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Twisted algebras of geometric algebras

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Abstract. A twisting system is one of the major tools to study graded algebras; however, it is often difficult to construct a (nonalgebraic) twisting system if a graded algebra is given by generators and relations. In this paper, we show that a twisted algebra of a geometric algebra is determined by a certain automorphism of its point variety. As an application, we classify twisted algebras of three-dimensional geometric Artin–Schelter regular algebras up to graded algebra isomorphism.

1 Introduction

The notion of twisting system was introduced by Zhang in [14]. If there is a twisting system $\theta = \{\theta_n\}_{n\in\mathbb{Z}}$ for a graded algebra A, then we can define a new graded algebra A^θ, called a twisted algebra. Zhang gave a necessary and sufficient algebraic condition via a twisting system when two categories of graded right modules are equivalent [14, Theorem 3.5]. Although a twisting system plays an important role to study a graded algebra, it is often difficult to construct a twisting system on a graded algebra if it is given by generators and relations.

Mori introduced the notion of geometric algebra $\mathcal{A}(E,\sigma)$, which is determined by geometric data which consist of a projective variety E, called the point variety, and its automorphism σ. For these algebras, Mori gave a necessary and sufficient geometric condition when two categories of graded right modules are equivalent [11, Theorem 4.7]. By using this geometric condition, we can easily construct a twisting system.

Cooney and Grabowski defined a groupoid whose objects are geometric noncommutative projective spaces and whose morphisms are isomorphisms between them. By studying a correspondence between the morphisms of this groupoid and a twisting system, they showed that the morphisms of this groupoid are parametrized by a set of certain automorphisms of the point variety [2, Theorem 28].

In this paper, we focus on studying a twisted algebra of a geometric algebra $A = \mathcal{A}(E,\sigma)$. For a twisting system θ on A, we set $\Phi(\theta) := \overline{(\theta_1|_{A_1})^*} \in \operatorname{Aut}_k \mathbb{P}(A_1^*)$ by dualization and projectivization. We find a subset $M(E,\sigma)$ of $\operatorname{Aut}_k \mathbb{P}(A_1^*)$ parametrizing twisted algebras of A up to isomorphism. As an application to three-dimensional geometric Artin–Schelter regular algebras, we will compute $M(E,\sigma)$ (see Theorems 4.8 and 4.11), which completes the classification of twisted algebras of three-dimensional geometric Artin–Schelter regular algebras.

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Itaba and the author showed that for any three-dimensional quadratic Artin–Schelter regular algebra B, there are a three-dimensional quadratic Calabi–Yau Artin–Schelter regular algebra S and a twisting system θ such that $B \cong S^\theta$ [8, Theorem 4.4]. Except for one case, a twisting system θ can be induced by a graded algebra automorphism of S. By using $M(E, \sigma)$, we can recover this fact in the case that B is geometric (see Corollary 4.12).

2 Preliminary

Throughout this paper, we fix an algebraically closed field k of characteristic zero and assume that a graded algebra is an $\mathbb{N}$-graded algebra $A = \bigoplus_{i \in \mathbb{N}} A_i$ over k. A graded algebra $A = \bigoplus_{i \in \mathbb{N}} A_i$ is called *connected* if $A_0 = k$. Let $\mathrm{GrAut}_k A$ denote the group of graded algebra automorphisms of A. We denote by $\mathrm{GrMod} A$ the category of graded right A-modules. We say that two graded algebras A and A' are *graded Morita equivalent* if two categories $\mathrm{GrMod} A$ and $\mathrm{GrMod} A'$ are equivalent.

2.1 Twisting systems and twisted algebras

Definition 2.1 Let A be a graded algebra. A set of graded k-linear automorphisms $\theta = \{\theta_n\}_{n \in \mathbb{Z}}$ of A is called a twisting system on A if

$$\theta_n(a\theta_m(b)) = \theta_n(a)\theta_{n+m}(b)$$

for any $l, m, n \in \mathbb{Z}$ and $a \in A_m$, $b \in A_l$. The twisted algebra of A by θ, denoted by A^θ, is a graded algebra A with a new multiplication $*$ defined by

$$a * b = a\theta_m(b)$$

for any $a \in A_m$, $b \in A_l$. A twisting system $\theta = \{\theta_n\}_{n \in \mathbb{Z}}$ is called *algebraic* if $\theta_{m+n} = \theta_m \circ \theta_n$ for every $m, n \in \mathbb{Z}$.

We denote by $\mathrm{TS}^{\mathbb{Z}}(A)$ the set of twisting systems on A. Zhang [14] found a necessary and sufficient algebraic condition for $\mathrm{GrMod} A \cong \mathrm{GrMod} A'$.

Theorem 2.2 [14, Theorem 3.5] *Let A and A' be graded algebras finitely generated in degree 1 over k. Then $\mathrm{GrMod} A \cong \mathrm{GrMod} A'$ if and only if A' is isomorphic to a twisted algebra A^θ by a twisting system $\theta \in \mathrm{TS}^{\mathbb{Z}}(A)$.*

Definition 2.3 For a graded algebra A, we define

$$\begin{aligned}
\mathrm{TS}_0^{\mathbb{Z}}(A) &:= \{\theta \in \mathrm{TS}^{\mathbb{Z}}(A) \mid \theta_0 = \mathrm{id}_A\},\\
\mathrm{TS}_{\mathrm{alg}}^{\mathbb{Z}}(A) &:= \{\theta \in \mathrm{TS}_0^{\mathbb{Z}}(A) \mid \theta \text{ is algebraic}\},\\
\mathrm{Twist}(A) &:= \{A^\theta \mid \theta \in \mathrm{TS}^{\mathbb{Z}}(A)\}/_{\cong},\\
\mathrm{Twist}_{\mathrm{alg}}(A) &:= \{A^\theta \mid \theta \in \mathrm{TS}_{\mathrm{alg}}^{\mathbb{Z}}(A)\}/_{\cong}.
\end{aligned}$$

Lemma 2.4 [14, Proposition 2.4] *Let A be a graded algebra. For every $\theta \in \mathrm{TS}^{\mathbb{Z}}(A)$, there exists $\theta' \in \mathrm{TS}_0^{\mathbb{Z}}(A)$ such that $A^\theta \cong A^{\theta'}$.*

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It follows from Lemma 2.4 that $\operatorname{Twist}(A) = \{A^{\theta} \mid \theta \in \mathrm{TS}_0^{\mathbb{Z}}(A)\}/_{\cong}$, so we may assume that $\theta \in \mathrm{TS}_0^{\mathbb{Z}}(A)$ to study $\operatorname{Twist}(A)$. By the definition of twisting system, it follows that $\theta \in \mathrm{TS}_{\mathrm{alg}}^{\mathbb{Z}}(A)$ if and only if $\theta_n = \theta_1^n$ for every $n \in \mathbb{Z}$ and $\theta_1 \in \operatorname{GrAut}_k A$, so

$$\operatorname{Twist}_{\mathrm{alg}}(A) = \{A^{\phi} \mid \phi \in \operatorname{GrAut}_k A\}/_{\cong},$$

where A^{ϕ} means the twisted algebra of A by $\{\phi^n\}_{n\in\mathbb{Z}}$.

2.2 Geometric algebra

Let V be a finite-dimensional k-vector space, and let $A = T(V)/(R)$ be a quadratic algebra where $T(V)$ is a tensor algebra over k and $R \subset V \otimes V$. Since an element of R defines a multilinear function on $V^* \times V^*$, we can define a zero set associated with R by

$$\mathcal{V}(R) = \{(p, q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid g(p, q) = 0 \text{ for any } g \in R\}.$$

Definition 2.5 Let $A = T(V)/(R)$ be a quadratic algebra. A geometric pair (E, σ) consists of a projective variety $E \subset \mathbb{P}(V^*)$ and $\sigma \in \operatorname{Aut}_k E$.

(1) We say that A satisfies (G1) if there exists a geometric pair (E, σ) such that

$$\mathcal{V}(R) = \{(p, \sigma(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E\}.$$

In this case, we write $\mathcal{P}(A) = (E, \sigma)$, and call E the *point variety* of A.

(2) We say that A satisfies (G2) if there exists a geometric pair (E, σ) such that

$$R = \{g \in V \otimes V \mid g(p, \sigma(p)) = 0 \text{ for all } p \in E\}.$$

In this case, we write $A = \mathcal{A}(E, \sigma)$.

(3) We say that A is a geometric algebra if it satisfies both (G1) and (G2) with $A = \mathcal{A}(\mathcal{P}(A))$.

For geometric algebras, Mori [11] found a necessary and sufficient geometric condition for $\operatorname{GrMod} A \cong \operatorname{GrMod} A'$.

Theorem 2.6 [11, Theorem 4.7] *Let $A = \mathcal{A}(E, \sigma)$ and $A' = \mathcal{A}(E', \sigma')$ be geometric algebras. Then $\operatorname{GrMod} A \cong \operatorname{GrMod} A'$ if and only if there exists a sequence of automorphisms $\{\tau_n\}_{n\in\mathbb{Z}}$ of $\mathbb{P}(V^*)$ for $n \in \mathbb{Z}$, each of which sends E isomorphically onto E', such that the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\ \tau_n|_E\ } & E' \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\sigma'} \\ E & \xrightarrow[\ \tau_{n+1}|_E\ ]{} & E' \end{array}$$

commutes for every $n \in \mathbb{Z}$.

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2.3 Artin–Schelter regular algebras and Calabi–Yau algebras

Definition 2.7 A connected graded algebra A is called a *d-dimensional Artin–Schelter regular algebra* (simply *AS-regular algebra*) if A satisfies the following conditions:

(1) $\operatorname{gldim} A = d < \infty$,
(2) $\operatorname{GKdim} A := \inf\{\alpha \in \mathbb{R} \mid \dim_k(\sum_{i=0}^{n} A_i) \leq n^\alpha \text{ for all } n \gg 0\} < \infty$, and
(3) $\operatorname{Ext}_A^i(k, A) = \begin{cases} k, & \text{if } i = d, \\ 0, & \text{if } i \neq d. \end{cases}$

Artin and Schelter proved that a three-dimensional AS-regular algebra A finitely generated in degree 1 is isomorphic to one of the following forms:

$$k\langle x, y, z\rangle/(f_1, f_2, f_3) \text{ or } k\langle x, y\rangle/(g_1, g_2),$$

where f_i are homogeneous polynomials of degree 2 (the quadratic case) and g_j are homogeneous polynomials of degree 3 (the cubic case; see [1, Theorem 1.5]).

We recall the definition of Calabi–Yau algebra introduced by [4].

Definition 2.8 [4] A k-algebra S is called *d-dimensional Calabi–Yau* if S satisfies the following conditions:

(1) $\operatorname{pd}_{S^e} S = d < \infty$, and
(2) $\operatorname{Ext}_{S^e}^i(S, S^e) \cong \begin{cases} S, & \text{if } i = d \\ 0, & \text{if } i \neq d \end{cases}$ (as right S^e-modules),

where $S^e = S^{\mathrm{op}} \otimes_k S$ is the enveloping algebra of S.

The following theorem tells us that we may assume that three-dimensional quadratic AS-regular algebra is Calabi–Yau up to graded Morita equivalence.

Theorem 2.9 [8, Theorem 4.4] *For every three-dimensional quadratic AS-regular algebra A, there exists a three-dimensional quadratic Calabi–Yau AS-regular algebra S such that* $\operatorname{GrMod} A \cong \operatorname{GrMod} S$.

Lemma 2.10 ([6, Lemma 2.8], [7, Theorem 3.2], and [12, Lemma 3.8]) *Every three-dimensional geometric AS-regular algebra A is graded Morita equivalent to* $S = k\langle x, y, z\rangle/(f_1, f_2, f_3) = \mathcal{A}(E, \sigma)$ *in Table 1.*

Remark 2.11 The original definition of geometric algebra given by Mori [11] is different from our definition. In the sense of Definition 2.5, there exists a three-dimensional quadratic AS-regular algebra which is not a geometric algebra. Strictly speaking, a three-dimensional quadratic AS-regular algebra is a geometric algebra in our sense if and only if the "point scheme" is reduced.

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Table 1: Defining relations and geometric pairs.

Type	f_1, f_2, f_3	E	σ
P	$\begin{cases} yz - zy \\ zx - xz \\ xy - yx \end{cases}$	$\mathbb{P}^2$	$\sigma(a,b,c) = (a,b,c)$
S	$\begin{cases} yz - \alpha zy \\ zx - \alpha xz \quad \alpha^3 \neq 0,1 \\ xy - \alpha yx \end{cases}$	$\mathcal{V}(x)$ $\cup\ \mathcal{V}(y)$ $\cup\ \mathcal{V}(z)$	$\begin{cases} \sigma(0,b,c) = (0,b,\alpha c) \\ \sigma(a,0,c) = (\alpha a,0,c) \\ \sigma(a,b,0) = (a,\alpha b,0) \end{cases}$
S′	$\begin{cases} yz - \alpha zy + x^2 \\ zx - \alpha xz \quad \alpha^3 \neq 0,1 \\ xy - \alpha yx \end{cases}$	$\mathcal{V}(x)$ $\cup\ \mathcal{V}(x^2 - \lambda yz)$ $\lambda = \frac{\alpha^3-1}{\alpha}$	$\begin{cases} \sigma(0,b,c) = (0,b,\alpha c) \\ \sigma(a,b,c) = (a,\alpha b,\alpha^{-1} c) \end{cases}$
T	$\begin{cases} yz - zy + x^2 \\ zx - xz + y^2 \\ xy - yx \end{cases}$	$\mathcal{V}(x+y)$ $\cup\ \mathcal{V}(\varepsilon x + y) \quad \varepsilon^3 = 1,\ \varepsilon, \varepsilon^2 \neq 1$ $\cup\ \mathcal{V}(\varepsilon^2 x + y)$	$\begin{cases} \sigma(a,-a,c) = (a,-a,a+c) \\ \sigma(a,-\varepsilon a,c) = (a,-\varepsilon a,\varepsilon^2 a + c) \\ \sigma(a,-\varepsilon^2 a,c) = (a,-\varepsilon^2 a,\varepsilon a + c) \end{cases}$
T′	$\begin{cases} yz - zy + xy + yx \\ zx - xz + x^2 - yz - zy + y^2 \\ xy - yx - y^2 \end{cases}$	$\mathcal{V}(x)$ $\cup\ \mathcal{V}(y^2 - xz)$	$\begin{cases} \sigma(0,b,c) = (0,b,b+c) \\ \sigma(a,b,c) = (a,-a+b,a-2b+c) \end{cases}$
NC	$\begin{cases} yz - \alpha zy + x^2 \\ zx - \alpha xz + y^2 \quad \alpha^3 \neq 0,1 \\ xy - \alpha yx \end{cases}$	$\mathcal{V}(x^3 + y^3 - \lambda xyz)$ $\lambda = \frac{\alpha^3-1}{\alpha}$	$\sigma(a,b,c) =$ $(a, \alpha b, -\frac{a^2}{b} + \alpha^2 c)$
CC	$\begin{cases} yz - zy + y^2 + 3x^2 \\ zx - xz + yx + xy - yz - zy \\ xy - yx - y^2 \end{cases}$	$\mathcal{V}(x^3 - y^2 z)$	$\sigma(a,b,c) =$ $(a-b, b, -3\frac{a^2}{b} + 3a - b + c)$
EC	$\begin{cases} \alpha yz + \beta zy + \gamma x^2 \\ \alpha zx + \beta xz + \gamma y^2 \quad (\alpha^3+\beta^3+\gamma^3)^3 \neq (3\alpha\beta\gamma)^3,\ \alpha\beta\gamma \neq 0 \\ \alpha xy + \beta yx + \gamma z^2 \end{cases}$	$\mathcal{V}(x^3 + y^3 + z^3 - \lambda xyz)$, $\lambda = \frac{\alpha^3+\beta^3+\gamma^3}{\alpha\beta\gamma}$	σ_p where $p = (\alpha,\beta,\gamma) \in E$

3 Twisted algebras of geometric algebras

In this section, we study twisted algebras of geometric algebras. Let $E \subset \mathbb{P}(V^*)$ be a projective variety where V is a finite-dimensional k-vector space. We use the following notations introduced in [2].

Definition 3.1 Let $E \subset \mathbb{P}(V^*)$ be a projective variety and $\sigma \in \operatorname{Aut}_k E$. We define

$$\begin{aligned}
&\operatorname{Aut}_k(E \uparrow \mathbb{P}(V^*)) := \{\tau \in \operatorname{Aut}_k E \mid \tau = \overline{\tau}|_E \text{ for some } \overline{\tau} \in \operatorname{Aut}_k \mathbb{P}(V^*)\},\\
&\operatorname{Aut}_k(\mathbb{P}(V^*) \downarrow E) := \{\tau \in \operatorname{Aut}_k \mathbb{P}(V^*) \mid \tau|_E \in \operatorname{Aut}_k E\},\\
&Z(E,\sigma) := \{\tau \in \operatorname{Aut}_k(\mathbb{P}(V^*) \downarrow E) \mid \sigma\tau|_E\sigma^{-1} = \tau|_E\},\\
&M(E,\sigma) := \{\tau \in \operatorname{Aut}_k(\mathbb{P}(V^*) \downarrow E) \mid (\tau|_E\sigma)^i\sigma^{-i} \in \operatorname{Aut}_k(E \uparrow \mathbb{P}(V^*)) \ \ \forall i \in \mathbb{Z}\},\\
&N(E,\sigma) := \{\tau \in \operatorname{Aut}_k(\mathbb{P}(V^*) \downarrow E) \mid \sigma\tau|_E\sigma^{-1} \in \operatorname{Aut}_k(E \uparrow \mathbb{P}(V^*))\}.
\end{aligned}$$

Note that $Z(E,\sigma) \subset M(E,\sigma) \subset N(E,\sigma) \subset \operatorname{Aut}_k(\mathbb{P}(V^*) \downarrow E)$, and $Z(E,\sigma)$, $N(E,\sigma)$ are subgroups of $\operatorname{Aut}_k(\mathbb{P}(V^*) \downarrow E)$.

Lemma 3.2 *Let $E \subset \mathbb{P}(V^*)$ be a projective variety and $\sigma \in \operatorname{Aut}_k E$. If $\sigma\operatorname{Aut}_k(E \uparrow \mathbb{P}(V^*)) = \operatorname{Aut}_k(E \uparrow \mathbb{P}(V^*))\sigma$, then $M(E,\sigma) = N(E,\sigma) = \operatorname{Aut}_k(\mathbb{P}(V^*) \downarrow E)$.*

Proof Since $M(E,\sigma) \subset N(E,\sigma) \subset \operatorname{Aut}_k(\mathbb{P}(V^*) \downarrow E)$ in general, it is enough to show that $\operatorname{Aut}_k(\mathbb{P}(V^*) \downarrow E) \subset M(E,\sigma)$. We will show that for any $\tau \in \operatorname{Aut}_k(\mathbb{P}(V^*) \downarrow E)$, $(\tau|_E\sigma)^i\sigma^{-i} \in \operatorname{Aut}_k(E \uparrow \mathbb{P}(V^*))$ for every $i \in \mathbb{Z}$ by induction so that $\tau \in M(E,\sigma)$. The claim is trivial for $i = 0$. If $(\tau|_E\sigma)^i\sigma^{-i} \in \operatorname{Aut}_k(E \uparrow \mathbb{P}(V^*))$ for some $i \geq 0$, then $(\tau|_E\sigma)^{i+1}\sigma^{-i-1} = \tau|_E\sigma((\tau|_E\sigma)^i\sigma^{-i})\sigma^{-1} \in \operatorname{Aut}_k(E \uparrow \mathbb{P}(V^*))$. If $(\tau|_E\sigma)^{-i}\sigma^i \in \operatorname{Aut}_k(E \uparrow \mathbb{P}(V^*))$ for some $i \geq 0$, then

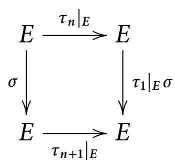
$$(\tau|_E\sigma)^{-(i+1)}\sigma^{i+1} = \sigma^{-1}\tau|_E^{-1}((\tau|_E\sigma)^{-i}\sigma^i)\sigma \in \operatorname{Aut}_k(E \uparrow \mathbb{P}(V^*)). \qquad \blacksquare$$

Let $A = \mathcal{A}(E,\sigma)$ be a geometric algebra. The map $\Phi : \mathrm{TS}_0^{\mathbb{Z}}(A) \to \operatorname{Aut}_k \mathbb{P}(A_1^*)$ is defined by $\Phi(\theta) := \overline{(\theta_1|_{A_1})^*}$.

Lemma 3.3 *Let $A = \mathcal{A}(E,\sigma)$ be a geometric algebra. Then*

$$\Phi(\mathrm{TS}_0^{\mathbb{Z}}(A)) = M(E,\sigma).$$

Proof Let $\theta \in \mathrm{TS}_0^{\mathbb{Z}}(A)$. We set $V := A_1 = (A^\theta)_1$. Then θ_n is also a graded k-linear isomorphism from A^θ to A and satisfies $\theta_n(a * b) = \theta_n(a)\theta_{n+m}(b)$ for every $n, m, l \in \mathbb{Z}$ and $a \in A_m^\theta$, $b \in A_l^\theta$. Let $\tau_n : \mathbb{P}(V^*) \to \mathbb{P}(V^*)$ be automorphisms induced by the duals of $\theta_n|_V : V \to V$. By [2, Remark 15], $\tau_n \in \operatorname{Aut}_k(\mathbb{P}(V^*) \downarrow E)$ and the diagram of automorphisms



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commutes for every $n \in \mathbb{Z}$. Then $(\tau_1|_E\sigma)^n\sigma^{-n} = \tau_n|_E \in \mathrm{Aut}_k(E \uparrow \mathbb{P}(V^*))$ for every $n \in \mathbb{Z}$, so it holds that $\Phi(\theta) = \tau_1 \in M(E,\sigma)$.

Conversely, let $\tau \in M(E,\sigma)$. Since $(\tau|_E\sigma)^n\sigma^{-n} \in \mathrm{Aut}_k(E \uparrow \mathbb{P}(V^*))$, there is an automorphism $\tau_n \in \mathrm{Aut}_k\,\mathbb{P}(V^*)$ such that $\tau_n|_E = (\tau|_E\sigma)^n\sigma^{-n}$ for every $n \in \mathbb{Z}$. By [2, Remark 15], there exists $\theta \in \mathrm{TS}_0^{\mathbb{Z}}(A)$ such that $\overline{(\theta_n|_{A_1})^*} = \tau_n$ for every $n \in \mathbb{Z}$. Hence, it follows that $\Phi(\theta) = \overline{(\theta_1|_{A_1})^*} = \tau$. ■

Let $A = T(V)/I$ be a connected graded algebra. Let $\Psi : \mathrm{GrAut}_k\,A \to \mathrm{PGL}(V)$ be a group homomorphism defined by $\Psi(\phi) = \overline{\phi|_V}$. We set

$$\mathrm{PGrAut}_k\,A := \mathrm{GrAut}_k\,A/\mathrm{Ker}\,\Psi.$$

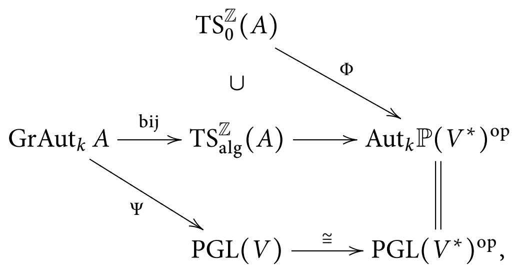
Lemma 3.4 *Let $A = \mathcal{A}(E,\sigma)$ be a geometric algebra.*

(1) $\Phi(\mathrm{TS}_{\mathrm{alg}}^{\mathbb{Z}}(A)) = Z(E,\sigma)$.
(2) $\mathrm{PGrAut}_k\,A \cong Z(E,\sigma)^{\mathrm{op}}$.

Proof (1) Let $\theta = \{\theta_1^n\}_{n\in\mathbb{Z}} \in \mathrm{TS}_{\mathrm{alg}}^{\mathbb{Z}}(A)$. We set $V := A_1 = (A^\theta)_1$. Let $\tau_n : \mathbb{P}(V^*) \to \mathbb{P}(V^*)$ be automorphisms induced by the duals of $\theta_1^n|_V : V \to V$. Then we can write $\tau_n = \tau_1^n$ for every $n \in \mathbb{Z}$. By the proof of Lemma 3.3, it follows that $(\tau_1|_E)^n = (\tau_1|_E\sigma)^n\sigma^{-n}$ for every $n \in \mathbb{Z}$. If $n = 2$, then $\tau_1|_E\sigma = \sigma\tau_1|_E$, so $\Phi(\theta) = \tau_1 \in Z(E,\sigma)$.

Conversely, let $\tau \in Z(E,\sigma)$. Since $\tau|_E\sigma = \sigma\tau|_E$, $(\tau|_E\sigma)^n\sigma^{-n} = (\tau|_E)^n$ for every $n \in \mathbb{Z}$. By [2, Remark 15], there exists $\theta = \{\theta_1^n\}_{n\in\mathbb{Z}} \in \mathrm{TS}_{\mathrm{alg}}^{\mathbb{Z}}(A)$ such that $\overline{(\theta_1|_V)^*}^{\,n} = \tau^n$ for every $n \in \mathbb{Z}$. Hence, it follows that $\Phi(\theta) = \tau$.

(2) By the following commutative diagram



it follows that $\mathrm{PGrAut}_k\,A \cong \Phi(\mathrm{TS}_{\mathrm{alg}}^{\mathbb{Z}}(A)) = Z(E,\sigma)^{\mathrm{op}}$. ■

Theorem 3.5 *Let $A = \mathcal{A}(E,\sigma)$ be a geometric algebra.*

(1) $\mathrm{Twist}(A) = \{\mathcal{A}(E,\tau|_E\sigma) \mid \tau \in M(E,\sigma)\}/_{\cong}$.
(2) $\mathrm{Twist}_{\mathrm{alg}}(A) = \{\mathcal{A}(E,\tau|_E\sigma) \mid \tau \in Z(E,\sigma)\}/_{\cong}$.

Proof By [2, Proposition 13], for every $\theta \in \mathrm{TS}_0^{\mathbb{Z}}(A)$, $A^\theta \cong \mathcal{A}(E,\Phi(\theta)|_E\sigma)$. By Lemma 3.3, it follows that

$$\begin{aligned}\mathrm{Twist}(A) &:= \{A^\theta \mid \theta \in \mathrm{TS}_0^{\mathbb{Z}}(A)\}/_{\cong}\\ &= \{\mathcal{A}(E,\Phi(\theta)|_E\sigma) \mid \theta \in \mathrm{TS}_0^{\mathbb{Z}}(A)\}/_{\cong}\end{aligned}$$

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$$= \{\mathcal{A}(E, \tau|_E \sigma) \mid \tau \in \Phi(\mathrm{TS}_0^{\mathbb{Z}}(A))\}/_{\cong}$$
$$= \{\mathcal{A}(E, \tau|_E \sigma) \mid \tau \in M(E, \sigma)\}/_{\cong}.$$

By Lemma 3.4, we can similarly show that

$$\mathrm{Twist}_{\mathrm{alg}}(A) = \{\mathcal{A}(E, \tau|_E \sigma) \mid \tau \in Z(E, \sigma)\}/_{\cong}. \qquad \blacksquare$$

4 Twisted algebras of three-dimensional geometric AS-regular algebras

In this section, we classify twisted algebras of three-dimensional geometric AS-regular algebras. We recall that for connected graded algebras A and A' generated in degree 1, $\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, A'$ if and only if $A' \in \mathrm{Twist}(A)$, so

$$\mathrm{Twist}(A) = \{A' \mid \mathrm{GrMod}\, A' \cong \mathrm{GrMod}\, A\}/_{\cong}.$$

By Lemma 2.10, we may assume that A is a three-dimensional geometric Calabi–Yau AS-regular algebra in Table 1 to compute $\mathrm{Twist}(A)$. The algebras in Table 1 are called *standard* in this paper. For three-dimensional standard AS-regular algebras, we will compute the subsets $Z(E, \sigma)$ and $M(E, \sigma)$ of $\mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$. We remark that some of the computations were given in [2, Section 4].

For a three-dimensional geometric AS-regular algebra $\mathcal{A}(E, \sigma)$, the map

$$\mathrm{Aut}_k(\mathbb{P}^2 \downarrow E) \to \mathrm{Aut}_k(E \uparrow \mathbb{P}^2); \tau \mapsto \tau|_E$$

is a bijection, so we identify $\tau \in \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$ with $\tau|_E \in \mathrm{Aut}_k(E \uparrow \mathbb{P}^2)$ if there is no potential confusion.

Let E be an elliptic curve in $\mathbb{P}^2$. We use a *Hesse form*

$$E = \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz),$$

where $\lambda \in k$ with $\lambda^3 \neq 1$. It is known that every elliptic curve in $\mathbb{P}^2$ can be written in this form (see [3, Corollary 2.18]). The j-invariant of a Hesse form E is given by $j(E) = \frac{27\lambda^3(\lambda^3+8)^3}{(\lambda^3-1)}$ (see [3, Proposition 2.16]). The j-invariant $j(E)$ classifies elliptic curves in $\mathbb{P}^2$ up to projective equivalence (see [5, Theorem IV.4.1(b)]). We fix the group structure on E with the zero element $o := (1, -1, 0) \in E$ (see [3, Theorem 2.11]). For a point $p \in E$, a *translation* by p, denoted by σ_p, is an automorphism of E defined by $\sigma_p(q) = p + q$ for every $q \in E$. We define $\mathrm{Aut}_k(E, o) := \{\sigma \in \mathrm{Aut}_k E \mid \sigma(o) = o\}$. It is known that $\mathrm{Aut}_k(E, o)$ is a finite cyclic subgroup of $\mathrm{Aut}_k E$ (see [5, Corollary IV.4.7]).

Lemma 4.1 [7, Theorem 4.6] *A generator of* $\mathrm{Aut}_k(E, o)$ *is given by:*

(1) $\tau_E(a, b, c) := (b, a, c)$ *if* $j(E) \neq 0, 12^3$,
(2) $\tau_E(a, b, c) := (b, a, \varepsilon c)$ *if* $\lambda = 0$ *(so that* $j(E) = 0$*),*
(3) $\tau_E(a, b, c) := (\varepsilon^2 a + \varepsilon b + c, \varepsilon a + \varepsilon^2 b + c, a + b + c)$ *if* $\lambda = 1 + \sqrt{3}$ *(so that* $j(E) = 12^3$*),*

where ε *is a primitive third root of unity. In particular,* $\mathrm{Aut}_k(E, o)$ *is a subgroup of* $\mathrm{Aut}_k(E \uparrow \mathbb{P}^2) = \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$.

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Table 2: The list of $Z(E)$ and $G(E)$.

Type	$Z(E)$	$G(E)$
P	$\mathrm{PGL}_3(k)$	$\{\mathrm{id}\}$
S	$\left\{\begin{pmatrix}1&0&0\\0&e&0\\0&0&i\end{pmatrix}\middle\vert ei\neq 0\right\}\rtimes\left\langle\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\right\rangle$	$\left\langle\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\right\rangle$
S'	$\left\{\begin{pmatrix}1&0&0\\0&e&0\\0&0&e^{-1}\end{pmatrix}\middle\vert e\neq 0\right\}$	$\left\langle\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\rangle$
T	$\left\{\begin{pmatrix}1&0&0\\0&e&0\\g&h&e^2\end{pmatrix}\middle\vert e^3=1\right\}\rtimes\left\langle\begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}\right\rangle$	$\left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&i\end{pmatrix}\middle\vert i\neq 0\right\}$
T'	$\left\{\begin{pmatrix}1&0&0\\d&1&0\\d^2&2d&1\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&0&0\\0&e&0\\0&0&e^2\end{pmatrix}\middle\vert e\neq 0\right\}$
NC	$\left\langle\begin{pmatrix}1&0&0\\0&\varepsilon&0\\0&0&\varepsilon^2\end{pmatrix}\right\rangle$	$\left\langle\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\right\rangle$
CC	$\{\mathrm{id}\}$	$\left\{\begin{pmatrix}1&0&0\\0&e&0\\0&0&e^2\end{pmatrix}\middle\vert e\neq 0\right\}$
EC	$T[3]$	$\mathrm{Aut}_k(E,o)$

Remark 4.2 When $j(E)=0,12^3$, we may fix $\lambda=0,1+\sqrt{3}$, respectively, as in Lemma 4.1, because if two elliptic curves E and E' in $\mathbb{P}^2$ are projectively equivalent, then for every $\mathcal{A}(E,\sigma)$, there exists an automorphism $\sigma'\in\mathrm{Aut}_k E'$ such that $\mathcal{A}(E,\sigma)\cong\mathcal{A}(E',\sigma')$ (see [12, Lemma 2.6]).

It follows from [7, Proposition 4.5] that every automorphism $\sigma\in\mathrm{Aut}_k E$ can be written as $\sigma=\sigma_p\tau_E^i$ where σ_p is a translation by a point $p\in E$, τ_E is a generator of $\mathrm{Aut}_k(E,o)$, and $i\in\mathbb{Z}_{|\tau_E|}$. For any $n\geq 1$, we call a point $p\in E$ *n-torsion* if $np=o$. We set $E[n]:=\{p\in E\mid np=o\}$ and $T[n]:=\{\sigma_p\in\mathrm{Aut}_k E\mid p\in E[n]\}$.

If $A=\mathcal{A}(E,\sigma)$ is a three-dimensional standard AS-regular algebra, we write

$$\mathrm{Aut}_k(\mathbb{P}^2\downarrow E)=Z(E)\rtimes G(E)$$

as in Table 2 where $Z(E)\leq\mathrm{Aut}_k(\mathbb{P}^2\downarrow E)$ with $Z(E)\subset Z(E,\sigma)$ and $G(E)\leq\mathrm{Aut}_k(\mathbb{P}^2\downarrow E)$ so that

$$N(E,\sigma)=Z(E)\rtimes(G(E)\cap N(E,\sigma)).$$

Table 2 can be checked by the following three steps:

- Step 1: Calculate $\mathrm{Aut}_k(\mathbb{P}^2\downarrow E)$.

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- Step 2: Find $Z(E) \leq \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$ with $Z(E) \subset Z(E,\sigma) \cong (\mathrm{PGrAut}_k A)^{\mathrm{op}}$ (see Lemma 3.4(2)).
- Step 3: Find $G(E) \leq \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$.

$\mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$ and $\mathrm{PGrAut}_k A$ were computed in [13]. We explain these steps for Type S. By Lemma 2.10, $E = \mathcal{V}(x) \cup \mathcal{V}(y) \cup \mathcal{V}(z)$ and

$$\begin{cases} \sigma(0,b,c) = (0,b,\alpha c), \\ \sigma(a,0,c) = (\alpha a,0,c), \\ \sigma(a,b,0) = (a,\alpha b,0), \end{cases}$$

where $\alpha^3 \neq 0,1$. By [13, Lemma 3.2.1],

$$\mathrm{Aut}_k(\mathbb{P}^2 \downarrow E) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{pmatrix} \middle| ei \neq 0 \right\} \rtimes \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

By [13, Theorem 3.3.1],

$$\begin{aligned} Z(E,\sigma) &= (\mathrm{PGrAut}_k A)^{\mathrm{op}} \\ &= \begin{cases} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{pmatrix} \middle| ei \neq 0 \right\} \rtimes \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle, & \text{if } \sigma^2 \neq \mathrm{id}, \\ \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E), & \text{if } \sigma^2 = \mathrm{id}, \end{cases} \end{aligned}$$

so we may take

$$Z(E) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{pmatrix} \middle| ei \neq 0 \right\} \rtimes \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle \text{ and } G(E) = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Remark 4.3 By Table 2:

(1) $|G(E)| < \infty$ if and only if A is of Types P, S, S', NC, and EC, and, in this case, there exists $\tau_E \in \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$ such that $G(E) = \langle \tau_E \rangle$ is a finite cyclic group.

(2) $|G(E)| < \infty$ but $|G(E)| \neq 2$ if and only if A is of Type P ($|G(E)| = 1$), or Type EC with $j(E) = 0$ ($|G(E) = 6|$), or Type EC with $j(E) = 12^3$ ($|G(E)| = 4$).

Theorem 4.4 *If $A = \mathcal{A}(E,\sigma)$ is a three-dimensional quadratic AS-regular algebra of Types T, T', and CC (so that $|\sigma| = \infty$ (cf. [9])), then $Z(E,\sigma) = M(E,\sigma) = N(E,\sigma)$.*

Proof Writing $\mathrm{Aut}_k(\mathbb{P}^2 \downarrow E) = Z(E) \rtimes G(E)$ as in Table 2, it is enough to show that $G(E) \cap N(E,\sigma) = \{\mathrm{id}\}$.

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Type T: For every $\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix} \in G(E)$,

$$\begin{cases} \sigma\tau|_E\sigma^{-1}(a,-a,c) = (a,-a,(1-i)a+ic), \\ \sigma\tau|_E\sigma^{-1}(a,-\varepsilon a,c) = (a,-\varepsilon a,\varepsilon^2(1-i)a+ic), \\ \sigma\tau|_E\sigma^{-1}(a,-\varepsilon^2 a,c) = (a,-\varepsilon^2 a,\varepsilon(1-i)a+ic). \end{cases}$$

If $\tau \in N(E,\sigma)$, then there exists $\overline{\tau} \in \operatorname{Aut}_k \mathbb{P}^2$ such that $\sigma\tau|_E\sigma^{-1} = \overline{\tau}|_E$. Then $1-i = \varepsilon^2(1-i) = \varepsilon(1-i)$, so $i=1$.

Type T′: For every $\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e^2 \end{pmatrix} \in G(E)$,

$$\begin{cases} \sigma\tau|_E\sigma^{-1}(0,b,c) = (0,eb,e(1-e)b+e^2c), \\ \sigma\tau|_E\sigma^{-1}(a,b,c) = (a,(e-1)a+eb,(e-1)^2a+2e(e-1)b+e^2c). \end{cases}$$

If $\tau \in N(E,\sigma)$, then there exists $\overline{\tau} \in \operatorname{Aut}_k \mathbb{P}^2$ such that $\sigma\tau|_E\sigma^{-1} = \overline{\tau}|_E$. Then $e(1-e) = 2e(e-1)$, so $e=1$.

Type CC: For every $\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e^2 \end{pmatrix} \in G(E) = \operatorname{Aut}_k(\mathbb{P}^2 \downarrow E)$,

$$\begin{aligned} &\sigma\tau|_E\sigma^{-1}(a,b,c) \\ &= \left(a+(1-e)b,\, eb,\, -3\frac{(a+b)^2}{eb} + 3(a+b) - eb + e^{-2}\left(3\frac{a^2}{b} + 3a + b + c\right)\right). \end{aligned}$$

If $\tau \in N(E,\sigma)$, then there exists $\overline{\tau} \in \operatorname{Aut}_k \mathbb{P}^2$ such that $\sigma\tau|_E\sigma^{-1} = \overline{\tau}|_E$. Then there exists $0 \neq e' \in k$ such that

$$(1+(1-e)b, eb) = (1, e'b) \text{ in } \mathbb{P}^1$$

for every $(1,b,c) \in E$, so $e' = e = 1$. ∎

Lemma 4.5 *Let $A = \mathcal{A}(E,\sigma)$ be a three-dimensional standard AS-regular algebra of Type S, S', NC, or EC. For every $i \geq 1$, $\sigma^i \in \operatorname{Aut}_k(E \uparrow \mathbb{P}^2) = \operatorname{Aut}_k(\mathbb{P}^2 \downarrow E)$ if and only if $\sigma^{3i} = \mathrm{id}$.*

Proof This lemma follows from [9, Theorem 3.4]. ∎

Lemma 4.6 *Let $A = \mathcal{A}(E,\sigma)$ be a three-dimensional standard AS-regular algebra. If $\sigma\tau\sigma^{-1}, \sigma^{-1}\tau\sigma \in \operatorname{Aut}_k(E \uparrow \mathbb{P}^2)$ for every $\tau \in G(E)$, then $M(E,\sigma) = N(E,\sigma) = \operatorname{Aut}_k(\mathbb{P}^2 \downarrow E)$.*

Proof Every $\tau \in \operatorname{Aut}_k(\mathbb{P}^2 \downarrow E)$ can be written as $\tau = \tau_1\tau_2$ where $\tau_1 \in Z(E)$, $\tau_2 \in G(E)$. Since $\sigma\tau\sigma^{-1} = \sigma\tau_1\tau_2\sigma^{-1} = \tau_1\sigma\tau_2\sigma^{-1}$, it holds that $\sigma\tau\sigma^{-1} \in \operatorname{Aut}_k(E \uparrow \mathbb{P}^2)$. Similarly, every $\tau \in \operatorname{Aut}_k(\mathbb{P}^2 \downarrow E)$ can be written as $\tau = \tau_2\tau_1$ where $\tau_1 \in Z(E)$, $\tau_2 \in G(E)$,

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so $\sigma^{-1}\tau\sigma = \sigma^{-1}\tau_2\tau_1\sigma = \sigma^{-1}\tau_2\sigma\tau_1$, and hence $\sigma^{-1}\tau\sigma \in \mathrm{Aut}_k(E \uparrow \mathbb{P}^2)$. The result now follows from Lemma 3.2. ■

Theorem 4.7 *Let $A = \mathcal{A}(E,\sigma)$ be a three-dimensional standard AS-regular algebra such that $|G(E)| = 2$.*

(1) *If $\sigma^2 = \mathrm{id}$, then $Z(E,\sigma) = M(E,\sigma) = N(E,\sigma) = \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$.*
(2) *If $\sigma^6 = \mathrm{id}$, then $M(E,\sigma) = N(E,\sigma) = \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$.*
(3) *If $\sigma^6 \neq \mathrm{id}$, then $Z(E) = Z(E,\sigma) = M(E,\sigma) = N(E,\sigma)$.*

Proof (1) We will give a proof for Type S′. The other types are proved similarly. By Lemma 2.10 and Table 2, $E = \mathcal{V}(x) \cup \mathcal{V}(x^2 - \lambda yz)$,

$$\begin{cases} \sigma(0,b,c) = (0,b,\alpha c), \\ \sigma(a,b,c) = (a,\alpha b,\alpha^{-1}c), \end{cases}$$

where $\lambda = \frac{\alpha^3-1}{\alpha}$ and $\alpha^3 \neq 0,1$, and $\tau_E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. In general, $Z(E) \subset Z(E,\sigma) \subset M(E,\sigma) \subset N(E,\sigma) \subset \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$. In this case, $\sigma^2 = \mathrm{id}$ if and only if $\alpha^2 = 1$. Since

$$\begin{cases} \sigma\tau_E\sigma^{-1}(0,b,c) = (0,c,\alpha^2 b), \\ \sigma\tau_E\sigma^{-1}(a,b,c) = (a,\alpha^2 c,\alpha^{-2}b), \end{cases}$$

if $\sigma^2 = \mathrm{id}$, then $\sigma\tau_E\sigma^{-1} = \tau_E$. Since $\tau_E \in Z(E,\sigma)$, $Z(E,\sigma) = \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$.

(2) By direct calculations, $(\tau_E\sigma)^2 = \mathrm{id}$, so $\sigma\tau_E\sigma^{-1}\tau_E^{-1} = \sigma^2 = \tau_E^{-1}\sigma^{-1}\tau_E\sigma$. By Lemma 4.5, $\sigma\tau_E\sigma^{-1}\tau_E^{-1}, \tau_E^{-1}\sigma^{-1}\tau_E\sigma \in \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$ if and only if $\sigma^6 = \mathrm{id}$. In particular, if $\sigma^6 = \mathrm{id}$, then $\sigma\tau_E\sigma^{-1}, \sigma^{-1}\tau_E\sigma \in \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$. By Lemma 4.6, $M(E,\sigma) = N(E,\sigma) = \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$, and hence (2) holds.

(3) If $\sigma^6 \neq \mathrm{id}$, then $\tau_E \notin N(E,\sigma)$. Since $G(E) \cap N(E,\sigma) = \{\mathrm{id}\}$, $N(E,\sigma) = Z(E)$, and hence (3) holds. ■

Theorem 4.8 *Let $A = \mathcal{A}(E,\sigma)$ be a three-dimensional standard AS-regular algebra except for Type EC. Then Table 3 gives $Z(E,\sigma)$ and $M(E,\sigma)$ for each type.*

Proof By Theorems 4.4 and 4.7, the result holds. ■

Definition 4.9 Let $E = \mathcal{V}(x^3 + y^3 + z^3) \subset \mathbb{P}^2$ so that $j(E) = 0$, and define

$$\mathcal{E} := \{(a,b,c) \in E \mid a^9 = b^9 = c^9\} \subset E[9] \backslash E[6].$$

In this paper, we say that a three-dimensional quadratic AS-regular algebra is *exceptional* if it is graded Morita equivalent to $\mathcal{A}(E,\sigma_p)$ for some $p \in \mathcal{E}$.

Lemma 4.10 *Let $A = \mathcal{A}(E,\sigma_p)$ be a three-dimensional standard AS-regular algebra of Type EC and $\sigma_q\tau_E^i \in \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$ where $q \in E[3]$, $i \in \mathbb{Z}_{|\tau_E|}$. Then:*

(1) *$\sigma_q\tau_E^i \in Z(E,\sigma_p)$ if and only if $p - \tau_E^i(p) = o$,*

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Table 3: $Z(E,\sigma)$ and $M(E,\sigma)$ except for Type EC.

Type	$Z(E,\sigma)$	$M(E,\sigma)$
P	$\mathrm{PGL}_3(k)$	$\mathrm{PGL}_3(k)$
T	$\left\{\begin{pmatrix}1&0&0\\0&e&0\\g&h&1\end{pmatrix} \middle\vert\, e^3=1\right\} \rtimes \left\langle \begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}\right\rangle$	$\left\{\begin{pmatrix}1&0&0\\0&e&0\\g&h&1\end{pmatrix} \middle\vert\, e^3=1\right\} \rtimes \left\langle \begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}\right\rangle$
T′	$\left\{\begin{pmatrix}1&0&0\\d&1&0\\d^2&2d&1\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&0&0\\d&1&0\\d^2&2d&1\end{pmatrix}\right\}$
CC	$\left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\}$
S	$\begin{cases} D \rtimes \left\langle \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\right\rangle \text{ if } \sigma^2 \neq \mathrm{id}, \\ \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E) \text{ if } \sigma^2 = \mathrm{id}. \end{cases}$	$\begin{cases} D \rtimes \left\langle \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\right\rangle \text{ if } \sigma^6 \neq \mathrm{id}, \\ \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E) \text{ if } \sigma^6 = \mathrm{id}. \end{cases}$
S′	$\begin{cases} \left\{\begin{pmatrix}1&0&0\\0&e&0\\0&0&e^{-1}\end{pmatrix} \middle\vert\, e \neq 0\right\} & \text{if } \sigma^2 \neq \mathrm{id} \\ \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E) & \text{if } \sigma^2 = \mathrm{id}. \end{cases}$	$\begin{cases} \left\{\begin{pmatrix}1&0&0\\0&e&0\\0&0&e^{-1}\end{pmatrix} \middle\vert\, e \neq 0\right\} & \text{if } \sigma^6 \neq \mathrm{id} \\ \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E) & \text{if } \sigma^6 = \mathrm{id}. \end{cases}$
NC	$\begin{cases} \left\langle\begin{pmatrix}1&0&0\\0&\varepsilon&0\\0&0&\varepsilon^2\end{pmatrix}\right\rangle & \text{if } \sigma^2 \neq \mathrm{id} \\ \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E) & \text{if } \sigma^2 = \mathrm{id} \end{cases}$	$\begin{cases} \left\langle\begin{pmatrix}1&0&0\\0&\varepsilon&0\\0&0&\varepsilon^2\end{pmatrix}\right\rangle & \text{if } \sigma^6 \neq \mathrm{id} \\ \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E) & \text{if } \sigma^6 = \mathrm{id} \end{cases}$
where $D := \left\{\begin{pmatrix}1&0&0\\0&e&0\\0&0&i\end{pmatrix} \middle\vert\, ei \neq 0\right\}.$		

(2) $\sigma_q\tau_E^i \in N(E,\sigma_p)$ *if and only if* $p - \tau_E^i(p) \in E[3]$, *and*
(3) $M(E,\sigma_p) = N(E,\sigma_p)$.

Proof (1) Since $\sigma_p(\sigma_q\tau_E^i)\sigma_p^{-1} = \sigma_{q+p-\tau_E^i(p)}\tau_E^i$, $\sigma_q\tau_E^i \in Z(E,\sigma_p)$ if and only if $p - \tau_E^i(p) = o$.

(2) Since $\sigma_p(\sigma_q\tau_E^i)\sigma_p^{-1} = \sigma_{q+p-\tau_E^i(p)}\tau_E^i$, by [11, Lemma 5.3], $\sigma_q\tau_E^i \in N(E,\sigma_p)$ if and only if $p - \tau_E^i(p) \in E[3]$.

(3) In general, $M(E,\sigma_p) \subset N(E,\sigma_p)$, so it is enough to show that $N(E,\sigma_p) \subset M(E,\sigma_p)$. Let $\sigma_q\tau_E^i \in N(E,\sigma_p) \subset \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$ where $q \in E[3]$ and $i \in \mathbb{Z}_{|\tau_E|}$. Since $\sigma_p(\sigma_q\tau_E^i)\sigma_p^{-1} = \sigma_{q+p-\tau_E^i(p)}\tau_E^i \in \mathrm{Aut}_k(\mathbb{P}^2 \downarrow E)$, $p - \tau_E^i(p) \in E[3]$. For any $j \geq 1$, we can

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Table 4: $Z(E,\sigma)$ and $M(E,\sigma)$ for Type EC.

Type	$j(E)$	$Z(E,\sigma_p)$	$M(E,\sigma_p)$
EC	$j(E)\neq 0, 12^3$	$\begin{cases} T[3], & \text{if } p\notin E[2] \\ \mathrm{Aut}_k(\mathbb{P}^2\downarrow E), & \text{if } p\in E[2] \end{cases}$	$\begin{cases} T[3], & \text{if } p\notin E[6] \\ \mathrm{Aut}_k(\mathbb{P}^2\downarrow E), & \text{if } p\in E[6] \end{cases}$
	$j(E)=0$	$\begin{cases} T[3], & \text{if } p\notin E[2] \\ T[3]\rtimes\langle\tau_E^3\rangle, & \text{if } p\in E[2] \end{cases}$	$\begin{cases} T[3], & \text{if } p\notin \mathcal{E}\cup E[6] \\ T[3]\rtimes\langle\tau_E^2\rangle, & \text{if } p\in \mathcal{E} \\ T[3]\rtimes\langle\tau_E^3\rangle, & \text{if } p\in E[6] \end{cases}$
	$j(E)=12^3$	$\begin{cases} T[3], & \text{if } p\notin E[2] \\ T[3]\rtimes\langle\tau_E^2\rangle, & \text{if } p\in E[2]\backslash\langle(1,1,\lambda)\rangle \\ \mathrm{Aut}_k(\mathbb{P}^2\downarrow E), & \text{if } p=(1,1,\lambda) \end{cases}$	$\begin{cases} T[3], & \text{if } p\notin E[6] \\ T[3]\rtimes\langle\tau_E^2\rangle, & \text{if } p\in E[6]\backslash\mathcal{F} \\ \mathrm{Aut}_k(\mathbb{P}^2\downarrow E), & \text{if } p\in\mathcal{F} \end{cases}$
where $\mathcal{F}:=\langle(1,1,\lambda)\rangle\oplus E[3]$.			

write

$$(\sigma_q\tau_E^i\sigma_p)^j(\sigma_p)^{-j}=\sigma_{r_j}\tau_E^{ji},$$

where $r_j=\sum_{l=0}^{j-1}\tau_E^{li}(q)+\sum_{l=1}^{j}\tau_E^{li}(p-\tau_E^{(j-l)i}(p))$, and

$$(\sigma_q\tau_E^i\sigma_p)^{-j}(\sigma_p)^j=\sigma_{s_j}\tau_E^{-ji},$$

where $s_j=\sum_{l=1}^{j}(-\tau_E^{-li}(q))+\sum_{l=0}^{j-1}\tau_E^{-li}(p-\tau_E^{(l-j)i}(p))$. By [7, Lemma 4.19], for any $j\geq 1$, $r_j,s_j\in E[3]$, so

$$(\sigma_q\tau_E^i\sigma_p)^j(\sigma_p)^{-j},(\sigma_q\tau_E^i\sigma_p)^{-j}(\sigma_p)^j\in\mathrm{Aut}_k(\mathbb{P}^2\downarrow E),$$

and hence (3) holds. ∎

Theorem 4.11 *Let $A=\mathcal{A}(E,\sigma_p)$ be a three-dimensional standard AS-regular algebra of Type EC. Then Table 4 gives $Z(E,\sigma_p)$ and $M(E,\sigma_p)$.*

Proof By Lemma 4.10(3), it is enough to calculate $Z(E,\sigma_p)$ and $N(E,\sigma_p)$. By Lemma 3.4(2), $Z(E,\sigma_p)$ was given in [10, Proposition 4.7]. The set of points satisfying $p-\tau_E^i(p)\in E[3]$ was given in [10, Theorem 3.8]. By Lemma 4.10(1) and (2), the result follows. ∎

Corollary 4.12 shows that in most cases a twisting system can be replaced by an automorphism to compute a twisted algebra.

Corollary 4.12 *Let $A=\mathcal{A}(E,\sigma)$ be a three-dimensional nonexceptional standard AS-regular algebra. If $\sigma^6\neq\mathrm{id}$ or $\sigma^2=\mathrm{id}$, then $Z(E,\sigma)=M(E,\sigma)$, so $\mathrm{Twist}_{\mathrm{alg}}(A)=\mathrm{Twist}(A)$.*

Proof By Theorems 4.4 and 4.7, it is enough to show the case that $A=\mathcal{A}(E,\sigma_p)$ is of Type EC such that $j(E)=0$, $p\notin\mathcal{E}$, or $j(E)=12^3$.

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(1) $j(E) = 0$, $p \notin \mathcal{E}$: Let $E = \mathcal{V}(x^3 + y^3 + z^3) \subset \mathbb{P}^2$. By Theorem 4.11, if $p \in E[2]$ or $p \notin E[6]$, then $N(E, \sigma_p) = M(E, \sigma_p) = Z(E, \sigma_p)$.

(2) $j(E) = 12^3$: Let $E = \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz) \subset \mathbb{P}^2$ where $\lambda = 1 + \sqrt{3}$. By Theorem 4.11, if $p \in E[2]$ or $p \notin E[6]$, then $N(E, \sigma_p) = M(E, \sigma_p) = Z(E, \sigma_p)$. ■

Let $E \subset \mathbb{P}^2$ be a projective variety. For $\tau \in \operatorname{Aut}_k E$, we define

$$||\tau|| := \inf\{i \in \mathbb{N}^+ \mid \tau^i \in \operatorname{Aut}_k(E \uparrow \mathbb{P}^2)\}.$$

Corollary 4.13 *For every three-dimensional nonexceptional geometric AS-regular algebra B, there exists a three-dimensional standard AS-regular algebra S such that* $\operatorname{Twist}(B) = \operatorname{Twist}_{\mathrm{alg}}(S)$.

Proof By Lemma 2.10, there exists a three-dimensional nonexceptional standard AS-regular algebra $A = \mathcal{A}(E, \sigma)$ such that $\operatorname{GrMod} B \cong \operatorname{GrMod} A$. If $\sigma^6 \neq \mathrm{id}$, then $\operatorname{Twist}(B) = \operatorname{Twist}(A) = \operatorname{Twist}_{\mathrm{alg}}(A)$ by Corollary 4.12, so we assume that $\sigma^6 = \mathrm{id}$. Set $\tau := \sigma^2 \in \operatorname{Aut}_k E$ and $S := \mathcal{A}(E, \sigma^3)$. Since $||\tau|| = ||\sigma^2|| = |\sigma^6| = 1$ by [9, Theorem 3.4], $\tau \in \operatorname{Aut}_k(E \uparrow \mathbb{P}^2)$. Since $\tau^{i+1}\sigma = \sigma^{2i+3} = \sigma^3\tau^i$ for every $i \in \mathbb{Z}$, $\operatorname{GrMod} A \cong \operatorname{GrMod} S$ by Theorem 2.6. Since $(\sigma^3)^2 = \mathrm{id}$, $\operatorname{Twist}(B) = \operatorname{Twist}(A) = \operatorname{Twist}(S) = \operatorname{Twist}_{\mathrm{alg}}(S)$ by Corollary 4.12. ■

Example 4.14 shows that even if $B \cong S^\theta$ for some three-dimensional quadratic Calabi–Yau AS-regular algebra S, there may be no $\phi \in \operatorname{GrAut}_k S$ such that $B \cong S^\phi$. We need to carefully choose S in order that $B \cong S^\phi$ for some $\phi \in \operatorname{GrAut}_k S$.

Example 4.14 Let $E \subset \mathbb{P}^2$ be an elliptic curve. Assume that $j(E) \neq 0, 12^3$. We set three geometric algebras of Type EC; $B := \mathcal{A}(E, \tau_E\sigma_p)$, $A := \mathcal{A}(E, \sigma_p)$, and $S := \mathcal{A}(E, \sigma_{3p})$, where $p \in E[6] \backslash (E[2] \cup E[3])$. By [8, Theorem 4.3], these algebras are three-dimensional quadratic AS-regular algebras. Moreover, A and S are standard. By [7, Theorem 4.20], $\operatorname{GrMod} B \cong \operatorname{GrMod} A \cong \operatorname{GrMod} S$. Since $|\sigma_p| = 6$ and $|\sigma_{3p}| = 2$, $M(E, \sigma_{3p}) = Z(E, \sigma_{3p}) \neq Z(E, \sigma_p)$ by Table 4, so $\operatorname{Twist}(B) = \operatorname{Twist}_{\mathrm{alg}}(S) \neq \operatorname{Twist}_{\mathrm{alg}}(A)$.

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