G}, N_{\mu_i})$ under a suitable map, so has cardinality at most κ, hence
$H^1_{\mathrm{def}}(\mathcal{G}, G)$ has cardinality at most κ.   ⊣

The following was proved in [1, Theorem 5.2], but some of the subtleties around
exact sequences of cohomology sets were overlooked. We take the opportunity to
give the proof again.

PROPOSITION 5.2.   *Suppose the field K is bounded and G is any algebraic group over
K. Then $H^1(K, G)$ is countable.*

PROOF.   The first point is to reduce to the case when G is connected. Let G^0 be the connected component of G. The exact sequence $1 \to G^0 \to G \to G/G^0 \to 1$ of algebraic groups over K induces an exact sequence

$$H^1(K, G_0) \to H^1(K, G) \to H^1(K, G/G^0).$$

The last term, $H^1(K, G/G^0)$, is finite by Proposition 8, Chapter III of [11]. Hence by Corollary 5.1, $H^1(K, G)$ is countable if and only if $H^1(K, {}_\mu G^0)$ is countable for all $\mu \in H^1(K, G)$. Since every K-form of G^0 is connected, the reduction is complete.

The Chevalley–Barsotti theorem (Theorem 16 of [10]) yields an exact sequence $1 \to L \to G \to A \to 1$ of connected algebraic groups over K where L is linear and A is an abelian variety. As shown in the proof in Theorem 5.2 of [1], $H^1(K, A)$ is countable (as it coincides with $H^1(K_0, A)$, where K_0 is a countable elementary substructure of K). It is proved in [11] that $H^1(K, L)$ is finite for a connected linear algebraic group L. Since every K-form of L is linear, each set $H^1(K, {}_\mu L)$ is finite. By Corollary 5.1, $H^1(K, G)$ is countable.   ⊣

We now pass to differential algebraic groups over differential fields. As in [3] we work with one derivation (so the ambient theory is DCF_0), although everything generalizes suitably to the case of a finite set Δ of commuting derivations (and the corresponding theory $DCF_{0,m}$). In [3, Theorem 4.1] we proved finiteness of the differential constrained cohomology $H^1_\partial(K, G)$ when G is a linear differential algebraic group over a bounded field K with (K, ∂) differentially large.

Here we aim towards proving:

THEOREM 5.3.   *Suppose that K is bounded* (*as a field*) *and* (K, ∂) *is differentially large. Then, for any differential algebraic group G over K, $H^1_\partial(K, G)$ is countable.*

We will go through a few lemmas. For us, G will denote a differential algebraic group over a differential field (K, ∂); equivalently, a group definable over K in a differentially closed field $(\mathcal{U}, \partial)$ containing K. There is no harm in taking $\mathcal{U}$ to be a differential closure K^{diff} of K.

LEMMA 5.4.   *Suppose G is finite and K is bounded* (*as a field*). *Then $H^1_\partial(K, G)$ is finite.*

PROOF.   We could consider G as an algebraic group and use Lemma 2.6(1) from [3] which says that for G definable over K in the field language, $H^1(K, G) = H^1_\partial(K, G)$ and the former is finite by Proposition 8 of Chapter III of [11].

Alternatively, note that the finitely many points of G are in K^{alg} so really $H^1_\partial(K, G) = H^1(Gal(K^{alg}/K), G)$ which is finite by Proposition 8 of Chapter III of [11] again.   ⊣

LEMMA 5.5.   *Suppose that G has finite Morley rank, K is bounded as a field, and (K, ∂) is differentially large. Then, $H^1_\partial(K, G)$ is countable.*

PROOF.   By the above lemma, together with Corollary 5.1, we may assume that G is *connected* as a differential algebraic group, namely has no proper definable subgroup of finite index.

This is an adaptation of Case 1 of the proof of Theorem 4.1 of [3]. We give some details. First (cf. Remark 2.4(1) of [3]), G is definably over K isomorphic to

the "sharp points" or "∂-points" of a connected algebraic ∂-group (H, s) over K. By a (connected) algebraic ∂-group (H, s) over K we mean a connected algebraic group H over K equipped with an extension of ∂ to a derivation of the structure sheaf of H; equivalently, a regular homomorphic section s (over K) of the surjective homomorphism $\tau(G) \to G$ of algebraic groups over K, τ being the first prolongation of G, which is a connected algebraic group over K as well as a torsor for the tangent bundle $T(G)$ of G. The group $(H, s)^{\partial}$ is the definable (in the differential field K^{diff}) over K group $\{a \in H(K) : \partial(a) = s(a)\}$. So we are assuming that $G = (H, s)^{\partial}$. Any definable over K principal homogeneous space (PHS) for G is of the form $(X, s_X)^{\partial}$ for some (algebraic) principal homogeneous space X over K for H, and regular section s_X over K for $\tau(X) \to X$ such that for all $h \in H$ and $x \in X$, $s_X(h \cdot x) = s(h) \cdot s_X(x)$ (see Remarks 2.2 and 2.3 of [3]). One of the main points of [3, Section 3] is Corollary 3.3 there which says that: given also (Y, s_Y), if X and Y are isomorphic over K as algebraic PHS's for H, then $(X, s_X)^{\partial}$ and $(Y, s_Y)^{\partial}$ are isomorphic over K as definable (differential algebraic) PHS's for $G = (H, s)^{\partial}$.

It follows that the cardinality of $H_{\partial}^1(K, G)$ is at most the cardinality of $H^1(K, H)$ which by Proposition 5.2 is countable as K is bounded. Recall that H^1 and H_{δ}^1 classify algebraic and differential algebraic PHSs [5].   ⊣

Before completing the proof of Theorem 5.3 let us recall the "Manin maps" and their properties (with precise references):

FACT 5.6. *Let A be an abelian variety over K. Then there is a definable (over K) surjective homomorphism μ from $A(\mathcal{U})$ to $(\mathcal{U}, +)^d$, where d is the dimension of A as an algebraic group, such that $ker(\mu)$ has finite Morley rank (and is connected).*

PROOF. A is an almost direct product of simple abelian varieties $A_1, \dots, A_m$ over K, where simple means having no proper infinite algebraic subgroups. For each i, there is by Fact 1.7(ii) of [4] a definable (over K) surjective homomorphism μ_i from $A_i(\mathcal{U})$ to $(\mathcal{U}^{d_i}, +)$ with $ker(\mu_i)$ connected of finite Morley rank, and where $d_i = dim(A_i)$. Then the μ_i's induce the required $\mu : A(\mathcal{U}) \to (\mathcal{U}^d, +)$. (Namely we have $\oplus_i \mu_i : \oplus_i A_i \to \mathcal{U}^d$, which is 0 on the torsion elements, so induces $\mu : A \to \mathcal{U}^d$.)   ⊣

We finish with the proof of Theorem 5.3.

PROOF OF THEOREM 5.3. We reduce to the connected case as in the beginning of Lemma 5.5. Let G be our connected definable (over K) group. By Corollary 4.2 of [6], we may assume that G is a subgroup of $G_1(\mathcal{U})$ for some connected algebraic group G_1 over K. The Chevalley–Barsotti theorem mentioned earlier gives an exact sequence $1 \to L_1 \to G_1 \to A_1 \to 1$ of connected algebraic groups over K where L_1 is linear and A_1 is an abelian variety. Let $L = G \cap L_1$. So we obtain an exact sequence $1 \to L \to G \to A \to 1$ of differential algebraic groups over K, where $L = G \cap L_1(\mathcal{U})$ and $A \subseteq A_1(\mathcal{U})$. Theorem 4.1 of [3] says that $H_{\partial}^1(K, L')$ is finite for any K-form L' of L (as L' will also be linear). So, by Corollary 5.1, it suffices to prove that $H_{\partial}^1(K, A)$ is countable. Let $\mu : A_1(\mathcal{U}) \to (\mathcal{U}^d, +)$ be the Manin map given by Fact 5.6. Then, restricting μ to A gives an exact sequence of definable over K groups

$$1 \to ker(\mu) \cap A \to A \to N \to 1,$$

where $ker(\mu) \cap A$ has finite Morley rank and N is linear.

By Lemma 5.5 (and 5.4), $H^1_\partial(K, ker(\mu) \cap A))$ is countable, and the same holds also for any K-form of $ker(\mu) \cap A$ (as it also has finite Morley rank). By Theorem 4.1 of [3], $H^1_\partial(K, N)$ is finite. Hence, by Corollary 5.1, $H^1_\partial(K, A)$ is countable. $\dashv$

REFERENCES

[1] M. KAMENSKY and A. PILLAY, *Interpretations and differential Galois extensions*. **International Mathematics Research Notices**, vol. 2016 (2016), pp. 7390–7413.

[2] E. KOLCHIN, **Differential Algebraic Groups**, Academic Press, Orlando, 1985.

[3] O. LEÓN SÁNCHEZ and A. PILLAY, *Differential Galois cohomology and parameterized Picard–Vessiot extensions*. **Communications in Contemporary Mathematics**, vol. 23 (2021), no. 8, pp. 1–25.

[4] D. MARKER and A. PILLAY, *Differential Galois theory III: Some inverse problems*. **Illinois Journal of Mathematics**, vol. 41 (1997), pp. 453–461.

[5] A. PILLAY, *Remarks on Galois cohomology and definability*, **Journal of Symbolic Logic**, vol. 62 (1997), no. 2, pp. 487–492.

[6] ———, *Some foundational questions concerning definable groups*. **q**, vol. 179 (1997), no. 1, pp. 179–200.

[7] ———, *The Picard–Vessiot theory, contrained cohomology, and linear differential algebraic groups*. **Journal de Mathématiques Pures et Appliquées**, vol. 108 (2017), pp. 809–817.

[8] V. PLATONOV and A. RAPINCHUK, **Algebraic Groups and Number Theory**, Academic Press, San Diego, 1994.

[9] B. POIZAT, **Groupes Stables**, Nur al-mantiq wal-ma'rifah, Villeurbanne, 1987.

[10] M. ROSENLICHT, *Some basic theorems on algebraic groups*. **American Journal of Mathematics**, vol. 78 (1956), no. 2, pp. 401–443.

[11] J. P. SERRE, **Galois Cohomology**, Springer Monographs in Mathematics, Springer, Berlin, 1997.

[12] ———, **Groupes de Congruence**, Séminaire Bourbaki, vol. 10 (1968), pp. 275–291.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MANCHESTER
OXFORD ROAD, MANCHESTER, M13 9PL, UK
E-mail: omar.sanchez@manchester.ac.uk

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NOTRE DAME
255 HURLEY, NOTRE DAME, IN 46556, USA
E-mail: dmeretzk@nd.edu

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NOTRE DAME
255 HURLEY, NOTRE DAME, IN 46556, USA
E-mail: apillay@nd.edu