On bounded skew-symmetric forms

By AUREL WINTNER, The Johns Hopkins University.

(*Received 7th October*, 1936. *Read 6th November*, 1936.)

It is well known that the real, skew-symmetric, non-singular, bilinear forms of $n + n$ variables have no invariants. In fact, each of these forms may be transformed into one and the same form, for instance into the one which occurs in the usual representation of the complex group. The standard proofs of this theorem break down in case of infinite forms which are bounded in the sense of Hilbert, one of the impediments being the possibility of a continuous spectrum[1]. The object of this note is to show that, while the usual proofs break down, the theorem itself is true in Hilbert's case also.

Let A, C, T, ... denote real infinite bounded matrices. Let E be the unit matrix. If C is such that $CD = E$ and $DC = E$ hold for a bounded D, one terms C non-singular and denotes D by C^{-1}. If C is non-singular and $C^{-1} = C'$, one calls C orthogonal. If C is symmetric and $\lambda E - C$ is non-singular for every $\lambda \leqq 0$, one calls $C = C'$ positive definite. On placing $\lambda = 0$, it is seen that every positive definite C is non-singular.

The theorem to be proved may be stated as follows: There exists for every pair of real, bounded, skew-symmetric, non-singular matrices, say S_1 and S_2, a real, bounded, non-singular matrix L such that $L'S_1L = S_2$. This theorem is clearly equivalent to the following one: There exists a universal matrix G with the property that one can find for every real, bounded, skew-symmetric, non-singular matrix S a real, bounded, non-singular matrix T such that $T'ST = G$.

In the proof use will be made of the following facts[2]:

---

[1] Simple examples of real, bounded, non-singular, skew-symmetric forms with continuous spectra may be deduced, according to Toeplitz, from his theory of L-forms. *Cf.* E. Hellinger and O. Toeplitz, *Encyklopädie der mathematischen Wissenschaften*, II C 13, § 44.

[2] A. Wintner, "On non-singular bounded matrices," *American Journal of Mathematics*, **54** (1932), 145-149. *Cf.* E. Hellinger and O. Toeplitz, *loc. cit.*, footnote 522a.

(i) There exists for every real, bounded, positive definite matrix P exactly one real, bounded, positive definite matrix Q such that $P = Q^2$.

(ii) The matrix Q mentioned under (i) has the property that every bounded matrix which is commutable with $P = Q^2$ is commutable with Q.

(iii) There exists for every real, bounded, non-singular matrix A exactly one pair of real, bounded, non-singular matrices P, O such that P is positive definite, O orthogonal and $A = PO$.

(iv) The matrices P, O mentioned under (iii) are commutable if and only if A is commutable with A'.

It may be mentioned[1] that (i) cannot be proved to-day by means of elementary methods, while (ii), (iii) and (iv) are, in the main, elementary consequences of (i).

Now let S be real, bounded, non-singular and skew-symmetric. Let P and O be the matrices which belong to $A = S$ in virtue of (iii). Since $S' = -S$ is commutable with S, it follows from (iv) that P is commutable with O. Hence, if Q denotes the matrix which belongs to P in virtue of (i), it is seen from (ii) that Q is commutable with O. Thus

$$S = PO = QQO = QOQ = Q'OQ,$$

since $Q = Q'$. Accordingly, $W'SW = O$, where W is the real, bounded, non-singular matrix Q^{-1}. Since $S' = -S$, it is clear from $W'SW = O$ that $O' = -O$, so that the spectrum of O consists of purely imaginary numbers. On the other hand, O is, by (iii), an orthogonal matrix, so that its spectrum consists of numbers of absolute value 1. Hence, the spectrum of O cannot contain numbers distinct from $\pm i$. In particular, O does not have a continuous spectrum. Since $O' = -O$, it follows[2] that there exists a real, bounded, non-singular matrix R which is orthogonal and such that $ROR^{-1} = (b_j a_{hj})$, where b_j is a real number and a_{hj} a two-rowed square matrix defined as follows:

$$a_{hj} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ if } h \neq j; \quad a_{hh} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

---

[1] *Cf.* A. Wintner, *loc. cit.*

[2] *Cf.* E. Hellinger and O. Toeplitz, *loc. cit.*, § 41. In the proof given there it is assumed that the matrix is completely continuous (vollstetig). Actually, the proof applies without change in all cases where there is no continuous spectrum.

Since ROR^{-1} has the same spectrum as O, the spectrum of every $b_j a_{hj}$ consists of $\pm i$. Hence $b_j = \pm 1$. Since the orthogonal matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

transforms a_{hh} into $-a_{hh}$, it is clear from $ROR^{-1} = (b_j a_{hj})$ and $b_j = \pm 1$ that one can choose the orthogonal matrix R such that $b_j = 1$ for every j. Then $ROR^{-1} = G$, where $G = (a_{hj})$. Since $W'SW = O$ and $R^{-1} = R'$, it follows that $T'ST = G$, where $T = WR^{-1}$. This completes the proof, since $G = (a_{hj})$ is independent of S.