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TWISTED HILBERT SPACES

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A Banach space X is called a twisted sum of the Banach spaces Y and Z if it has a subspace isomorphic to Y such that the corresponding quotient is isomorphic to Z. A twisted Hilbert space is a twisted sum of Hilbert spaces. We prove the following tongue-twister: there exists a twisted sum of two subspaces of a twisted Hilbert space that is not isomorphic to a subspace of a twisted Hilbert space. In other words, being a subspace of a twisted Hilbert space is not a three-space property.

INTRODUCTION

A Banach space X is called a twisted sum of the Banach spaces Y and Z if it has a subspace isomorphic to Y whose corresponding quotient is isomorphic to Z, or else, if there exists an exact sequence

$$0 \to Y \to X \to Z \to 0,$$

where the arrows represent bounded linear operators. This note is about "twisted" Hilbert spaces (that is, twisted sums of Hilbert spaces). We construct a twisted sum of two subspaces of twisted Hilbert spaces that cannot be embedded in a twisted Hilbert space, thus answering in part to a question of Castillo, González and Yost [**1**, p.95].

THE EXAMPLE

The example is based on the space Z_2 of Kalton and Peck [**4**] whose construction we briefly sketch. Consider the homogeneous map $F : l_2 \to l_2$ defined as

$$F\Big(\sum_{i=1}^{n} x_i e_i\Big) = \sum_{i=1}^{n} \big(\log\|x\| - \log|x_i|\big) x_i e_i.$$

It can be proved that for $x, y \in l_2$ one has

$$\big\|F(x+y) - Fx - Fy\big\| \leqslant K\big(\|x\| + \|y\|\big),$$

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so that F is quasi-linear. Observe that F is only defined for finitely supported sequences. It can be extended to the whole of l_2 keeping quasi-linearity ([**4**], [**1**, p.90]). The space Z_2 is $l_2 \oplus_F l_2$, that is, the product space $l_2 \times l_2$ equipped with the quasi-norm

$$\|(y,z)\|_F = \|y - Fz\|_2 + \|z\|_2 .$$

Actually this is only a quasi-norm, but it is equivalent to a norm by results of Kalton ([**3**], [**1**, p.19]). Observe that Z_2 contains an isometric copy of l_2 (the subspace $\{(y,0) : y \in l_2\}$) whose corresponding quotient is also isometric to l_2, so that there is an exact sequence

$$0 \to l_2 \to Z_2 \to l_2 \to 0.$$

Kalton and Peck proved that this sequence does not split and therefore Z_2 is a twisted Hilbert space but not itself a Hilbert space. (An earlier example was given by Enflo, Lindenstrauss and Pisier [**2**], [**1**, p.82].)

Consider now the subspace Z of Z_2 spanned by the sequence $\{(0,e_i)\}$, where $\{e_i\}$ is the standard basis of l_2. (This space is isomorphic to the Orlicz sequence space l_M, being $M(t) = (t\log t)^2$, [**4**, Lemma 5.3] but the following description of Z will simplify the exposition.) We want to see that $(0,e_i)$ is a symmetric basis. That $\{(0,e_i)\}$ is a basic sequence is an immediate consequence of Nikolskii's criterion since, for $n < m$, one has

$$\begin{aligned}\Big\|\sum_{i=1}^{n} x_i(0,e_i)\Big\|_F &= \Big\|F\Big(\sum_{i=1}^{n} x_i e_i\Big)\Big\|_2 + \Big\|\sum_{i=1}^{n} x_i e_i\Big\|_2 \\ &\leqslant \Big\|F\Big(\sum_{i=1}^{m} x_i e_i\Big)\Big\|_2 + \Big\|\sum_{i=1}^{m} x_i e_i\Big\|_2 \\ &= \Big\|\sum_{i=1}^{m} x_i(0,e_i)\Big\|_F .\end{aligned}$$

Moreover, for every permutation π of the integers and every choice of signs $\sigma_i = \pm 1$, one has

$$\Big\|\sum_{i=1}^{n} x_i(0,e_i)\Big\|_F = \Big\|\sum_{i=1}^{n} \sigma_i x_i(0,e_{\pi(i)})\Big\|_F$$

since l_2 has symmetric norm:

$$\Big\|\sum_{i=1}^{n} x_i e_i\Big\|_2 = \Big\|\sum_{i=1}^{n} \sigma_i x_i e_{\pi(i)}\Big\|_2$$

and also

$$\Big\|F\Big(\sum_{i=1}^{n} x_i e_i\Big)\Big\|_2 = \Big\|F\Big(\sum_{i=1}^{n} \sigma_i x_i e_{\pi(i)}\Big)\Big\|_2 .$$

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Hence $\{(0, e_i)\}$ is a symmetric basis (even with symmetric norm). We identify Z with a sequence space via the basis which we denote by $\{\nu_n\}$ (instead of $\{(0, e_n)\}$). The (quasi)-norm of Z will be denoted by $\|\cdot\|_Z$. It is not hard to verify that Z satisfies the following three conditions:

(a) $\|\nu_n\|_Z = 1$ for all n;
(b) $\|z\|_\infty \leqslant C\|z\|_Z$ for some C and all $z \in Z$;
(c) $\|sz\|_Z \leqslant M\|s\|_\infty \|z\|_Z$ for some M and all $s \in l_\infty$, $z \in Z$,

so that the method of [4] still works. Define for $z = \sum_{i=1}^{n} z_i \nu_i$

$$G(z) = \sum_{i=1}^{n} \bigl(\log \|z\|_Z - \log |z_i|\bigr) z_i \nu_i.$$

Then G is quasi-linear on the finitely supported sequences of Z and can be extended to a quasi-linear map $G : Z \to Z$. Let $Z \oplus_G Z$ be the twisted sum induced by G, that is, the algebraic product space $Z \times Z$ endowed with the quasi-norm

$$\|(y, z)\|_B = \|y - Gz\|_Z + \|z\|_Z$$

which is always equivalent to a norm.

CLAIM. The space $Z \oplus_G Z$ cannot be embedded in a twisted Hilbert space.

PROOF: Let us estimate the cotype 2 constants which are the least numbers $a_{n,2}$ such that, for $x_1, \ldots, x_n$ in $Z \oplus_G Z$,

$$\left[\int_0^1 \left\|\sum_{i=1}^{n} r_i(t) x_i\right\|^2 dt\right]^{1/2} \leqslant a_{n,2} \left[\sum_{i=1}^{n} \|x_i\|^2\right]^{1/2},$$

where r_i is the sequence of Rademacher functions. Take $x_i = (0, \nu_i)$. Then

$$\sum_{i=1}^{n} \|x_i\|_G^2 = n$$

while

$$\begin{aligned}
\int_0^1 \Big\| \sum_{i=1}^n r_i(t)x_i \Big\|_G^2 dt &= \int_0^1 \Big\| \Big(0, \sum_{i=1}^n r_i(t)\nu_i\Big) \Big\|_G^2 dt \\
&\geqslant \int_0^1 \Big\| G\Big(\sum_{i=1}^n r_i(t)\nu_i\Big) \Big\|_Z^2 dt \\
&= \int_0^1 \Big\| \sum_{i=1}^n \Big\{ \log \Big\| \sum_{j=1}^n r_j(t)\nu_j \Big\|_Z \Big\} r_i(t)\nu_i \Big\|_Z^2 dt \\
&= \int_0^1 \Big[\Big\| \sum_{i=1}^n r_i(t)\nu_i \Big\|_Z \log \Big\| \sum_{j=1}^n r_j(t)\nu_j \Big\|_Z \Big]^2 dt \\
&= \Big[\Big\| \sum_{i=1}^n \nu_i \Big\|_Z \log \Big\| \sum_{i=1}^n \nu_i \Big\|_Z \Big]^2 \\
&\geqslant \frac{1}{16} n \log^4(n),
\end{aligned}$$

since a straightforward computation shows that $\Big\| \sum_{i=1}^n \nu_i \Big\|_Z \geqslant (n^{1/2} \log n)/2$. This obviously implies that $a_{n,2}(Z \oplus_G Z) \geqslant c \log^2 n$. But Kalton and Peck proved in [**4**, Theorem 6.2(a)] that for a twisted Hilbert space T one has $a_{n,2}(T) \leqslant C \log n$. Hence $Z \oplus_G Z$ is not a subspace of a twisted Hilbert space and the proof is complete.

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