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Abstract

We study Cox rings of K3 surfaces. A first result is that a K3 surface has a finitely generated Cox ring if and only if its effective cone is rational polyhedral. Moreover, we investigate degrees of generators and relations for Cox rings of K3 surfaces of Picard number two, and explicitly compute the Cox rings of generic K3 surfaces with a non-symplectic involution that have Picard number 2 to 5 or occur as double covers of del Pezzo surfaces.

1. Introduction

The Cox ring $\mathcal{R}(X)$ of a normal complete algebraic variety X with a finitely generated divisor class group $\mathrm{Cl}(X)$ is the multigraded algebra

$$\mathcal{R}(X) := \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

For a toric variety X, the Cox ring $\mathcal{R}(X)$ is a polynomial ring and its multigrading can be explicitly determined in terms of a defining fan; see [Cox95]. Moreover, for del Pezzo surfaces X there are recent results on generators and relations of the Cox ring; see [BP04, Der, LV09, STV07, SX, TVV]. In the present paper, we investigate Cox rings of K3 surfaces X, i.e. smooth complex projective surfaces X with $b_1(X) = 0$ that admit a nowhere vanishing holomorphic 2-form ω_X.

A first basic problem is to decide if the Cox ring $\mathcal{R}(X)$ is finitely generated. In § 2, we first discuss this question in general, and extend results of Hu and Keel on $\mathbb{Q}$-factorial projective varieties [HK00] to normal complete ones; a consequence is that every normal complete surface with finitely generated Cox ring is $\mathbb{Q}$-factorial and projective; see Theorem 2.5. For K3 surfaces, we obtain the following; see Theorem 2.7.

Theorem 1. *A K3 surface has finitely generated Cox ring if and only if its cone of effective rational divisor classes is polyhedral.*

The same characterization holds for Enriques surfaces; see Theorem 2.10. The second basic problem is to describe the Cox ring $\mathcal{R}(X)$ in terms of generators and relations. We first consider K3 surfaces X having Picard number $\varrho(X) = 2$; see § 3. In this setting, if the effective cone is polyhedral, then it is known that its generators are of self intersection zero or minus two; see § 2 for this and some more background. For the case that both generators are of self intersection zero, we obtain the following; see Proposition 3.1 and Theorem 3.2.

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Theorem 2. *Let X be a K3 surface with $\mathrm{Cl}(X) \cong \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$, where w_1, w_2 are effective, and intersection form given by $w_1^2 = w_2^2 = 0$ and $w_1 \cdot w_2 = k \geqslant 3$.*

(i) *The effective cone of X is generated by w_1 and w_2 and it coincides with the semiample cone of X.*

(ii) *The Cox ring $\mathcal{R}(X)$ is generated in degrees w_1, w_2, $w_1 + w_2$, and one has*

$$\dim(\mathcal{R}(X)_{w_i}) = 2, \quad \dim(\mathcal{R}(X)_{w_1+w_2}) = k + 2.$$

Moreover, every minimal system of generators of $\mathcal{R}(X)$ has $k + 2$ members.

(iii) *For $k = 3$, the Cox ring $\mathcal{R}(X)$ is of the form $\mathbb{C}[T_1, \ldots, T_5]/\langle f \rangle$ and the degrees of the generators and the relation are given by*

$$\deg(T_1) = \deg(T_2) = w_1, \quad \deg(T_4) = \deg(T_5) = w_2,$$
$$\deg(T_3) = w_1 + w_2, \quad \deg(f) = 3w_1 + 3w_2.$$

(iv) *For $k \geqslant 4$, any minimal ideal $\mathcal{I}(X)$ of relations of $\mathcal{R}(X)$ is generated in degree $2w_1 + 2w_2$, and we have*

$$\dim(\mathcal{I}(X)_{2w_1+2w_2}) = \frac{k(k-3)}{2}.$$

The statements on the generators are directly obtained and, for the relations, we use the techniques developed in [LV09]. When at least one of the generators of the effective cone is a (-2)-curve, then the semiample cone is a proper subset of the effective cone. We show that in this case the number of degrees needed to generate the Cox ring can be arbitrarily big; Propositions 3.6 and 3.7 give a lower bound for this number in terms of the intersection form of $\mathrm{Cl}(X)$.

For the K3 surfaces X with Picard number $\varrho(X) \geqslant 3$, we use a different approach. Many K3 surfaces X with $\varrho(X) \geqslant 3$ and polyhedral effective cone admit a non-symplectic involution, i.e. an automorphism $\sigma\colon X \to X$ of order two with $\sigma^*\omega_X \neq \omega_X$. The associated quotient map $\pi\colon X \to Y$ is a double cover. If it is unramified then $Y := X/\langle\sigma\rangle$ is an Enriques surface, otherwise Y is a smooth rational surface. In the latter case, one may use known results and techniques to obtain the Cox ring of Y.

This observation suggests studying the behavior of Cox rings under double coverings $\pi\colon X \to Y$. As it may be of independent interest, we consider in § 4 more general, e.g. cyclic, coverings $\pi\colon X \to Y$ of arbitrary normal varieties X and Y. We relate finite generation of the Cox rings of X and Y to each other; Propositions 4.6 and 4.3 provide generators and relations for the Cox ring of X in terms of π and the Cox ring of Y for the case that π induces an isomorphism on the level of divisor class groups. This enables us to compute Cox rings of K3 surfaces that are general double covers of $\mathbb{F}_0$ or of del Pezzo surfaces.

Besides $\mathbb{F}_0$ and the del Pezzo surfaces, other rational surfaces $Y = X/\langle\sigma\rangle$ can occur. For $2 \leqslant \varrho(X) \leqslant 5$, these turn out to be blow ups of the fourth Hirzebruch surface $\mathbb{F}_4$ in at most three general points, and we are in this setting if and only if the branch divisor of the covering $\pi\colon X \to Y$ has two components. Then, in order to determine the Cox ring of X, we have to solve two problems. Firstly, the computation of the Cox ring of Y. While blowing up one or two points gives a toric surface, the blow up of $\mathbb{F}_4$ in three general points is non-toric; we compute its Cox ring in § 5 using the technique of toric ambient modifications developed in [Hau08]. The second problem is that $\pi\colon X \to Y$ no longer induces an isomorphism on the divisor class groups. Here, Proposition 6.4 provides a general result.

Putting all this together, we obtain the following results in the case of Picard number $2 \leqslant \varrho(X) \leqslant 5$; see Propositions 6.5–6.8 for the complete statements.

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Theorem 3. *Let X be a generic K3 surface with a non-symplectic involution and associated double cover $X \to Y$ and Picard number $2 \leqslant \varrho(X) \leqslant 5$. Then the Cox ring $\mathcal{R}(X)$ is given as follows.*

(i) *For $\varrho(X) = 2$ one has $\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_5]/\langle T_5^2 - f\rangle$ and the degree of T_i is the ith column of*

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix} \quad \text{if } Y = \mathbb{F}_0,$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 3 \end{bmatrix} \quad \text{if } Y = \mathbb{F}_1,$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 4 & 1 & 6 \end{bmatrix} \quad \text{if } Y = \mathbb{F}_4.$$

(ii) *For $\varrho(X) = 3$ one has $\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_6]/\langle T_6^2 - f\rangle$ and the degree of T_i is the ith column of*

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{bmatrix} \quad \text{if } Y = \mathrm{Bl}_1(\mathbb{F}_0),$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 & 1 & 5 \end{bmatrix} \quad \text{if } Y = \mathrm{Bl}_1(\mathbb{F}_4).$$

(iii) *For $\varrho(X) = 4$ one has $\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_7]/\langle T_7^2 - f\rangle$ and the degree of T_i is the ith column of*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 \end{bmatrix} \quad \text{if } Y = \mathrm{Bl}_2(\mathbb{F}_0),$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 & 3 & 1 & 5 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 4 \end{bmatrix} \quad \text{if } Y = \mathrm{Bl}_2(\mathbb{F}_4).$$

(iv) *For $\varrho(X) = 5$ one has the following two cases.*

(a) *The surface Y is the blow up of $\mathbb{F}_0$ at three general points. Then the Cox ring $\mathcal{R}(X)$ of X is*

$$\mathbb{C}[T_1, \ldots, T_{11}]/\langle f_1, \ldots, f_5, T_{11}^2 - g\rangle,$$

where $f_1, \ldots, f_5, g \in \mathbb{C}[T_1, \ldots, T_{10}]$ and $f_1, \ldots, f_5$ are the Plücker relations in $T_1, \ldots, T_{10}$. The degree of T_i is the ith column of

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & -3 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 1 \end{bmatrix}.$$

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(b) *The surface Y is the blow up of $\mathbb{F}_4$ at three general points. Then the Cox ring $\mathcal{R}(X)$ of X is*

$$\mathbb{C}[T_1,\dots,T_9]/\langle T_2T_5+T_4T_6+T_7T_8, T_9^2-f\rangle,$$

where $f\in\mathbb{C}[T_1,\dots,T_8]$ is a prime polynomial and the degree of $T_i\in\mathcal{R}(X)$ is the ith column of

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 1\\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & 3 & 4\\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 2 & 4\\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 1 \end{bmatrix}.$$

If $Y=X/\langle\sigma\rangle$ is a del Pezzo surface, then the Cox ring of Y is known, and we obtain the following for the Cox ring of X; see Proposition 6.9.

Theorem 4. *Let X be a generic K3 surface with a non-symplectic involution, associated double cover $\pi\colon X\to Y$ and intersection form $U(2)\oplus A_1^{k-2}$, where $5\leqslant k\leqslant 9$. Then Y is a del Pezzo surface of Picard number k and:*

(i) *the Cox ring $\mathcal{R}(X)$ is generated by the pull-backs of the (-1)-curves of Y, the section T defining the ramification divisor and, for $k=9$, the pull-back of an irreducible section of $H^0(Y,-K_Y)$;*

(ii) *the ideal of relations of $\mathcal{R}(X)$ is generated by quadratic relations of degree $\pi^*(D)$, where $D^2=0$ and $D\cdot K_Y=-2$, and the relation T^2-f in degree $-2\pi^*(K_Y)$, where f is the pull-back of the canonical section of the branch divisor.*

2. Finite generation of the Cox ring

The first result of this section is Theorem 2.3, which characterizes finite generation of the Cox ring for normal complete varieties in a similar way as Hu and Keel did for $\mathbb{Q}$-factorial projective varieties in [HK00]. In the case of surfaces, we obtain statements extending and complementing results of Galindo and Monserrat [GM05]; a consequence is that every normal complete surface with finitely generated Cox ring is projective and $\mathbb{Q}$-factorial; see Theorem 2.5. Our main application is Theorem 2.7, which characterizes finite generation of the Cox ring of a K3 surface.

Let X be a normal complete algebraic variety defined over some algebraically closed field $\mathbb{K}$ of characteristic zero and assume that the divisor class group $\mathrm{Cl}(X)$ is finitely generated. The *Cox ring* of X is the ring $\mathcal{R}(X)$ of global sections of a sheaf $\mathcal{R}$ of $\mathrm{Cl}(X)$-graded $\mathbb{K}$-algebras. We briefly recall the definition for the case where $\mathrm{Cl}(X)$ is free and refer to [Hau08] for the case of torsion: take a subgroup $K\subseteq\mathrm{WDiv}(X)$ of the group of Weil divisors such that the canonical morphism $K\to\mathrm{Cl}(X)$ is an isomorphism and set

$$\mathcal{R}(X):=\Gamma(X,\mathcal{R}),\quad \mathcal{R}:=\bigoplus_{D\in K}\mathcal{O}_X(D),$$

where multiplication is defined via multiplying homogeneous sections in the field of rational functions of X. Up to isomorphy, this definition does not depend on the choice of the group $K\subseteq\mathrm{WDiv}(X)$. We will also identify K with $\mathrm{Cl}(X)$. So, for $w\in\mathrm{Cl}(X)$ represented by $D\in K$, the homogeneous component $\mathcal{R}(X)_w$ is just $H^0(X,\mathcal{O}_X(D))$.

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If the divisor class group $\mathrm{Cl}(X)$ is free, then the Cox ring $\mathcal{R}(X)$ admits unique factorization; see [BH03]. If $\mathrm{Cl}(X)$ has torsion, then the unique factorization property is replaced by a graded version: every non-trivial homogeneous non-unit is a product of homogeneous primes, where the latter refers to non-trivial homogeneous non-units f such that $f|gh$ with g, h homogeneous implies that $f|g$ or $f|h$; see [Hau08]. If the Cox ring $\mathcal{R}(X)$ is finitely generated as a $\mathbb{K}$-algebra, then one may define the *total coordinate space* $\overline{X} = \operatorname{Spec} \mathcal{R}(X)$ and realize X as the quotient of an open subset $\widehat{X} = \operatorname{Spec}_X \mathcal{R}$ of $\overline{X}$ by the action of the diagonalizable group $\operatorname{Spec} \mathrm{Cl}(X)$ defined by the $\mathrm{Cl}(X)$-grading of the sheaf $\mathcal{R}$ of $\mathcal{O}_X$-algebras. For smooth X, the map $\widehat{X} \to X$ is also known as the *universal torsor* of X.

Let $\mathrm{Cl}_{\mathbb{Q}}(X) = \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ denote the rational divisor class group of X. A first step is to give descriptions of the cones $\mathrm{Eff}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ of effective classes and $\mathrm{Mov}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ of movable classes, i.e. classes having a stable base locus of codimension at least two. We call a cone in a rational vector space V polyhedral if it is the positive hull $\mathrm{cone}(v_1, \ldots, v_r)$ of finitely many vectors $v_i \in V$. The following statement generalizes part of [Hau08, Proposition 4.1].

Proposition 2.1. *Let X be a normal complete variety with finitely generated Cox ring $\mathcal{R}(X)$. Then the cones of effective and movable divisor classes in $\mathrm{Cl}_{\mathbb{Q}}(X)$ are polyhedral. Moreover, if $f_1, \ldots, f_r \in \mathcal{R}(X)$ is any system of pairwise non-associated homogeneous prime generators, then one has*

$$\mathrm{Eff}(X) = \mathrm{cone}(\deg(f_i); i = 1, \ldots, r),$$
$$\mathrm{Mov}(X) = \bigcap_{i=1}^{r} \mathrm{cone}(\deg(f_j); j \neq i).$$

For the proof and also for later use, we have to fix some notation. On a normal variety X, let a class $w \in \mathrm{Cl}(X)$ be represented by a divisor $D \in \mathrm{WDiv}(X)$. Then, as usual, we write

$$H^i(D) := H^i(X, D) := H^i(X, \mathcal{O}_X(D)), \quad h^i(w) := h^i(D) := \dim_{\mathbb{K}}(H^i(D)).$$

Lemma 2.2. *Let X be a normal complete variety with $\mathrm{Cl}(X)$ finitely generated and let $w \in \mathrm{Cl}(X)$ be effective. Then the following two statements are equivalent.*

(i) *The stable base locus of the class $w \in \mathrm{Cl}(X)$ contains a divisor.*

(ii) *There exists a class $w_0 \in \mathrm{Cl}(X)$ with the following properties:*
- *the class w_0 generates an extremal ray of $\mathrm{Eff}(X)$ and $h^0(nw_0) = 1$ holds for any $n \in \mathbb{N}$;*
- *there is an $f_0 \in \mathcal{R}(X)_{w_0}$ such that for any $n \in \mathbb{N}$ and $f \in \mathcal{R}(X)_{nw}$ one has $f = f' f_0$ with some $f' \in \mathcal{R}(X)_{nw - w_0}$.*

Proof. The implication '(ii) $\Rightarrow$ (i)' is obvious. So, assume that (i) holds. The class $w \in \mathrm{Cl}(X)$ is represented by some non-negative divisor D. Let D_0 be a prime component of D, which occurs in the fixed part of any positive multiple nD, and let $w_0 \in \mathrm{Cl}(X)$ be the class of D_0. Then the canonical section of D_0 defines an element $f_0 \in \mathcal{R}(X)_{w_0}$, which divides any $f \in \mathcal{R}(X)_{nw}$. Note that $h^0(nD_0) = 1$ holds for every $n \in \mathbb{Z}_{\geqslant 0}$, because otherwise $H^0(na_0D_0) \subseteq \mathcal{R}(X)_{nw}$, where $a_0 > 0$ is the multiplicity of D_0 in D, would provide enough sections to move na_0D_0. Moreover, $\mathrm{cone}(w_0)$ is an extremal ray of $\mathrm{Eff}(X)$, because otherwise we would have $nD_0 \sim D_1 + D_2$ with some $n \in \mathbb{Z}_{\geqslant 0}$ and non-negative divisors D_1, D_2, none of which is a multiple of D_0; this contradicts $h^0(nD_0) = 1$. □

Proof of Proposition 2.1. Only for the description of the moving cone there is something to show. For this, set $w_i := \deg(f_i)$ and note that the extremal rays of the effective cone occur

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among the $\mathbb{Q}_{\geqslant 0} \cdot w_i$. By suitably renumbering, we achieve that the indices $1 \leqslant i \leqslant d$ are precisely those with $h^0(nw_i) \leqslant 1$ for all $n \in \mathbb{N}$.

Let $w \in \mathrm{Mov}(X)$. Then Lemma 2.2 tells us that for any $i = 1, \ldots, d$ there must be a monomial of the form $\prod_{j \neq i} f_j^{n_j}$ in some $\mathcal{R}(X)_{nw}$. Consequently, w lies in the cone of the right-hand side. Conversely, consider an element w of the cone of the right-hand side. Then, for every $i = 1, \ldots, d$, a product $\prod_{j \neq i} f_j^{n_j}$ belongs to some $\mathcal{R}(X)_{nw}$. Hence, none of the $f_1, \ldots, f_d$ divides all elements of $\mathcal{R}(X)_{nw}$. Again by Lemma 2.2, we conclude that $w \in \mathrm{Mov}(X)$. □

We characterize finite generation of the Cox ring. By $\mathrm{SAmple}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ we denote the cone of semiample divisor classes of a variety X, i.e. classes having a base point free positive multiple. Moreover, by a small birational map $X \to Y$, we mean a rational map that defines an isomorphism $U \to V$ of open subsets $U \subseteq X$ and $V \subseteq Y$ such that the respective complements $X \backslash U$ and $Y \backslash V$ are of codimension at least two.

Theorem 2.3. *Let X be a normal complete variety with finitely generated divisor class group. Then the following statements are equivalent.*

(i) *The Cox ring $\mathcal{R}(X)$ is finitely generated.*

(ii) *The effective cone $\mathrm{Eff}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ is polyhedral and there are small birational maps $\pi_i \colon X \to X_i$, where $i = 1, \ldots, r$, such that each semiample cone $\mathrm{SAmple}(X_i) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X_i)$ is polyhedral and one has*

$$\mathrm{Mov}(X) = \pi_1^*(\mathrm{SAmple}(X_1)) \cup \cdots \cup \pi_r^*(\mathrm{SAmple}(X_r)).$$

Moreover, if one of these two statements holds, then there is a small birational map $X \to X'$ with a $\mathbb{Q}$-factorial projective variety X'.

In the proof we use the fact that the moving cone of any normal complete variety is of full dimension; we are grateful to Jenia Tevelev for providing us with the following statement and proof.

Lemma 2.4. *Let X be a normal complete variety with $\mathrm{Cl}(X)$ finitely generated. Then the moving cone $\mathrm{Mov}(X)$ is of full dimension in the rational divisor class group $\mathrm{Cl}_{\mathbb{Q}}(X)$.*

Proof. Using Chow's lemma and resolution of singularities, we obtain a birational morphism $\pi \colon X' \to X$ with a smooth projective variety X'. Let $D_1, \ldots, D_r \in \mathrm{WDiv}(X)$ be prime divisors generating $\mathrm{Cl}(X)$, and consider their proper transforms $D_1', \ldots, D_r' \in \mathrm{WDiv}(X')$. Moreover, let $E' \in \mathrm{CaDiv}(X')$ be very ample such that all $E' + D_i'$ are also very ample, and denote by $E \in \mathrm{WDiv}(X)$ its push-forward. Then the classes E and $E + D_i$ generate a full-dimensional cone $\tau \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ and, since E' and the $E' + D_i'$ are movable, we have $\tau \subseteq \mathrm{Mov}(X)$. □

Proof of Theorem 2.3. Suppose that (i) holds. Then Proposition 2.1 tells us that $\mathrm{Eff}(X)$ is polyhedral. Let $\mathfrak{F} = (f_1, \ldots, f_r)$ be a system of pairwise non-associated homogeneous prime generators of $R := \mathcal{R}(X)$ and set $w_i := \deg(f_i)$.

By [Hau08, Proposition 2.2], the group $H = \mathrm{Spec}\, \mathbb{K}[\mathrm{Cl}(X)]$ acts freely on an open subset $W \subseteq \overline{X}$ of $\overline{X} = \mathrm{Spec}\, \mathcal{R}(X)$ such that $\overline{X} \backslash W$ is of codimension at least two in $\overline{X}$. Thus, we can choose a point $z \in W$ with $f_i(z) = 0$ and $f_j(z) \neq 0$ for $j \neq i$. Consequently, the weights w_j, where $j \neq i$, generate $\mathrm{Cl}(X)$ and hence the system of generators $\mathfrak{F}$ is admissible in the sense of [Hau08, Definition 3.4]. Moreover, by Lemma 2.4, the moving cone of X is of full dimension and, by

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Proposition 2.1, it is given as

$$\mathrm{Mov}(X) = \bigcap_{i=1}^{r} \mathrm{cone}(w_j; j \neq i).$$

Thus, we are in the setting of [Hau08, Corollary 4.3]. That means that $\mathrm{Mov}(X)$ is a union of full-dimensional GIT chambers $\lambda_1, \ldots, \lambda_r$, the relative interiors of which are contained in the relative interior of $\mathrm{Mov}(X)$ and the associated projective varieties $X_i := \widehat{X}_i /\!/ H$, where $\widehat{X}_i := \overline{X}^{ss}(\lambda_i)$ are $\mathbb{Q}$-factorial and have $\mathcal{R}(X)$ as their Cox ring and λ_i as their semiample cone.

Moreover, if $q\colon \widehat{X} \to X$ and $q_i\colon \widehat{X}_i \to X_i$ denote the associated universal torsors, then the desired small birational maps $\pi_i\colon X \to X_i$ are obtained as follows. Let $X' \subseteq X$ and $X_i' \subseteq X_i$ be the respective sets of smooth points. Then, by [Hau08, Proposition 2.2], the sets $q^{-1}(X')$ and $q_i^{-1}(X_i')$ have a small complement in $\overline{X}$ and thus we obtain open embeddings with a small complement

$$X \xleftarrow{\quad} (q^{-1}(X') \cap q_i^{-1}(X_i')) /\!/ H \xrightarrow{\quad} X_i\ .$$

Now suppose that (ii) holds. Let $w_1, \ldots, w_d \in \mathrm{Eff}(X)$ be those primitive generators of extremal rays of $\mathrm{Eff}(X)$ that satisfy $h^0(nw_i) \leqslant 1$ for any $n \in \mathbb{Z}_{\geqslant 0}$ and fix $0 \neq f_i \in \mathcal{R}(X)_{n_i w_i}$ with n_i minimal. Then we have

$$\bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} \mathcal{R}(X)_{nw_i} = \mathbb{K}[f_i].$$

Set $\lambda_i := \pi_i^*(\mathrm{SAmple}(X_i))$. Then, by Gordon's lemma and [HK00, Lemma 1.8], we have another finitely generated subalgebra of the Cox ring, namely

$$\mathcal{S}(X) := \bigoplus_{w \in \mathrm{Mov}(X)} \mathcal{R}(X)_w = \bigoplus_{i=1}^{r} \Bigg(\bigoplus_{w \in \lambda_i} \mathcal{R}(X)_w \Bigg).$$

We show that $\mathcal{R}(X)$ is generated by $\mathcal{S}(X)$ and the $f_i \in \mathcal{R}(X)_{n_i w_i}$. Consider any $0 \neq f \in \mathcal{R}(X)_w$ with $w \notin \mathrm{Mov}(X)$. Then, by Lemma 2.2, we have $f = f^{(1)} f_i$ for some $1 \leqslant i \leqslant d$ and some $f^{(1)} \in \mathcal{R}(X)$ homogeneous of degree $w(1) := w - n_i w_i$. If $w(1) \notin \mathrm{Mov}(X)$ holds, then we repeat this procedure with $f^{(1)}$ and obtain $f = f^{(2)} f_i f_j$ with $f^{(2)}$ homogeneous of degree $w(2)$. At some point, we must end with $w(n) = \deg(f^{(n)}) \in \mathrm{Mov}(X)$, because otherwise the sequence of the $w(n)$ would leave the effective cone. □

Theorem 2.5. *Let X be a normal complete surface with finitely generated divisor class group $\mathrm{Cl}(X)$. Then the following statements are equivalent.*

(i) *The Cox ring $\mathcal{R}(X)$ is finitely generated.*

(ii) *The effective cone $\mathrm{Eff}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ and the moving cone $\mathrm{Mov}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ are polyhedral and $\mathrm{Mov}(X) = \mathrm{SAmple}(X)$ holds.*

Moreover, if one of these two statements holds, then the surface X is $\mathbb{Q}$-factorial and projective.

Proof. We verify the implication '(i) ⇒ (ii)'. By Proposition 2.1, we only have to show that the moving cone coincides with the semiample cone. Clearly, we have $\mathrm{SAmple}(X) \subseteq \mathrm{Mov}(X)$. Suppose that $\mathrm{SAmple}(X) \neq \mathrm{Mov}(X)$ holds. Then $\mathrm{Mov}(X)$ is properly subdivided into GIT chambers; see [Hau08, Corollary 4.3]. In particular, we find two chambers λ' and λ both intersecting the relative interior of $\mathrm{Mov}(X)$ such that λ' is a proper face of λ. The associated GIT quotients Y' and Y of the total coordinate space $\overline{X}$ have λ' and λ as their respective

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semiample cones. Moreover, the inclusion $\lambda' \subseteq \lambda$ gives rise to a proper morphism $Y \to Y'$, which is an isomorphism in codimension one. As Y and Y' are normal surfaces, we obtain $Y \cong Y'$, which contradicts the fact that the semiample cones of Y and Y' are of different dimension.

The verification of '(ii) $\Rightarrow$ (i)' runs as in the preceding proof; this time one uses the finitely generated subalgebra

$$\mathcal{S}(X) := \bigoplus_{w \in \mathrm{Mov}(X)} \mathcal{R}(X)_w = \bigoplus_{w \in \mathrm{SAmple}(X)} \mathcal{R}(X)_w.$$

Moreover, by Theorem 2.3, there is a small birational map $X \to X'$ with X' projective and $\mathbb{Q}$-factorial. As X and X' are complete surfaces, this map already defines an isomorphism. □

In the case of a $\mathbb{Q}$-factorial surface X, we obtain the following simpler characterization involving the cone $\mathrm{Nef}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ of numerically effective divisor classes; note that the implication '(ii) $\Rightarrow$ (i)' was obtained for smooth surfaces in [GM05, Corollary 1].

Corollary 2.6. *Let X be a $\mathbb{Q}$-factorial projective surface with finitely generated divisor class group $\mathrm{Cl}(X)$. Then the following statements are equivalent.*

(i) *The Cox ring $\mathcal{R}(X)$ is finitely generated.*

(ii) *The effective cone $\mathrm{Eff}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ is polyhedral and $\mathrm{Nef}(X) = \mathrm{SAmple}(X)$ holds.*

Proof of Corollary 2.6. If (i) holds, then we infer from [BH07, Corollary 7.4] that the semiample cone and the nef cone of X coincide. Now suppose that (ii) holds. From

$$\mathrm{SAmple}(X) \subseteq \mathrm{Mov}(X) \subseteq \mathrm{Nef}(X),$$

we then conclude that $\mathrm{Mov}(X) = \mathrm{Nef}(X)$. Moreover, since $\mathrm{Eff}(X)$ is polyhedral, $\mathrm{Nef}(X)$ is given by a finite number of inequalities and hence is also polyhedral. Thus, we can apply Theorem 2.5. □

We turn to K3 surfaces X. Recall that, by definition, X is a smooth complete complex surface with $b_1(X) = 0$ and trivial canonical class. We always assume a K3 surface X to be *algebraic*. As a sublattice of $H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{22}$, the divisor class group $\mathrm{Cl}(X)$ is finitely generated and free. In particular, we can define a Cox ring $\mathcal{R}(X)$ as above. Our first result characterizes finite generation of $\mathcal{R}(X)$.

Theorem 2.7. *For any complex algebraic K3 surface X, the following statements are equivalent.*

(i) *The Cox ring $\mathcal{R}(X)$ is finitely generated.*

(ii) *$\mathrm{Eff}(X)$ is polyhedral.*

Lemma 2.8. *Let X be a K3 surface and D be a non-principal divisor on X. If we have $h^0(D) = 1$, then $D^2 < 0$ holds.*

Proof. Since the canonical divisor of X is principal, Serre's duality theorem gives us $h^2(D) = h^0(-D) = 0$. The Riemann–Roch theorem then yields $1 \geqslant D^2/2 + 2$. The assertion follows. □

Proof of Theorem 2.7. Only for '(ii) $\Rightarrow$ (i)' there is something to show. So, assume that $\mathrm{Eff}(X)$ is polyhedral. Then, by Corollary 2.6, we have to show that for every numerically effective divisor D on X, some positive multiple is semiample.

By a result of Kleiman, the class $[D] \in \mathrm{Cl}_{\mathbb{Q}}(X)$ lies in the closure of the cone of ample divisor classes. Since any ample class is effective and $\mathrm{Eff}(X)$ as a polyhedral cone is closed in $\mathrm{Cl}_{\mathbb{Q}}(X)$, we obtain $[D] \in \mathrm{Eff}(X)$. Thus, we may assume that D is non-negative.

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Since D is numerically effective, we have $D^2 \geqslant 0$. If $D^2 > 0$ holds, then [May72, Corollary, p. 11] tells us that the linear system $|3D|$ is base point free, i.e. that $3D$ is semiample. If we have $D^2 = 0$, then we write $D = D_0 + D_1$, where D_0 denotes the fixed part of D and D_0, D_1 are non-negative. Then we have

$$0 = D^2 = D \cdot D_0 + D \cdot D_1.$$

Since D_0 and D_1 are non-negative, we conclude that $D \cdot D_0 = 0$ and $D \cdot D_1 = 0$. We show that $D_0 = 0$ must hold. Otherwise, using Lemma 2.8, we obtain

$$D_1^2 = (D - D_0)^2 < 0.$$

On the other hand, D_1 has no fixed components. Thus, according to [Sai74, Corollary 3.2], the divisor D_1 is base point free and hence numerically effective, a contradiction. Thus, we see that $D = D_1$ holds, and thus D is semiample. □

Corollary 2.9. *Let X be a K3 surface. If the cone $\mathrm{Eff}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ of effective divisor classes is polyhedral, then the cone $\mathrm{SAmple}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ of semiample divisor classes is also polyhedral.*

For Enriques surfaces, i.e. smooth projective surfaces X with $q(X) = 0$ and $2K_X$ trivial but K_X non-trivial, we obtain the following analogue of Theorem 2.7.

Theorem 2.10. *For any Enriques surface X, the following statements are equivalent.*

(i) *The Cox ring $\mathcal{R}(X)$ is finitely generated.*

(ii) *$\mathrm{Eff}(X)$ is polyhedral.*

Proof. Only for '(ii) $\Rightarrow$ (i)' is there something to show. So, assume that $\mathrm{Eff}(X)$ is polyhedral. Then, by Corollary 2.6, we have to show that for every given numerically effective divisor D on X, some positive multiple is semiample. Since $\mathrm{Eff}(X)$ is polyhedral, we obtain $\mathrm{Nef}(X) \subseteq \mathrm{Eff}(X)$ and hence we may assume that D is non-negative.

Let $\pi\colon S \to X$ be the universal covering. Then S is a K3 surface and π is an unramified double covering. The pull-back π^*D on S is effective and numerically effective. As in the proof of Theorem 2.7, we see that some positive multiple π^*nD is semiample. From [BHPV04, Lemma 17.2], we infer that

$$H^0(S, \pi^*nD) = \pi^*H^0(X, nD) + \pi^*H^0(X, nD + K_X).$$

If $x \in X$ is a base point of nD, then there is a $g \in H^0(X, nD + K_X)$ such that $g(x) \neq 0$ holds; otherwise $\pi^{-1}(x)$ would be in the base locus of π^*nD, which is a contradiction. Since $2K_X$ is trivial, we deduce that $2nD$ has no base points. □

We conclude the section with recalling some classical statements on algebraic K3 surfaces X characterizing the case of a polyhedral effective cone and thus providing further criteria for finite generation of the Cox ring. Consider the lattice $\mathrm{Cl}(X) = \mathrm{Pic}(X)$ with the intersection pairing; denote by $\mathrm{O}(\mathrm{Cl}(X))$ the group of its isometries and by $W(\mathrm{Cl}(X))$ the Weyl group, i.e. the subgroup generated by reflections with respect to $\delta \in \mathrm{Cl}(X)$ with $\delta^2 = -2$.

Theorem 2.11. *See [Kov94, Theorem 2, Remark 7.2] and [PS71, § 7, Corollary]. For any algebraic K3 surface X, the following statements are equivalent.*

(i) *The cone $\mathrm{Eff}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ is polyhedral.*

(ii) *The set $\mathrm{O}(\mathrm{Cl}(X))/W(\mathrm{Cl}(X))$ is finite.*

(iii) *The automorphism group $\mathrm{Aut}(X)$ is finite.*

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Moreover, if the Picard number is at least three, then (i) is equivalent to the property that X contains only finitely many smooth rational curves. In this case, the classes of such curves generate the effective cone.

The hyperbolic lattices satisfying (ii) have been classified in [PS71] and a series of papers by Nikulin; see [Nik79, Nik83, Nik85]. In particular, it has been proved that there are only finitely many of them having rank at least three. The results of these papers, together with Theorem 2.11, give the following.

Theorem 2.12. *See [Nik79, Nik83, Nik85]. Let X be an algebraic K3 surface with Picard number $\varrho(X)$.*

(i) *Suppose that $\varrho(X) = 2$ holds. Then $\mathrm{Eff}(X)$ is polyhedral if and only if $\mathrm{Cl}(X)$ contains a class of self intersection 0 or -2.*

(ii) *Suppose that $\varrho(X) \geqslant 3$ holds. Then $\mathrm{Eff}(X)$ is polyhedral if and only if $\mathrm{Cl}(X)$ belongs to a finite list of hyperbolic lattices. The following table gives the number n of these lattices for any Picard number.*

$\varrho(X)$	3	4	5–6	7	8	9	10	11–12	13–14	15–19	20
n	27	17	10	9	12	10	9	4	3	1	0

The following statement is a consequence of the results mentioned above or of [Kov94, Theorem 2].

Proposition 2.13. *Let X be an algebraic K3 surface such that $\mathrm{Eff}(X)$ is polyhedral. Then the generators of $\mathrm{Eff}(X)$ are described in the following table.*

$\varrho(X)$	$\mathrm{Eff}(X)$	*Type of generators*
1	$\mathbb{Q}_+[H]$	*Ample divisor*
2	$\mathbb{Q}_+[E_1] + \mathbb{Q}_+[E_2]$	*(-2)- or (0)-curves*
3–19	$\sum \mathbb{Q}_+[E_i]$	*(-2)-curves*

Note that for $\varrho(X) = 1$, the Cox ring of X coincides with its usual homogeneous coordinate ring, whose generators have been studied in [Sai74].

3. K3 surfaces of Picard number two

We consider (complex algebraic) K3 surfaces X with divisor class group $\mathrm{Cl}(X) = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$, where $w_i^2 \in \{0, -2\}$ and, as we may assume then, $w_1 \cdot w_2 \geqslant 1$ hold; recall from [Mor84, Corollary 2.9(i)] that any even lattice of rank two with signature $(1, 1)$ is the Picard lattice of an algebraic K3 surface. According to Theorem 2.7 and the characterization of $\mathrm{Eff}(X)$ being polyhedral provided in Theorem 2.12, such surfaces X have a finitely generated Cox ring $\mathcal{R}(X)$. We investigate the possible degrees of generators and relations for $\mathcal{R}(X)$. An explicit computation of $\mathcal{R}(X)$ for the cases $w_1 \cdot w_2 = 1, 2$ is given in § 6. A first observation concerns the effective cone.

Proposition 3.1. *Let X be a K3 surface with $\mathrm{Cl}(X) = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$, where w_1, w_2 are effective such that $w_i^2 \in \{0, -2\}$ and $w_1 \cdot w_2 \geqslant 2$ hold. Then w_1 and w_2 generate $\mathrm{Eff}(X)$ as a cone.*

Proof. Suppose that $\mathrm{cone}(w_1, w_2) \subsetneq \mathrm{Eff}(X)$ holds. Then we may assume that w_1 does not lie on the boundary of $\mathrm{Eff}(X)$. Thus, one of the generators of $\mathrm{Eff}(X)$ is of the form $w = aw_1 - bw_2$ for

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some $a, b \in \mathbb{N}$, where $a > 0$. Proposition 2.13 gives

$$w^2 = a^2 w_1^2 + b^2 w_2^2 - 2ab\, w_1 \cdot w_2 \in \{0, -2\}.$$

This can only be realized for $b = 0$, because we assumed that $w_i^2 \in \{0, -2\}$ and $w_1 \cdot w_2 \geqslant 2$. Thus, $w = aw_1$ holds and, consequently, w_1 lies on the boundary of $\mathrm{Eff}(X)$; a contradiction. □

Note that the assumption of w_1 and w_2 being effective in Proposition 3.1 can always be achieved: the Riemann–Roch theorem and $w_i^2 \in \{0, -2\}$ show that either w_i or $-w_i$ is effective.

Our next result settles the case $w_i^2 = 0$ and $w_1 \cdot w_2 \geqslant 3$. In order to state it, we first have to fix our usage. Consider any finitely generated $\mathbb{C}$-algebra R, graded by a lattice K. We say that a system of homogeneous generators $f_1, \ldots, f_r$ of R is *minimal* if no f_i can be expressed as a polynomial in the remaining f_j. Moreover, we say that R *has a generator* in degree $w \in K$ if any minimal system of generators for R contains a non-trivial element of R_w. Given a system $f_1, \ldots, f_r$ of generators, we have the surjection

$$\mathbb{C}[T_1, \ldots, T_r] \to R, \quad T_i \mapsto f_i.$$

The *ideal of relations* determined by $f_1, \ldots, f_r$ is the kernel $I \subseteq \mathbb{C}[T_1, \ldots, T_r]$ of this map; it is homogeneous with respect to the K-grading of $\mathbb{C}[T_1, \ldots, T_r]$ defined by $\deg(T_i) := \deg(f_i)$. By a *minimal ideal* of relations, we mean the ideal of relations determined by a minimal system of generators.

Theorem 3.2. *Let X be a K3 surface with $\mathrm{Cl}(X) \cong \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$, where w_1, w_2 are effective, and intersection form given by $w_1^2 = w_2^2 = 0$ and $w_1 \cdot w_2 = k \geqslant 3$.*

(i) *The semiample cone of X coincides with its effective cone.*

(ii) *The Cox ring $\mathcal{R}(X)$ is generated in degrees w_1, w_2, $w_1 + w_2$, and one has*

$$\dim(\mathcal{R}(X)_{w_i}) = 2, \quad \dim(\mathcal{R}(X)_{w_1+w_2}) = k + 2.$$

Moreover, any minimal system of generators of $\mathcal{R}(X)$ has $k + 2$ members.

(iii) *For $k = 3$, the Cox ring $\mathcal{R}(X)$ is of the form $\mathbb{C}[T_1, \ldots, T_5]/\langle f \rangle$ and the degrees of the generators and the relation are given by*

$$\deg(T_1) = \deg(T_2) = w_1, \quad \deg(T_4) = \deg(T_5) = w_2,$$
$$\deg(T_3) = w_1 + w_2, \quad \deg(f) = 3w_1 + 3w_2.$$

(iv) *For $k \geqslant 4$, any minimal ideal $\mathcal{I}(X)$ of relations of $\mathcal{R}(X)$ is generated in degree $2w_1 + 2w_2$, and we have*

$$\dim(\mathcal{I}(X)_{2w_1+2w_2}) = \frac{k(k-3)}{2}.$$

Note that for $k = 3, 4$ the Cox ring of X is a complete intersection, while for $k \geqslant 5$ this no longer holds. Before giving the proof of the above theorem, we briefly provide the necessary ingredients.

Lemma 3.3. *Let X be a smooth surface, assume that $D, D_1, D_2 \in \mathrm{WDiv}(X)$ satisfy $D_1 \cdot D_2 = 0$ and $h^1(D - D_1 - D_2) = 0$, and let $0 \neq f_i \in H^0(D_i)$ be such that $\mathrm{div}(f_1) + D_1$ and $\mathrm{div}(f_2) + D_2$ have no common components. Then one has a surjection*

$$H^0(D - D_1) \oplus H^0(D - D_2) \to H^0(D),$$
$$(g_1, g_2) \mapsto g_1 f_1 + g_2 f_2.$$

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Proof. First note that due to the assumptions, the assignments $\imath\colon h \mapsto (hf_2, -hf_1)$ and $\varphi\colon (h_1, h_2) \mapsto h_1 f_1 + h_2 f_2$ give rise to an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X(-D_1 - D_2) \xrightarrow{\imath} \mathcal{O}_X(-D_1) \oplus \mathcal{O}_X(-D_2) \xrightarrow{\varphi} \mathcal{O}_X \longrightarrow 0.$$

Tensoring this sequence with $\mathcal{O}_X(D)$ and looking at the associated cohomology sequence, we obtain the assertion. □

Proposition 3.4. *Let X be a K3 surface, $w \in \mathrm{Cl}(X)$ be the class of a smooth irreducible curve $D \subseteq X$ of genus g and consider the Veronese algebra*

$$\mathcal{R}(X, w) := \bigoplus_{n \in \mathbb{N}} \mathcal{R}(X)_{nw}.$$

Then the algebra $\mathcal{R}(X, w)$ is generated in degree one if D is not hyperelliptic or $g \leqslant 1$, in degrees one and three if $g = 2$ and in degrees one and two if D is hyperelliptic of genus $g \geqslant 3$.

Proof. If $g = 0$ holds, then $\mathcal{R}(X, w) = \mathbb{C}[s]$ with $s \in H^0(D)$, since D is irreducible with negative self intersection. Thus, $\mathcal{R}(X, w)$ is generated in degree one.

For non-rational D, the canonical algebra $\oplus H^0(D, nK_D)$ is generated in degree one if D is not hyperelliptic or $g = 1$, in degrees one and three if $g = 2$ and in degrees one and two if D is hyperelliptic and $g \geqslant 3$; see [ACGH85, p. 117]. By the adjunction formula we have $\mathcal{O}_X(D)_{|D} \cong K_D$. Thus, we obtain the exact sequence

$$0 \longrightarrow H^0(X, (n-1)D) \longrightarrow H^0(X, nD) \longrightarrow H^0(D, nK_D) \longrightarrow 0,$$

where the last zero is due to the Kawamata–Viehweg vanishing theorem. This gives the assertion. □

In order to prove Theorem 3.2(iii), we use the techniques introduced in [LV09]. We say that a degree $w \in R$ is *not essential* for a minimal ideal I of relations of a K-graded algebra R if no minimal system of homogeneous generators of I has members of degree w.

Theorem 3.5. *See [LV09]. Let $f_1, \ldots, f_r \in \mathcal{R}(X)$ be a minimal system of generators for the Cox ring of a surface X and set $w_i := \deg(f_i) \in \mathrm{Cl}(X)$. Consider the maps*

$$\varphi_{w,i}\colon \mathcal{R}(X)_{w-w_1-w_i} \oplus \mathcal{R}(X)_{w-w_2-w_i} \to \mathcal{R}(X)_{w-w_i},$$
$$(g_1, g_2) \mapsto g_1 f_1 + g_2 f_2,$$

where $w \in \mathrm{Cl}(X)$ and $i = 3, \ldots, r$. If $w_1 \cdot w_2 = 0$ holds and $\varphi_{w,i}$ is surjective for $i = 3, \ldots, r$, then w is not essential for the ideal of relations arising from $f_1, \ldots, f_r$.

Proof of Theorem 3.2. Let $D_i \in \mathrm{WDiv}(X)$ represent $w_i \in \mathrm{Cl}(X)$. Then $D_i^2 = 0$ implies that the complete linear system of D_i defines a fibration, which in turn gives $h^0(w_i) = 2$. In particular, we have bases (f_{i1}, f_{i2}) for $\mathcal{R}(X)_{w_i}$, where $i = 1, 2$. Moreover, applying the Riemann–Roch theorem yields $h^1(w_i) = 0$.

By Proposition 3.1, the classes w_1 and w_2 generate the effective cone. Moreover, $h^0(w_i) = 2$ and $w_i^2 = 0$ show that w_i is semiample, i.e. we have $\mathrm{Eff}(X) = \mathrm{SAmple}(X)$. Consequently, any divisor class $w = aw_1 + bw_2$ with $a, b > 0$ is ample and the Kawamata–Viehweg vanishing theorem gives $h^1(w) = 0$.

We show now that $\mathcal{R}(X)$ is generated in degrees w_1, w_2 and $w_1 + w_2$. Consider a class $w = aw_1 + bw_2$. If $a \geqslant 3$ and $b \geqslant 1$ or $(a, b) = (2, 1)$ holds, then we have $h^1(w - 2w_1) = 0$.

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Thus, Lemma 3.3 provides a surjective map

$$\varphi\colon \mathcal{R}(X)_{w-w_1} \oplus \mathcal{R}(X)_{w-w_1} \to \mathcal{R}(X)_w, \quad (g_1, g_2) \mapsto g_1 f_{11} + g_2 f_{12}.$$

If $b = 0$ holds, then the complete linear system of any representative of w is composed with a pencil. This implies again surjectivity of the above map φ.

Iterating this procedure, we see that for any $w = aw_1 + bw_2$ with $a \geqslant 3$ and $b \geqslant 1$ or $(a, b) = (2, 1)$ or $b = 0$, the elements of $\mathcal{R}(X)_w$ are polynomials in f_{11}, f_{12} and the elements of $\mathcal{R}(X)_u$, where

$$u = 2w_1 + bw_2 \text{ if } b \geqslant 2, \quad u = w_1 + bw_2 \text{ if } b = 0, 1.$$

Interchanging the roles of a and b, in this reasoning, we finally see that any element of $\mathcal{R}(X)_w$ is a polynomial in $f_{11}, f_{12}, f_{21}, f_{22}$ and elements of $\mathcal{R}(X)_{nu}$, where $u := w_1 + w_2$ and $n \leqslant 2$.

Thus, we are left with describing the elements of $\mathcal{R}(X)_{nu}$, where $u = w_1 + w_2$. Observe that no complete linear system on X has fixed components, because, by Lemma 2.8 and the adjunction formula, any such component would be a (-2)-curve and, by our assumptions, there are no classes of self intersection -2 in $\mathrm{Cl}(X)$. Moreover, note that w_1 and w_2 are the only classes of elliptic curves in $\mathrm{Cl}(X)$.

It follows that $u = w_1 + w_2$ is represented by a smooth irreducible curve $D \subseteq X$ of genus $u^2/2 + 1 > 3$. Since $u^2 \geqslant 6$ holds, u is a primitive class in $\mathrm{Cl}(X)$ and we have $u \cdot w_i \geqslant 3$: the curve D is not hyperelliptic, see [Sai74, Theorem 5.2]. According to Proposition 3.4, the elements of $\mathcal{R}(X)_{nu}$ are polynomials in those of $\mathcal{R}(X)_u$.

Thus, we obtained that $\mathcal{R}(X)$ is generated in the degrees w_1, w_2 and $u := w_1 + w_2$. Moreover, the Riemann–Roch theorem gives us

$$\dim(\mathcal{R}(X)_{w_i}) = 2, \quad \dim(\mathcal{R}(X)_{w_1+w_2}) = k + 2.$$

We now turn to the relations. First note that any minimal system of generators must comprise a basis (f_{11}, f_{12}) of $\mathcal{R}(X)_{w_1}$ and a basis (f_{21}, f_{22}) of $\mathcal{R}(X)_{w_2}$. Now, consider any degree $w = aw_1 + bw_2$. If $a \geqslant 4$ and $b \geqslant 2$ or $(a, b) = (3, 2)$ holds, then we have

$$h^1(w - 2w_1 - w_2) = h^1(w - 3w_1 - w_2) = 0.$$

Thus, taking $f_1 = f_{11}$ and $f_2 = f_{12}$ in Theorem 3.5 and using Lemma 3.3, we see that w is not essential for any minimal ideal of relations of $\mathcal{R}(X)$. If $b = 1$ holds, then

$$\mathcal{R}(X)_{(c-1)w_1} \oplus \mathcal{R}(X)_{(c-1)w_1} \to \mathcal{R}(X)_{cw_1}, \quad (g_1, g_2) \mapsto g_1 f_{11} + g_2 f_{12}$$

is surjective for $c = a, a - 1$, because $\mathcal{R}(X)$ is generated in degrees w_1, w_2 and $w_1 + w_2$. Thus, Theorem 3.5 shows that w is not essential for $b = 1$. Eventually, there are no relations of degree w for $b = 0$. In fact, then $\mathcal{R}(X)_w$ is generated by f_{11} and f_{12} and hence any such relation defines a relation among f_{11} and f_{12}, which contradicts the fact that f_{11}, f_{12} define a surjection $X \to \mathbb{P}_1$.

Exchanging the roles of w_1 and w_2 in this consideration, we obtain that essential relations can only occur in degrees $2u$ and $3u$, where $u = w_1 + w_2$.

In the case $k = 3$, the statements proven so far give that any minimal system of generators has five members and their degrees are w_1, w_1, w_2, w_2 and u. Hence, there must be exactly one relation in $\mathcal{R}(X)$. The degree of this relation minus the sum of the degrees of the generators gives the canonical class, see [BH07, Proposition 8.5], and hence vanishes. Thus, our relation must have degree $3u$.

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Finally, let $k \geqslant 4$. As observed before, $\mathcal{R}(X)_{nu}$ is generated by $\mathcal{R}(X)_u$. Hence, any relation in degree nu is also a relation of

$$\bigoplus_{n\in\mathbb{N}} \mathcal{R}(X)_{nu}.$$

Since $u \cdot w_i > 3$ holds and X does not contain smooth rational curves, [Sai74, Theorem 7.2] tells us that the ideal of relations of this algebra is generated in degree two. Thus, there are only essential relations of degree $2u$ in $\mathcal{R}(X)$.

In order to determine the dimension of $\mathcal{I}(X)_{2u}$ for a minimal ideal of relations $\mathcal{I}(X)$, note that we have the four generators f_{ij}, where $1 \leqslant i, j \leqslant 2$, of degree w_i, and $k-2$ generators of degree $u = w_1 + w_2$. Using the Riemann–Roch theorem, we obtain that $\mathcal{R}(X)_{2u}$ is of dimension $4k+2$. Thus, denoting by $\mathbb{C}[T]$ the polynomial ring in the above generators and by $V \subseteq \mathcal{R}(X)_u$ the vector space spanned by the $k-2$ generators of degree u, we obtain

$$\begin{aligned}\dim(\mathcal{I}(X)_{2u}) &= \dim(\mathbb{C}[T]_{2u}) - \dim(\mathcal{R}(X)_{2u}) \\ &= \dim(\operatorname{Sym}^2 V) + 4(k-2) + 9 \\ &= \frac{k(k-3)}{2}.\end{aligned}$$

□

We now turn to the cases $w_1^2 = -2$ and $w_2^2 = 0, -2$. In contrast to the previous cases, the number of degrees occurring in a minimal system of generators for the Cox ring becomes arbitrarily large when $w_1 \cdot w_2$ increases.

Proposition 3.6. *Let X be a K3 surface with $\operatorname{Cl}(X) \cong \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$ and intersection form given by $w_1^2 = -2$, $w_2^2 = 0$ and $w_1 \cdot w_2 = k \in \mathbb{N}$.*

(i) *The semiample cone $\operatorname{SAmple}(X)$ of X is generated by the classes $kw_1 + 2w_2$ and w_2.*

(ii) *The Cox ring $\mathcal{R}(X)$ has generators in degrees w_1 and $aw_1 + w_2$, where $0 \leqslant a \leqslant \lfloor k/2 \rfloor$.*

(iii) *If $k > 1$ holds and k is odd, then the Cox ring $\mathcal{R}(X)$ has, in addition to those of (ii), generators in degree $kw_1 + 2w_2$.*

Proof. To verify (i), note that $(aw_1 + bw_2) \cdot w_1 = -2a + kb$ and $(aw_1 + bw_2) \cdot w_2 = ka$ hold. These intersection products are both non-negative if $0 \leqslant a \leqslant kb/2$ holds. So, the nef cone of X is generated by the classes $kw_1 + 2w_2$ and w_2. Since $\operatorname{SAmple}(X)$ is polyhedral, the claim follows.

We prove (ii). By Proposition 3.1, the classes w_1 and w_2 generate the effective cone. Thus, $\mathcal{R}(X)$ has generators in the degrees w_1 and w_2. Note that $h^0(w_1) = 1$ and, fixing a generator $f_1 \in \mathcal{R}(X)_{w_1}$, we obtain $\mathcal{R}(X)_{nw_1} = \mathbb{C}f_1^n$. Moreover, up to a constant, f_1 occurs in any minimal system of generators of $\mathcal{R}(X)$; we fix such a system $f_1, \ldots, f_r$.

We now show that $\mathcal{R}(X)$ has generators in degree $aw_1 + w_2$ for any $1 \leqslant a \leqslant [k/2]$. By assertion (i), the class $aw_1 + w_2$ is big and nef for $1 \leqslant a \leqslant [k/2]$. Using the Riemann–Roch theorem and the Kawamata–Viehweg vanishing theorem, we obtain

$$h^0((a-1)w_1 + w_2) < h^0(aw_1 + w_2) \quad \text{for } 1 < a \leqslant \lfloor k/2 \rfloor.$$

This implies that there exists an $f \in \mathcal{R}(X)_{aw_1+w_2}$, which is not a multiple of $f_1 \in \mathcal{R}(X)_{w_1}$. The same holds for $a = 1$, because then we have ($k = 1$ implies that $a = 0$)

$$h^0(w_2) = 2, \quad h^0(w_1 + w_2) = k + 1 > 2.$$

Suppose that every monomial $m \in \mathcal{R}(X)_{aw_1+w_2}$ in the f_i is a product $m = m_1 m_2$ of non-constant m_i. Then m_1 or m_2 belongs to $\mathcal{R}(X)_{bw_1} = \mathbb{C}f_1^b$, where $1 \leqslant b \leqslant a$. Then

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every $f \in \mathcal{R}(X)_{aw_1+w_2}$ is a multiple of f_1, contradicting the previous statement. Hence, some f_i, where $i = 2, \ldots, r$, has degree $aw_1 + w_2$.

We turn to (iii). Then $kw_1 + 2w_2$ is big and nef. Reasoning as before, we obtain that there is an $f \in \mathcal{R}(X)_{kw_1+2w_2}$, which is not a multiple of f_1. Suppose that every monomial $m \in \mathcal{R}(X)_{kw_1+2w_2}$ in the f_i is a product $m = m_1 m_2$ of non-constant m_i. Then m_1 or m_2 belongs to $\mathcal{R}(X)_{bw_1+cw_2}$, where $b/c \geqslant \lfloor k/2 \rfloor + 1$. Since $(bw_1 + cw_2) \cdot w_1 = -2b + kc < 0$ holds, every element of $\mathcal{R}(X)_{bw_1+cw_2}$ is divisible by f_1; a contradiction. Hence, some f_i, where $i = 2, \ldots, r$, has degree $kw_1 + 2w_2$. □

Proposition 3.7. *Let X be a K3 surface with $\mathrm{Cl}(X) \cong \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$ and intersection form given by $w_1^2 = w_2^2 = -2$ and $w_1 \cdot w_2 = k \in \mathbb{N}$.*

(i) *We have $k \geqslant 3$ and the semiample cone $\mathrm{SAmple}(X)$ is generated by the classes $kw_1 + 2w_2$ and $2w_1 + kw_2$.*

(ii) *The Cox ring $\mathcal{R}(X)$ has generators in degrees $aw_1 + w_2$ and $w_1 + aw_2$, where $0 \leqslant a \leqslant \lfloor k/2 \rfloor$.*

(iii) *If $k > 1$ holds and k is odd, then the Cox ring $\mathcal{R}(X)$ has, in addition to those of (ii), generators in degrees $kw_1 + 2w_2$ and $2w_1 + kw_2$.*

Proof. By the Hodge index theorem, $\mathrm{Cl}(X)$ has signature $(1, 1)$, which implies that $k \geqslant 3$. Determining the semiample cone runs as in the proof of Proposition 3.6.

As to the remaining statements, note that the semigroup $\mathrm{SAmple}(X) \cap \mathbb{Z}^2$ is generated by $aw_1 + w_2$ and $w_1 + aw_2$ with $0 \leqslant a \leqslant \lfloor k/2 \rfloor$ if k is even, and by the same classes plus the two extremal rays if k is odd. Reasoning as in the proof of Proposition 3.6, we obtain

$$h^0((a-1)w_1 + w_2) < h^0(aw_1 + w_2)$$

whenever $0 \leqslant a \leqslant \lfloor k/2 \rfloor$ holds. This formula also holds when w_1 and w_2 are exchanged. Now the same arguments as used in the proof of Proposition 3.6 give the assertion. □

Example 3.8. Let X be a K3 surface with $\mathrm{Cl}(X) \cong \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$ and intersection form given by $w_1^2 = w_2^2 = -2$ and $w_1 \cdot w_2 = 3$. Then the Cox ring $\mathcal{R}(X)$ has generators

$$f_{1,0}, f_{0,1}, f_{1,1}, g_{1,1}, f_{2,3}, f_{3,2}$$

in the corresponding degrees by Proposition 3.7 and its proof. A monomial basis of $\mathrm{Sym}^3 \mathcal{R}(X)_{w_1+w_2}$, plus $f_{2,3}f_{0,1}$ and $f_{3,2}f_{1,0}$, give 12 linearly dependent elements of $\mathcal{R}(X)_{3w_1+3w_2}$, since this space has dimension 11 by the Riemann–Roch theorem. This means that $\mathcal{R}(X)$ has a relation in degree $3w_1 + 3w_2$.

Similarly, a monomial basis of $\mathrm{Sym}^5 \mathcal{R}(X)_{w_1+w_2}$, plus $f_{2,3}f_{3,2}$ and the product of $f_{2,3}f_{1,0}$ for a monomial basis of $\mathrm{Sym}^2 \mathcal{R}(X)_{w_1+w_2}$, give 28 monomials. These are linearly dependent, since the dimension of $\mathcal{R}(X)_{5w_1+5w_2}$ is 27 by the Riemann–Roch theorem. This means that $\mathcal{R}(X)$ has a relation in degree $5w_1 + 5w_2$.

We now give a geometric interpretation for generators and relations. The map $\pi\colon X \to \mathbb{P}^2$ associated with $w_1 + w_2$ is a double cover branched along a smooth plane sextic; see [Sai74]. Observe that $f_{1,0}f_{0,1} = \pi^*(s)$ and $f_{2,3}f_{3,2} = \pi^*(t)$, where $s = 0$ is a line and $t = 0$ is a quintic in $\mathbb{P}^2$. The second equality gives a relation in degree $5w_1 + 5w_2$.

The assumptions $w_i^2 \in \{0, -2\}$ made in Theorem 3.2 imply that the primitive generators of the effective cone form a basis of the divisor class group. However, the techniques of its proof

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allow us as well to treat, for example, the following case, where the primitive generators of the effective cone span a sublattice of index two in the divisor class group.

Proposition 3.9. *Let X be a K3 surface with $\mathrm{Cl}(X) \cong \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$ and intersection form given by $w_1^2 = 4$, $w_2^2 = -4$ and $w_1 \cdot w_2 = 0$.*

(i) *The effective cone of X is generated by $u_1 := w_1 + w_2$ and $u_2 := w_1 - w_2$.*

(ii) *The Cox ring $\mathcal{R}(X)$ is generated in degrees u_1, u_2 and w_1.*

(iii) *Any minimal ideal of relations of $\mathcal{R}(X)$ is generated in degree $2w_1$.*

Proof. Note that we have $u_1^2 = u_2^2 = 0$. Thus, by the Riemann–Roch theorem, we can assume that u_1 and u_2 are effective. Moreover, Theorem 2.12 tells us that $\mathrm{Eff}(X)$ is polyhedral; we denote by v_1 and v_2 its primitive generators. Then we have $v_i^2 \in 4\mathbb{Z}$ and thus Proposition 2.13 gives $v_1^2 = v_2^2 = 0$. Choosing presentations $u_i = a_i v_1 + b_i v_2$ with non-negative $a_i, b_i \in \mathbb{Q}$, we obtain

$$8 = u_1 \cdot u_2 = (a_1 v_1 + b_1 v_2) \cdot (a_2 v_1 + b_2 v_2) = (a_1 b_2 + a_2 b_1) \cdot v_1 \cdot v_2,$$
$$0 = u_i^2 = (a_i v_1 + b_i v_2)^2 = 2 a_i b_i \, v_1 \cdot v_2.$$

The first identity gives $v_1 \cdot v_2 \neq 0$ and, thus, the second one shows that $a_i b_i = 0$. As a consequence, we obtain $\{u_1, u_2\} = \{v_1, v_2\}$. This proves the first assertion.

As to the second one, note that any effective divisor class $w \in \mathrm{Cl}(X)$ can be written as

$$w = a u_1 + b u_2 + c w_1, \quad \text{where } a, b \in \mathbb{N}, c = 0, 1.$$

Observe that u_1 and u_2 are classes of elliptic curves. Moreover, we have $h^0(u_i) = 2$ and thus $\mathcal{R}(X)_{u_i}$ has a basis of the form (f_{i1}, f_{i2}).

We now proceed as in the proof of Theorem 3.2. If $a \geqslant 3$ and $b \geqslant 1$ hold, then $w - 2u_1$ is nef and big, and thus we have $h^1(w - 2u_1) = 0$. Since $u_1^2 = 0$ holds, Lemma 3.3 shows that the sections of w are polynomials in f_{11}, f_{12} and elements of $\mathcal{R}(X)_{w-u_1}$.

Iterating this procedure and interchanging the roles of a and b, we reduce to the study of classes w with $a + b \leqslant 4$. A case by case analysis now shows that $\mathcal{R}(X)$ is generated in degrees u_1, u_2 and w_1. In the following table we briefly provide the reason why $h^1(w - 2u_1) = 0$ holds, when $a \geqslant b$ and $w - 2u_1$ is not nef and big.

a	b	c	$h^1(w - 2w_1) = 0$ because
1	1	0	$\mathcal{R}(X, w_1)$ is 1-generated
1	0	1	$h^1(-u_2) = 0$
1	1	1	$\mathcal{R}(X, w_1)$ is 1-generated
2	0	0	$\mathcal{R}(X, u_1)$ is 1-generated
2	1	0	$h^1(u_2) = 0$

In a similar way, Theorem 3.5 and Lemma 3.3 imply that w is not essential unless $a = b = 1$. Since w_1 is the class of a smooth irreducible curve of genus at most one, Proposition 3.4 yields that $\mathcal{R}(X)_{w_1}$ is generated in degree one. This implies that the monomials $f_{1i} f_{2k}$ are quadratic functions in the $f_i \in \mathcal{R}(X)_{w_1}$:

$$f_{1i} f_{2k} = q_{ik}(f_0, f_1, f_2, f_3),$$

where q_{ik} is a homogeneous polynomial of degree two. This gives four independent relations in degree $(2, 0)$, as can be checked since $h^0(2w_1) = 10$ and the number of monomials of type $f_i f_j$ and $f_{1i} f_{2k}$ is 14. □

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4. Cox rings and coverings

In this section, we investigate the effect of certain, e.g. cyclic, coverings $\pi\colon X \to Y$ on the Cox ring. Among other things, we obtain that finite generation of the Cox ring is preserved, provided that $\pi^*(\mathrm{Cl}(Y))$ is of finite index in $\mathrm{Cl}(X)$; see Proposition 4.6. In the whole section, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. First, we make precise which type of coverings we will treat.

Construction 4.1. Let Y be a normal variety and $D_1, \ldots, D_r \in \mathrm{CaDiv}(Y)$ be a linearly independent collection of Cartier divisors. Denote by $M^+ \subseteq \mathrm{CaDiv}(Y)$ the semigroup generated by the divisors $D_1, \ldots, D_r$ and set

$$Y(D_1, \ldots, D_r) := \mathrm{Spec}_Y(\mathcal{A}), \quad \mathcal{A} := \bigoplus_{D \in M^+} \mathcal{O}_Y(-D).$$

Then the inclusion $\mathcal{O}_Y \to \mathcal{A}$ defines a morphism $\alpha\colon Y(D_1, \ldots, D_r) \to Y$, which is a (split) vector bundle of rank r over Y. Similarly, with $n_1, \ldots, n_r \in \mathbb{Z}_{>0}$ and $E_i := n_i D_i$, denote by $N^+ \subseteq \mathrm{CaDiv}(Y)$ the semigroup generated by $E_1, \ldots, E_r$. Setting

$$Y(E_1, \ldots, E_r) := \mathrm{Spec}_Y(\mathcal{B}), \quad \mathcal{B} := \bigoplus_{E \in N^+} \mathcal{O}_Y(-E)$$

gives a further (split) vector bundle $\beta\colon Y(E_1, \ldots, E_r) \to Y$ of rank r over Y. The inclusion $\mathcal{B} \subseteq \mathcal{A}$ defines a morphism $\kappa\colon Y(D_1, \ldots, D_r) \to Y(E_1, \ldots, E_r)$. Now, let $\sigma\colon Y \to Y(E_1, \ldots, E_r)$ be a section such that all projections of σ to the factors $Y(E_i)$ are non-trivial. Then we define

$$X := \kappa^{-1}(\sigma(Y)) \subseteq Y(D_1, \ldots, D_r).$$

Restricting α gives a morphism $\pi\colon X \to Y$, which we call an *abelian covering* of Y. Note that $\pi\colon X \to Y$ is the quotient for the action of the abelian group $\mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r\mathbb{Z}$ on X defined by the inclusion $\mathcal{B} \subseteq \mathcal{A}$ of graded algebras.

Remark 4.2. For a smooth variety Y, Cartier divisors D and $E := nD$ on Y and a section $\sigma\colon Y \to Y(E)$ with non-trivial reduced divisor B, Construction 4.1 gives a branched n-cyclic covering of Y; see [BHPV04, § 1.17].

In order to formulate our first result, we need the following pull-back construction for Weil divisors under an abelian covering $\pi\colon X \to Y$ of normal varieties. Given $D \in \mathrm{WDiv}(Y)$, consider the restriction D' of D to the set $Y' \subseteq Y$ of smooth points and take the usual pull-back $\pi^*(D')$ on $\pi^{-1}(Y')$. Since π is finite, the complement $X \setminus \pi^{-1}(Y')$ is of codimension at least two in X and hence $\pi^*(D')$ uniquely extends to a Weil divisor $\pi^*(D)$ on X.

Proposition 4.3. *Let $\pi\colon X \to Y$ be an abelian covering as in Construction 4.1, assume that X is normal and let $K \subseteq \mathrm{WDiv}(Y)$ be a subgroup containing $D_1, \ldots, D_r$ of Construction 4.1. Set*

$$\mathcal{S}_Y := \bigoplus_{D \in K} \mathcal{O}_Y(D), \quad \mathcal{S}_X := \bigoplus_{D \in K} \mathcal{O}_X(\pi^* D).$$

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Then, setting $\deg(T_i) := D_i$ *turns* $\mathcal{S}_Y[T_1, \ldots, T_r]$ *into a* K*-graded sheaf of* $\mathcal{O}_Y$*-algebras and there is a* K*-graded isomorphism of sheaves*

$$\pi_*\mathcal{S}_X \cong \mathcal{S}_Y[T_1, \ldots, T_r]/\langle T_1^{n_1} - g_1, \ldots, T_r^{n_r} - g_r\rangle,$$

where $g_i \in \Gamma(Y, \mathcal{O}(E_i))$ *are sections such that the branch divisor* B *of the covering* $\pi\colon X \to Y$ *is given as*

$$B = \operatorname{div}(g_1) + \cdots + \operatorname{div}(g_r).$$

Proof. Note that for any open set $V \subseteq Y$ and its intersection $V' := V \cap Y'$ with the set $Y' \subseteq Y$ of smooth points, the sections of $\mathcal{S}_Y$ over V and V' coincide and also the sections of $\pi_*\mathcal{S}_X$ over V and V' coincide. Hence, we may assume that $K \subseteq \operatorname{WDiv}(Y)$ consists of Cartier divisors.

A first step is to express the direct image $\pi_*\mathcal{S}_X$ in terms of $\pi_*\mathcal{O}_X$ and data living on Y. Using the projection formula, we obtain

$$\pi_*\mathcal{S}_X = \pi_* \bigoplus_{D\in K} \mathcal{O}_X(\pi^*D) \cong \bigoplus_{D\in K} \mathcal{O}_Y(D) \otimes_{\mathcal{O}_Y} \pi_*\mathcal{O}_X \cong \mathcal{S}_Y \otimes_{\mathcal{O}_Y} \pi_*\mathcal{O}_X. \tag{1}$$

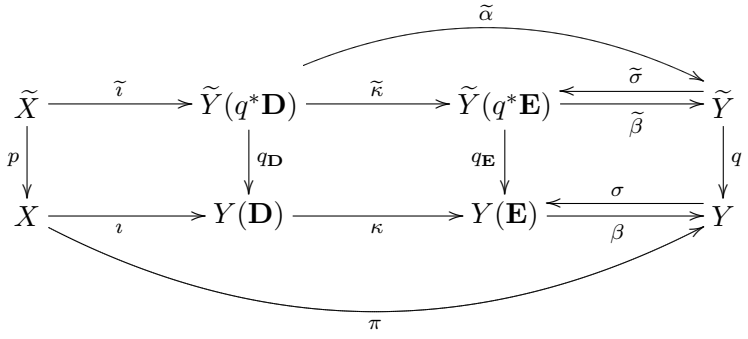
Now we have to investigate $\pi_*\mathcal{O}_X$. Denote by $q\colon \widetilde{Y} \to Y$ the torsor associated with $\mathcal{S}_Y$, i.e. we have $\widetilde{Y} = \operatorname{Spec}_Y(\mathcal{S}_Y)$. Moreover, in Construction 4.1 we constructed the rank r vector bundles

$$\alpha\colon Y(D_1, \ldots, D_r) \to Y, \quad \beta\colon Y(E_1, \ldots, E_r) \to Y.$$

Using the pull-back divisors $q^*(D_i)$ and $q^*(E_i)$, we obtain the respective pull-back bundles

$$\widetilde{\alpha}\colon \widetilde{Y}(q^*D_1, \ldots, q^*D_r) \to \widetilde{Y}, \quad \widetilde{\beta}\colon \widetilde{Y}(q^*E_1, \ldots, q^*E_r) \to \widetilde{Y}.$$

Set for short $Y(\mathbf{D}) := Y(D_1, \ldots, D_r)$ and $\widetilde{Y}(q^*\mathbf{D}) := \widetilde{Y}(q^*D_1, \ldots, q^*D_r)$. Similarly, define $Y(\mathbf{E})$ and $\widetilde{Y}(q^*\mathbf{E})$. Then we have a commutative diagram



where p, $q_{\mathbf{D}}$ and $q_{\mathbf{E}}$ are the canonical morphisms, we set $\widetilde{X} := q_{\mathbf{D}}^{-1}(X)$ and $\widetilde{\sigma} := q^*\sigma$ is the pull-back section.

Recall that $q\colon \widetilde{Y} \to Y$ is the quotient for the free action of the torus $H := \operatorname{Spec}(\mathbb{K}[K])$ defined by the grading of $q_*\mathcal{O}_{\widetilde{Y}} = \mathcal{S}_Y$. Thus, $\widetilde{Y}(\mathbf{D})$ and $\widetilde{X}$ inherit free H-actions having $q_{\mathbf{D}}$ and p as their respective quotients. Moreover, let $\widetilde{\mathcal{I}}$ denote the ideal sheaf of $\widetilde{X}$ in $\widetilde{Y}(q^*\mathbf{D})$. Then $\widetilde{\mathcal{I}}$ is homogeneous, and we have

$$\pi_*\mathcal{O}_X \cong (\pi_*p_*\mathcal{O}_{\widetilde{X}})_0 \cong (\pi_*p_*\widetilde{\imath}^*(\mathcal{O}_{\widetilde{Y}(q^*\mathbf{D})}/\widetilde{\mathcal{I}}))_0 = (q_*\widetilde{\alpha}_*(\mathcal{O}_{\widetilde{Y}(q^*\mathbf{D})}/\widetilde{\mathcal{I}}))_0. \tag{2}$$

To proceed, we need a suitable trivialization of the bundle $\widetilde{\alpha}\colon \widetilde{Y}(q^*\mathbf{D}) \to \widetilde{Y}$. For this, consider an open affine subset $V \subseteq Y$ such that on V we have $D_i = \operatorname{div}(h_{i,V})$ for $1 \leqslant i \leqslant r$.

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This gives us sections

$$\eta_{i,V} := h_{i,V}^{-1} \in \Gamma(V, \mathcal{O}_Y(D_i)) \subseteq \Gamma(q^{-1}(V), \mathcal{O}_{\widetilde{Y}})_{D_i},$$
$$q_{\mathbf{D}}^*(h_{i,V}) \in q_{\mathbf{D}}^*(\Gamma(\alpha^{-1}(V), \mathcal{O}_{Y(\mathbf{D})})) \subseteq \Gamma(\widetilde{\alpha}^{-1}(q^{-1}(V)), \mathcal{O}_{\widetilde{Y}(q^*\mathbf{D})})_0.$$

Given another open affine subset $W \subseteq Y$ such that on W we have $D_i = \operatorname{div}(h_{i,W})$ for $1 \leqslant i \leqslant r$, we obtain over $V \cap W$ for the corresponding sections:

$$\frac{\widetilde{\alpha}^*(\eta_{i,W})}{\widetilde{\alpha}^*(\eta_{i,V})} = \widetilde{\alpha}^* q^* \left(\frac{\eta_{i,W}}{\eta_{i,V}}\right) = q_{\mathbf{D}}^* \alpha^* \left(\frac{h_{i,V}}{h_{i,W}}\right) = \frac{q_{\mathbf{D}}^*(h_{i,V})}{q_{\mathbf{D}}^*(h_{i,W})}.$$

Covering Y with V's as above, we obtain that the functions $\widetilde{\alpha}^*(\eta_{i,V}) \cdot q_{\mathbf{D}}^*(h_{i,V})$ living on $\widetilde{\alpha}^{-1}(q^{-1}(V))$ glue together to a global regular function f_i of degree D_i on $\widetilde{Y}(q^*D_i)$ generating $\mathcal{O}_{\widetilde{Y}}(q^*D_i)$ over $\mathcal{O}_{\widetilde{Y}}$. Thus, the f_i define a trivialization,

$$\begin{array}{ccccccc}
 & & \widetilde{Y} \times \mathbb{K}^r & \xrightarrow{(\widetilde{y},z)\mapsto(\widetilde{y},z^n)} & \widetilde{Y} \times \mathbb{K}^r & \xleftarrow{\quad\mathrm{id}\times g\quad} & \\
 & & \widetilde{\alpha}\times f \uparrow \cong & & \widetilde{\beta}\times f^n \uparrow \cong & \searrow^{\mathrm{pr}_{\widetilde{Y}}} & \\
\widetilde{X} & \xrightarrow{\widetilde{\imath}} & \widetilde{Y}(q^*\mathbf{D}) & \xrightarrow[\widetilde{\kappa}]{} & \widetilde{Y}(q^*\mathbf{E}) & \underset{\widetilde{\beta}}{\overset{\widetilde{\sigma}}{\rightleftarrows}} & \widetilde{Y}
\end{array}$$

where we write z for $(z_1, \ldots, z_r)$ and z^n for $(z_1^{n_1}, \ldots, z_r^{n_r})$, etc. Since $\widetilde{\sigma} = q^*\sigma$ is H-equivariant, each component g_i of g is homogeneous with $\deg(g_i) = E_i$. Note that the divisors $\operatorname{div}(g_i)$ describe the branch divisor, as claimed.

Denote by $\widetilde{\mathcal{J}} \subseteq \mathcal{O}_{\widetilde{Y}}[T_1, \ldots, T_r]$ the ideal sheaf of the image of $\widetilde{X}$ in $\widetilde{Y} \times \mathbb{K}^r$. Then, using $\widetilde{X} = \widetilde{\kappa}^{-1}(\widetilde{\sigma}(\widetilde{Y}))$, we obtain

$$\widetilde{\mathcal{I}} = \langle f_1^{n_1} - \widetilde{\alpha}^* g_1, \ldots, f_r^{n_r} - \widetilde{\alpha}^* g_r \rangle, \quad \widetilde{\mathcal{J}} = \langle T_1^{n_1} - g_1, \ldots, T_r^{n_r} - g_r \rangle.$$

Thus, using the isomorphism $q_* \widetilde{\alpha}_* \mathcal{O}_{\widetilde{Y}(q^*\mathbf{D})} \cong \mathcal{S}_Y[T_1, \ldots, T_r]$ established by the above commutative diagram, we may continue Equations (2) as

$$\pi_* \mathcal{O}_X \cong (\mathcal{S}_Y[T_1, \ldots, T_r]/\widetilde{\mathcal{J}})_0 \cong \mathcal{S}_Y[T_1, \ldots, T_r]_0/\widetilde{\mathcal{J}}_0. \tag{3}$$

The homogeneous ideal sheaf $\widetilde{\mathcal{I}}$ is locally, over Y, generated in degree zero in the sense that we have $\widetilde{\mathcal{I}} = \widetilde{\alpha}^* \mathcal{O}_{\widetilde{Y}} \cdot \widetilde{\mathcal{I}}_0$. The same holds for $\widetilde{\mathcal{J}}$, and we obtain

$$\pi_* \mathcal{S}_X \cong \mathcal{S}_Y \otimes_{\mathcal{O}_Y} \pi_* \mathcal{O}_X \cong \mathcal{S}_Y \otimes_{\mathcal{O}_Y} \mathcal{S}_Y[T_1, \ldots, T_r]_0/\widetilde{\mathcal{J}}_0 \cong \mathcal{S}_Y[T_1, \ldots, T_r]/\widetilde{\mathcal{J}}. \qquad \square$$

Proposition 4.4. *Consider a normal variety X, a finitely generated subgroup $K \subseteq \mathrm{WDiv}(X)$ mapping onto $\mathrm{Cl}(X)$, a subgroup $L \subseteq K$ and the algebras*

$$R := \bigoplus_{D \in K} \Gamma(X, \mathcal{O}_X(D)), \quad A := \bigoplus_{D \in L} \Gamma(X, \mathcal{O}_X(D)).$$

If the subgroup $L \subseteq K$ is of finite index and the algebra A is finitely generated, then the algebra R is also finitely generated.

Lemma 4.5. *Let X be a normal variety, K be a finitely generated abelian group, $\mathcal{R}$ be a quasicoherent sheaf of normal K-graded $\mathcal{O}_X$-algebras and Z be its relative spectrum,*

$$\mathcal{R} = \bigoplus_{w \in K} \mathcal{R}_w, \quad Z := \operatorname{Spec}_X(\mathcal{R}),$$

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where we assume $\mathcal{R}$ to be locally of finite type. Let $L \subseteq K$ be a subgroup of finite index and consider the associated Veronese subalgebra of the algebra of global sections

$$A := \bigoplus_{w \in L} \Gamma(X, \mathcal{R}_w) \subseteq \bigoplus_{w \in K} \Gamma(X, \mathcal{R}_w) =: R.$$

Suppose that there are homogeneous sections $f_1, \ldots, f_r \in A$ such that each R_{f_i} is finitely generated, each $Z_{f_i} = Z \setminus V(Z, f_i)$ is an affine variety and we have

$$Z = Z_{f_1} \cup \cdots \cup Z_{f_r}.$$

If A is finitely generated and for $Y := \operatorname{Spec}(A)$ the complement $Y \setminus (Y_{f_1} \cup \cdots \cup Y_{f_r})$ is of codimension at least two in Y, then R is finitely generated.

Proof. First note that Z is a variety with $\Gamma(Z, \mathcal{O}) = R$ and that $\Gamma(Z_{f_i}, \mathcal{O}) = R_{f_i}$ holds. Since each R_{f_i} is finitely generated, we may construct a finitely generated K-graded subalgebra $S \subseteq R$ with

$$A \subseteq S, \quad S_{f_i} = R_{f_i} \text{ for } 1 \leqslant i \leqslant r.$$

Set $Z' := \operatorname{Spec}(S)$. Then the inclusion $S \subseteq R$ defines a canonical morphism $\imath\colon Z \to Z'$. Moreover, S is canonically graded by the factor group K/L and hence Z' comes with an action of the finite abelian group $G := \operatorname{Spec}(\mathbb{K}[K/L])$. The inclusion $A \subseteq S$ defines the quotient map $\pi\colon Z' \to Y$ for the action of G. By construction, we have

$$Z_{f_i} = \imath^{-1}(Z'_{f_i}) \cong Z'_{f_i}, \quad Z'_{f_i} = \pi^{-1}(Y_{f_i}).$$

Consequently, $\imath\colon Z \to Z'$ is an open embedding, and we may regard Z as a subset of Z'. By our assumptions, setting $V := Y_{f_1} \cup \cdots \cup Y_{f_r}$, we obtain $Z = \pi^{-1}(V)$. Since $\pi\colon Z' \to Y$ is a finite map, we can conclude that

$$\dim(Z' \setminus Z) = \dim(Y \setminus V) \leqslant \dim(Y) - 2 = \dim(Z') - 2.$$

Let $Z'' \to Z'$ be the normalization. Since Z is normal, we have $Z \subseteq Z''$. We conclude that $R = \Gamma(Z, \mathcal{O}) = \Gamma(Z'', \mathcal{O})$ holds, and thus R is finitely generated. □

Proof of Proposition 4.4. First note that the rings R and A do not change if we replace X with the set of its smooth points. Thus, we may assume that X is smooth. Then we obtain graded sheaves of normal $\mathcal{O}_X$-algebras

$$\mathcal{R} = \bigoplus_{w \in K} \mathcal{R}_w, \quad \mathcal{A} = \bigoplus_{w \in L} \mathcal{R}_w,$$

which are locally of finite type. Our task is to verify the assumptions of Lemma 4.5 for $\mathcal{R}$ and $A = \Gamma(X, \mathcal{A})$. Setting $Z := \operatorname{Spec}_X(\mathcal{R})$ and $\widetilde{X} := \operatorname{Spec}_X(\mathcal{A})$, we obtain normal varieties, and we have a commutative diagram of affine morphisms

$$\begin{array}{ccc} Z & \xrightarrow{\kappa} & \widetilde{X} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array}$$

where κ is the quotient for the action of the finite abelian group $G := \operatorname{Spec}(\mathbb{K}[K/L])$ on Z defined by the canonical (K/L)-grading of $\mathcal{R}$. Moreover, we have an affine variety $Y := \operatorname{Spec}(A)$, and there is a canonical morphism $\imath\colon \widetilde{X} \to Y$.

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To obtain the desired sections $f_i \in A$, cover X by affine open subsets $U_1, \ldots, U_r$. Then each $X\setminus U_i$ is the set of zeroes of a suitable homogeneous section $f_i \in R_i$. Replacing f_i with a suitable power, we achieve $f_i \in A_i$. Thus, we can cover $\widetilde{X}$ by the affine open subsets

$$q^{-1}(U_i) = \widetilde{X}_{f_i} = \imath^{-1}(Y_{f_i});$$

use e.g. [Hau08, Lemma 2.3]. Now, $\Gamma(\widetilde{X}, \mathcal{O}) = A = \Gamma(Y, \mathcal{O})$ implies that $\widetilde{X}_{f_i} \cong Y_{f_i}$. Thus, each restriction $\imath\colon \widetilde{X}_{f_i} \to Y_{f_i}$ is an isomorphism and $\imath\colon \widetilde{X} \to Y$ is an open embedding. Moreover, using again $\Gamma(\widetilde{X}, \mathcal{O}) = \Gamma(Y, \mathcal{O})$, we see that we have a small complement

$$Y\setminus\widetilde{X} = Y\setminus(\widetilde{X}_{f_1} \cup \cdots \cup \widetilde{X}_{f_r}) = Y\setminus(Y_{f_1} \cup \cdots \cup Y_{f_r}). \qquad \square$$

Proposition 4.6. *Let $\pi\colon X \to Y$ be an abelian covering of normal varieties with finitely generated free divisor class groups such that $\pi^*(\mathrm{Cl}(Y))$ is of finite index in $\mathrm{Cl}(X)$. Then the following statements are equivalent.*

(i) *The Cox ring $\mathcal{R}(X)$ is a finitely generated $\mathbb{K}$-algebra.*

(ii) *The Cox ring $\mathcal{R}(Y)$ is a finitely generated $\mathbb{K}$-algebra.*

Proof. Let $M \subseteq \mathrm{WDiv}(Y)$ and $K \subseteq \mathrm{WDiv}(X)$ be subgroups mapping isomorphically to the respective divisor class groups $\mathrm{Cl}(Y)$ and $\mathrm{Cl}(X)$. Then the Cox rings are given as

$$\mathcal{R}(Y) = \bigoplus_{E\in M} \Gamma(Y, \mathcal{O}_Y(E)), \quad \mathcal{R}(X) = \bigoplus_{D\in K} \Gamma(X, \mathcal{O}_X(D)).$$

Since $\mathrm{Cl}(Y)$ is free and $\pi\colon X \to Y$ is the quotient for a finite group action, the pull-back $\pi^*\colon \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$ is injective; see [Ful98, Example 1.7.6]. Consequently, there are a unique subgroup $L \subseteq K$ and an isomorphism $\pi^*(M) \to L$ inducing the identity on $\pi^*(\mathrm{Cl}(Y))$. By our assumption, L is of finite index in K. Moreover, we have canonical identifications

$$\mathcal{R}(Y) \subseteq \bigoplus_{E\in M} \Gamma(X, \mathcal{O}_X(\pi^*(E))) = S := \bigoplus_{D\in L} \Gamma(X, \mathcal{O}_X(D)) \subseteq \mathcal{R}(X).$$

Suppose that $\mathcal{R}(X)$ is finitely generated over $\mathbb{K}$. Then the Veronese subalgebra $S \subseteq \mathcal{R}(X)$ is also finitely generated over $\mathbb{K}$. Moreover, by Proposition 4.3, the algebra S is a finite module over $\mathcal{R}(Y)$. Thus, the tower $\mathbb{K} \subseteq \mathcal{R}(Y) \subseteq S$ fulfills the assumptions of the Artin–Tate lemma [AM69, Proposition 7.8], and we obtain that $\mathcal{R}(Y)$ is a finitely generated $\mathbb{K}$-algebra.

Now let $\mathcal{R}(Y)$ be finitely generated over $\mathbb{K}$. Then Proposition 4.3 tells us that S is finitely generated over $\mathbb{K}$. Thus, Proposition 4.4 shows that $\mathcal{R}(X)$ is finitely generated over $\mathbb{K}$. $\square$

5. Cox rings and blowing up

In this section, we compute the Cox ring of the fourth Hirzebruch surface blown up at three general points. As in the preceding section, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. We use the technique of toric ambient modifications provided in [Hau08], and begin with giving a short outline of this technique. A basic ingredient is the following construction of the Cox ring and the universal torsor of a toric variety given in [Cox95].

Construction 5.1. Let Z be the toric variety arising from a complete fan Σ in a lattice N, and suppose that the primitive generators $v_1, \ldots, v_r$ of Σ span the lattice N. Then we have mutually

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dual exact sequences

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^r \xrightarrow{P\colon e_i \mapsto v_i} N \longrightarrow 0,$$

$$0 \longleftarrow K \underset{Q}{\longleftarrow} \mathbb{Z}^r \longleftarrow M \longleftarrow 0,$$

where the lattice K is isomorphic to the divisor class group $\mathrm{Cl}(Z)$. The Cox ring of Z is the polynomial ring $\mathcal{R}(Z) = \mathbb{K}[T_1, \ldots, T_r]$ with the K-grading defined by $\deg(T_i) := Q(e_i)$. Moreover, denoting by $\delta \subseteq \mathbb{Q}^r$ the positive orthant, we obtain a fan in $\mathbb{Z}^r$ consisting of certain faces of δ, namely

$$\widehat{\Sigma} := \{\widehat{\sigma} \preceq \delta;\, P(\widehat{\sigma}) \subseteq \sigma \text{ for some } \sigma \in \Sigma\}.$$

The associated toric variety $\widehat{Z}$ is an open toric subvariety of $\overline{Z} := \mathbb{K}^r$. The toric morphism $p\colon \widehat{Z} \to Z$ defined by $P\colon \mathbb{Z}^r \to N$ is a universal torsor; it is a quotient for the action of the torus $\mathrm{Spec}(\mathbb{K}[K])$ on $\overline{Z}$ defined by the K-grading of $\mathcal{R}(Z)$.

Given a variety X_0, the rough idea of [Hau08] is to work with a suitable embedding $X_0 \subseteq Z_0$ into a toric variety, consider the proper transform $X_1 \subseteq Z_1$ under suitable toric modifications $Z_1 \to Z_0$ and then compute the Cox ring $\mathcal{R}(X_1)$ in terms of $\mathcal{R}(X_0)$ using the toric universal torsors over Z_1 and Z_0. In our outline, we restrict to the case of blowing up a smooth projective surface X_0 with divisor class group $K_0 \cong \mathbb{Z}^{k_0}$ and a Cox ring, which admits a representation

$$\mathcal{R}(X_0) = \mathbb{K}[T_1, \ldots, T_r]/\langle f_0 \rangle, \quad \deg(T_i) = w_i \in K_0,$$

where $f_0 \in \mathbb{K}[T_1, \ldots, T_r]$ is a homogeneous polynomial and the T_i define pairwise non-associated prime elements in $\mathcal{R}(X_0)$. We use this presentation to embed X_0 into a toric variety. First note that we have mutually dual sequences

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^r \xrightarrow{Q_0\colon e_i \mapsto w_i} K_0, \longrightarrow 0$$

$$0 \longleftarrow N \underset{P_0}{\longleftarrow} \mathbb{Z}^r \longleftarrow L_0 \longleftarrow 0.$$

Consider any complete simplicial fan Σ_0 in N having the images $v_i := P_0(e_i) \in N$ of the canonical base vectors $e_i \in \mathbb{Z}^r$ as the generators of its rays. Let $\widehat{\Sigma}_0$ be the fan consisting of faces of the positive orthant $\delta \subseteq \mathbb{Q}^r$ provided by Construction 5.1. Then the toric variety $\widehat{Z}_0$ is an open toric subvariety of $\overline{Z}_0 := \mathbb{K}^r$, and the toric morphism $p_0\colon \widehat{Z}_0 \to Z_0$ defined by $P_0\colon \mathbb{Z}^r \to N$ is a universal torsor. Moreover, setting

$$\overline{X}_0 := V(\overline{Z}_0, f_0), \quad \widehat{X}_0 := \overline{X}_0 \cap \widehat{Z}_0,$$

we obtain $X_0 \cong p_0(\widehat{X}_0)$, which allows us to view X_0 as a closed subvariety of the toric variety Z_0; see [Hau08, Proposition 3.14]. Note that our freedom in choosing the fan Σ_0 essentially relies on the assumption that X_0 is a surface; in general one has to proceed more carefully, as X_0 and $p_0(\widehat{X}_0)$ may differ by a small birational transformation.

Now we perform a toric modification. Suppose that for some $d \geqslant 2$ the cone σ_0 spanned by $v_1, \ldots, v_d$ belongs to Σ_0 and that we have a primitive lattice vector

$$v_\infty := v_1 + \cdots + v_d \in N.$$

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Recall that σ_0 corresponds to a toric orbit $T_0 \cdot z_0 \subseteq Z_0$. Moreover, the stellar subdivision Σ_1 of Σ_0 at v_∞ defines a modification $Z_1 \to Z_0$ of toric varieties having the closure of the toric orbit $T_0 \cdot z_0 \subseteq Z_0$ as its center. Then we have commutative diagrams

$$\begin{array}{ccc} \overline{Z}_1 & \xrightarrow{\overline{\pi}} & \overline{Z}_0 \\ \uparrow & & \uparrow \\ \widehat{Z}_1 & \longrightarrow & \widehat{Z}_0 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_0} \\ Z_1 & \xrightarrow[\pi]{} & Z_0 \end{array} \qquad \begin{array}{ccc} \overline{X}_1 & \longrightarrow & \overline{X}_0 \\ \uparrow & & \uparrow \\ \widehat{X}_1 & \longrightarrow & \widehat{X}_0 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_0 \end{array}$$

where $p_1 \colon \widehat{Z}_1 \to Z_1$ denotes the toric universal torsor, $X_1 \subseteq Z_1$ the proper transform of $X_0 \subseteq Z_0$ and we write $\widehat{X}_1 = p_1^{-1}(X_1)$ for the inverse image and $\overline{X}_1$ for the closure of $\widehat{X}_1$ in $\overline{Z}_1 = \mathbb{K}^{r+1}$. Note that we have

$$\overline{\pi}(z_1, \ldots, z_\infty) = (z_1 z_\infty, \ldots, z_d z_\infty, z_{d+1}, \ldots, z_r)$$

for the lifting $\overline{\pi} \colon \overline{Z}_1 \to \overline{Z}_0$ of the toric modification $\pi \colon Z_1 \to Z_0$ to the total coordinate spaces; see [Hau08, Lemma 5.3]. Moreover, $p_1 \colon \widehat{Z}_1 \to Z_1$ defines another pair of dual sequences

$$0 \longrightarrow L_1 \longrightarrow \mathbb{Z}^{r+1} \xrightarrow{P_1 \colon e_i \mapsto v_i} N \longrightarrow 0,$$

$$0 \longleftarrow K_1 \underset{Q_1}{\longleftarrow} \mathbb{Z}^{r+1} \longleftarrow M \longleftarrow 0.$$

Now, the basic observation is that under some mild assumptions, $\overline{X}_1$ is the total coordinate space of X_1 and the explicit description of $\overline{\pi}$ given above enables us to compute moreover the Cox ring. For the precise statement, consider the $\mathbb{Z}_{\geqslant 0}$-grading of $\mathbb{K}[T_1, \ldots, T_r]$ given by

$$\deg(T_i) := \begin{cases} 1, & 1 \leqslant i \leqslant d, \\ 0, & d+1 \leqslant i \leqslant r. \end{cases}$$

Then we can write $f_0 = g_{k_0} + \cdots + g_{k_m}$ with g_{k_i} homogeneous of degree $k_i \in \mathbb{Z}_{\geqslant 0}$ and $k_0 < \cdots < k_m$. We call $f_0 \in \mathbb{K}[T_1, \ldots, T_r]$ *admissible*, if g_{k_0} is an irreducible polynomial in at least two variables and, moreover, $\overline{X}_0 = V(f_0)$ intersects the toric orbit $0 \times \mathbb{T}^{r-d}$ of $\mathbb{K}^r$. Then [Hau08, Proposition 7.2] says the following.

Proposition 5.2. *Suppose that the polynomial $f_0 \in \mathbb{K}[T_1, \ldots, T_r]$ is admissible. Then the proper transform X_1 has $\overline{X}_1$ as its total coordinate space, and the Cox ring of X_1 is given as*

$$\mathcal{R}(X_1) = \mathbb{K}[T_1, \ldots, T_r, T_\infty]/\langle f_1 \rangle, \quad f_1 := \frac{f_0(T_1 T_\infty, \ldots, T_d T_\infty, T_{d+1}, \ldots, T_r)}{T_\infty^a},$$

where $a \in \mathbb{N}$ is maximal such that f_1 stays a polynomial. The divisor class group of X_1 is given by $\mathrm{Cl}(X_1) \cong K_1$ and the degrees of the variables $T_1, \ldots, T_r, T_\infty$ are given by $\deg(T_i) = Q_1(e_i)$.

We turn to the fourth Hirzebruch surface $\mathbb{F}_4$. If $q \colon \mathbb{F}_4 \to \mathbb{P}_1$ denotes the bundle projection, we write $C_1, C_2, C_3, C_5 \subseteq \mathbb{F}_4$ for the section at infinity, the fiber $q^{-1}(0)$, the fiber $q^{-1}(\infty)$ and the zero section, respectively. As a toric variety, $\mathbb{F}_4$ arises from the fan

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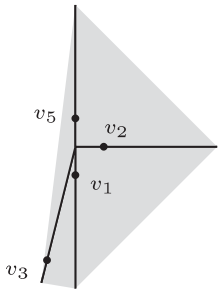


and the curves C_1, C_2, C_3, C_5 are the toric curves corresponding to the rays through v_1, v_2, v_3, v_5, respectively. In the following, we consider blow ups of $\mathbb{F}_4$ and we will denote any proper transform of some C_i again by C_i.

Proposition 5.3. *Let X be the blow up of $\mathbb{F}_4$ at points $c_0, c_\infty, c_1 \in \mathbb{F}_4 \backslash C_1$, no two of them lying in a common fiber of $q\colon \mathbb{F}_4 \to \mathbb{P}_1$, and let $C_4 \subseteq X$ be the exceptional divisor over c_0. Then X is a smooth surface with*

$$\mathrm{Cl}(X) \cong \mathbb{Z} \cdot w_1 \oplus \cdots \oplus \mathbb{Z} \cdot w_5,$$

where $w_i \in \mathrm{Cl}(X)$ denotes the class of the curve $C_i \subseteq X$. Moreover, the Cox ring of X is the polynomial ring

$$\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_8] / \langle T_2T_4 + T_3T_6 + T_7T_8 \rangle$$

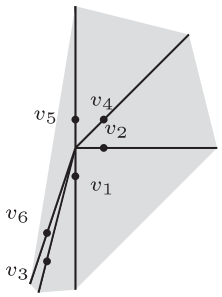
and, with respect to the basis $(w_1, \ldots, w_5)$ of $\mathrm{Cl}(X)$, the degree of the generator $T_i \in \mathcal{R}(X)$ is the ith column of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}.$$

Proof. We first reduce to the case that our three points c_0, c_1, c_∞ belong to the zero section $C_5 \subseteq \mathbb{F}_4$ and within that c_0, c_∞ are toric fixed points, whereas c_1 is the distinguished point of the non-trivial toric orbit; note that C_5 is the closure of the toric orbit corresponding to $\mathrm{cone}(v_5)$.

We proceed in two steps. First choose an automorphism of $\mathbb{F}_4$ that moves c_0, c_1 and c_∞ into the fibers over $0, 1$ and ∞, respectively. Then construct a section $s\colon \mathbb{P}_1 \to \mathbb{F}_4$ mapping $0, 1$ and ∞ to c_0, c_1 and c_∞, and apply the automorphism $x \mapsto x - s(\pi(x))$, where $\pi\colon \mathbb{F}_4 \to \mathbb{P}_1$ denotes the projection.

Blowing up the Hirzebruch surface $\mathbb{F}_4$ at the points c_0 and c_∞ gives a toric variety X_0; its fan looks as follows.



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The matrix P having $v_1, \ldots, v_6$ as its columns defines a surjection $\mathbb{Z}^6 \to \mathbb{Z}^2$ and hence an exact sequence. As a matrix for the projection $\mathbb{Z}^6 \to \mathbb{Z}^4$ in the dual sequence, we may take

$$Q := \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix}.$$

Assigning to $T_1, \ldots, T_6$ the columns $w_1, \ldots, w_6$ of Q as their degrees, we obtain an action of the four torus $\mathbb{T}^4 = (\mathbb{K}^*)^4$ on $\mathbb{K}^6$. Note that X_0 is obtained as the GIT quotient of the set of $\mathbb{T}^4$-semistable points associated with the $\mathbb{T}^4$-linearization of the trivial bundle given by the weight $(3, 11, 2, 10) \in \mathbb{Z}^4$; in fact, by [Hau08, Corollary 4.3], we could take any weight from the relative interior of the moving cone inside the moving cone

$$\mathrm{Mov}(X) = \bigcap_{i=1}^{6} \mathrm{cone}(w_j; j \neq i).$$

In order to blow up the point $c_1 \in C_5$, we first embed X_0 in a suitable toric variety Z_0. Consider the polynomial ring $\mathbb{K}[T_1, \ldots, T_7]$ with the $\mathbb{Z}^4$-grading given by

$$\deg(T_i) := w_i \text{ for } 1 \leqslant i \leqslant 6, \quad \deg(T_7) := (0, 1, 0, 1),$$

and let Q_0 denote the matrix having these degrees as its columns. Then we have a surjection $\mathbb{K}[T_1, \ldots, T_7] \to \mathbb{K}[T_1, \ldots, T_6]$ of $\mathbb{Z}^4$-graded rings, defined by

$$T_i \mapsto T_i \text{ for } 1 \leqslant i \leqslant 6, \quad T_7 \mapsto T_2T_4 - T_3T_6.$$

This gives a $\mathbb{T}^4$-equivariant embedding of $\overline{X}_0 := \mathbb{K}^6$ into $\overline{Z}_0 := \mathbb{K}^7$. Note that the vanishing ideal of $\overline{X}_0$ in $\overline{Z}_0$ is generated by the polynomial

$$f_0 := T_7 - T_2T_4 + T_3T_6.$$

Consider the linearization of the trivial bundle on $\overline{Z}_0 = \mathbb{K}^7$ given by the weight $(3, 11, 2, 10) \in \mathbb{Z}^4$. Then the corresponding set of semistable points $\widehat{Z}_0 \subseteq \overline{Z}_0$ is an open toric subvariety. The quotient $Z_0 := \widehat{Z}_0/\mathbb{T}^4$ is a smooth projective toric variety; its fan Σ_0 has the columns $v_1', \ldots, v_7'$ of the matrix

$$P_0 = \begin{bmatrix} 0 & -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 1 \\ -1 & -2 & -1 & -1 & 1 & 0 & -1 \end{bmatrix}$$

corresponding to Q_0 as the primitive generators of its rays, and the 10 maximal cones of the fan Σ_0 are given as

$$\mathrm{cone}(v_1', v_2', v_3'), \mathrm{cone}(v_1', v_2', v_7'), \mathrm{cone}(v_1', v_3', v_7'), \mathrm{cone}(v_2', v_3', v_6'), \mathrm{cone}(v_2', v_4', v_6'),$$

$$\mathrm{cone}(v_2', v_4', v_7'), \mathrm{cone}(v_3', v_6', v_7'), \mathrm{cone}(v_4', v_5', v_6'), \mathrm{cone}(v_4', v_5', v_7'), \mathrm{cone}(v_5', v_6', v_7').$$

The closed embedding $\overline{X}_0 \subseteq \overline{Z}_0$ induces a closed embedding $X_0 \to Z_0$ of the quotient spaces, and this is a neat embedding in the sense of [Hau08]; see [Hau08, Proposition 3.14]. By our choice of the embedding, the curve C_5 intersects the toric orbit corresponding to $\mathrm{cone}(v_5', v_7') \in \Sigma_0$ exactly in the point c_1.

Moreover, the polynomial f_0 is admissible, the g_{k_0}-term is just $T_2T_4 + T_3T_6$ and thus we may perform a toric ambient blow up $Z_1 \to Z_0$ at the toric orbit closure corresponding to

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$\mathrm{cone}(v_5', v_7') \in \Sigma_0$. Adding the column $v_5' + v_7' = (0,0,1)$ to the matrix P_0 gives the matrix P_1 describing the quotient presentation $\widehat{Z}_1 \to Z_1$.

With the proper transform $X_1 \subseteq Z_1$ we obtain a modification $X_1 \to X_0$ of X_0 centered at the point $c_1 \in X_0$. According to Proposition 5.2, the Cox ring of X_1 is given as

$$\mathcal{R}(X_1) = \mathbb{K}[T_1, \ldots, T_8]/\langle T_7T_8 - T_2T_4 + T_3T_6\rangle,$$

where the $\mathbb{Z}^5$-grading assigns to the generator T_i the ith column of the matrix Q_1 corresponding to P_1; a direct computation shows that Q_1 is the matrix given in the assertion.

We still have to check that $X_1 = X$ holds; that means that the modification $X_1 \to X_0$ is indeed a blow up of the point $c_1 \in X_0$. For this, note first that, according to [Hau08, Corollary 4.13], the variety X_1 inherits smoothness from its toric ambient variety Z_1. Secondly, the exceptional curve over c_1 is smooth and rational, and thus it must be a (-1)-curve. □

6. K3 surfaces with a non-symplectic involution

We now take a closer look at (complex algebraic) K3 surfaces X admitting a non-symplectic involution, i.e. an automorphism $\sigma\colon X \to X$ of order two such that $\sigma^*\omega_X = -\omega_X$ holds, where ω_X is a non-zero holomorphic 2-form of X. Since $\mathrm{Cl}(X) = H^2(X,\mathbb{Z}) \cap \omega_X^{\perp}$ holds, and σ is non-symplectic, one has

$$L^\sigma := \{u \in H^2(X,\mathbb{Z});\, \sigma^*(u) = u\} \subseteq \mathrm{Cl}(X)$$

for the fixed lattice. The K3 surface X is called *generic* if $\mathrm{Cl}(X) = L^\sigma$ holds: for fixed $\mathrm{Cl}(X) = L^\sigma$, these K3 surfaces form a family of dimension $20 - \mathrm{rk}(L^\sigma)$; see [Nik83]. Our aim is to determine the Cox ring for the generic K3 surfaces with Picard number $2 \leqslant \varrho(X) \leqslant 5$, see Propositions 6.5–6.8, and for those that are generic double covers of del Pezzo surfaces, see Proposition 6.9.

For any K3 surface with a non-symplectic involution $\sigma\colon X \to X$, one has a quotient surface $Y := X/\langle\sigma\rangle$ and the quotient map $\pi\colon X \to Y$. We will use the following basic facts.

Proposition 6.1. *Let X be a generic K3 surface with a non-symplectic involution $\sigma\colon X \to X$. Then the quotient map $\pi\colon X \to Y$ is a double cover and:*

(i) *if $\pi\colon X \to Y$ is unramified then the quotient surface Y is an Enriques surface;*

(ii) *if $\pi\colon X \to Y$ is ramified, then Y is a smooth rational surface and the following statements hold:*

  (a) *the branch divisor $B \in \mathrm{WDiv}(Y)$ of π is smooth and, denoting by K_Y the canonical divisor of Y, we have*

$$\pi^*(B) = 2\pi^{-1}(B), \quad B \sim -2K_Y;$$

  (b) *the pull-back $\pi^*\colon \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$ is injective and $\pi^*(\mathrm{Cl}(Y))$ is of index 2^{n-1} in $\mathrm{Cl}(X)$, where n is the number of components of B.*

Proof. The fact that $\pi\colon X \to Y$ is a double cover and (i) as well as (ii) up to part (b) are known; see [Zha98, Lemma 1.2]. In order to show part (b) of (ii), note first that $\mathrm{Cl}(Y)$ is free, because Y arises by blowing up points from $\mathbb{P}_2$ or a Hirzebruch surface. According to [Ful98, Ex. 1.7.6], we have

$$\pi^*(\mathrm{Cl}_\mathbb{Q}(Y)) = \mathrm{Cl}_\mathbb{Q}(X)^\sigma, \quad \pi_*\pi^*(\mathrm{Cl}(Y)) = 2\mathrm{Cl}(Y).$$

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Since X is generic, the first equation tells us that $\pi^*(\mathrm{Cl}(Y))$ is of finite index in $\mathrm{Cl}(X)$. The second one shows that π^* is injective. Moreover, by [BHPV04, Lemma 2.1], we have

$$[\pi^*(\mathrm{Cl}(Y)) : \mathrm{Cl}(X)]^2 = \frac{\det \pi^*(\mathrm{Cl}(Y))}{\det \mathrm{Cl}(X)}.$$

Since $\mathrm{Cl}(Y)$ is unimodular, see [BHPV04], the numerator equals $2^{\varrho(Y)}$. By [Nik83, Theorem 4.2.2], the lattice $L^\sigma = \mathrm{Cl}(X)$ has determinant 2^l and the difference $\varrho(Y) - l$ equals $2(n-1)$, where n is the number of connected components of the branch divisor. □

Lemma 6.2. *Let X be a generic K3 surface with a non-symplectic involution such that the associated double cover $\pi\colon X \to Y$ has branch divisor $B = C_1 + C_B$, where $C_1 \subseteq Y$ is a smooth rational curve and $C_B \subseteq Y$ is an irreducible curve.*

(i) *Let $w_1 \in \mathrm{Cl}(Y)$ be the class of $C_1 \subseteq Y$. Then $(w_1, w_2, \ldots, w_r)$ is a basis of $\mathrm{Cl}(Y)$ if and only if $(\pi^*(w_1)/2, \pi^*(w_2), \ldots, \pi^*(w_r))$ is a basis of $\mathrm{Cl}(X)$.*

(ii) *With respect to bases as in (i), the homomorphism $\pi^*\colon \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$ is given by the matrix*

$$A := \begin{bmatrix} 2 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}.$$

(iii) *Let $C \subseteq X$ be any smooth rational curve and let $w \in \mathrm{Cl}(X)$ be its class. Then precisely one of the following statements holds:*

(a) *$\pi(C)$ is a component of B and $\pi(C)^2 = -4$;*
(b) *$w \in \pi^*(\mathrm{Cl}(Y))$ and $\pi(C)^2 = -1$.*

Proof. Since $\pi^*\colon \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$ is injective, $(w_1, \ldots, w_r)$ is a basis of $\mathrm{Cl}(Y)$ if and only if $(\pi^*(w_1), \ldots, \pi^*(w_r))$ is a basis of $\pi^*(\mathrm{Cl}(Y))$. By Proposition 6.1(ii), we have $\pi^*(w_1) = 2u_1$ with some $u_1 \in \mathrm{Cl}(X)$. Moreover, also by Proposition 6.1(ii), the pull-back $\pi^*(\mathrm{Cl}(Y))$ is of index two in $\mathrm{Cl}(X)$. This gives (i) and (ii).

To prove (iii), let $w \in \mathrm{Cl}(X)$ denote the class of $C \subseteq X$. The adjunction formula and the Riemann–Roch theorem give $w^2 = -2$ and $h^0(w) = 1$. Since the elements of $\mathrm{Cl}(X)$ are fixed under the involution $\sigma\colon X \to X$, we can conclude that $\sigma(C) = C$. If $\sigma = \mathrm{id}$ holds on C, then C is contained in the ramification divisor. By Proposition 6.1(ii), we have $2C = \pi^{-1}(\pi(C))$, which implies that $\pi(C)^2 = -4$. If $\sigma \neq \mathrm{id}$ on C, then the restriction $\pi\colon C \to \pi(C)$ is a double cover. This implies that $C = \pi^{-1}(\pi(C))$ and $\pi(C)^2 = 1/2 \cdot C^2 = -1$. □

We are ready to describe the quotient surfaces of generic K3 surfaces with small Picard number. In the following, we denote by $\mathrm{Bl}_k(Z)$ the blow up of a variety Z in k general points. Moreover, we adopt the standard notation for integral lattices, see [BHPV04, § 2, ch. I], and $L(k)$ denotes the lattice obtained from L by multiplying the intersection matrix by k.

Proposition 6.3. *Let X be a generic K3 surface X with a non-symplectic involution and associated double cover $\pi\colon X \to Y$. For $2 \leqslant \varrho(X) \leqslant 5$, the table*

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$\varrho(X)$	$\mathrm{Cl}(X)$	Y	B
2	U, $U(2)$, $(2)\oplus A_1$	$\mathbb{F}_4, \mathbb{F}_0, \mathrm{Bl}_1(\mathbb{P}^2)$	$\mathbb{P}_1 + C_{10}$, C_9, C_9
$3 \leqslant k \leqslant 5$	$U \oplus A_1^{k-2}$, $U(2) \oplus A_1^{k-2}$	$\mathrm{Bl}_{k-2}(\mathbb{F}_4), \mathrm{Bl}_{k-2}(\mathbb{F}_0)$	$\mathbb{P}_1 + C_{12-k}$, C_{11-k}

describes the intersection form of X, the quotient surface Y and the branch divisor B of π, where C_g denotes a smooth irreducible curve of genus g.

Proof. According to [Nik83, § 4], a lattice L of rank at most five is the fixed lattice L^σ of an involution σ on a K3 surface if and only if it is an even lattice of signature $(1, k-1)$ which is 2-elementary, i.e. satisfies $\mathrm{Hom}(L,\mathbb{Z})/L = \mathbb{Z}_2^a$, where $2^a = |\det(L)|$. Such lattices are classified up to isometries by three invariants: the rank k, the integer a and an invariant δ defined as

$$\delta(L) = \begin{cases} 0 & \text{if } u^2 \in \mathbb{Z} \text{ for all } u \in \mathrm{Hom}(L,\mathbb{Z}), \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to check that the lattices in the table are the only 2-elementary even lattices of signature $(1, k-1)$ with $2 \leqslant k \leqslant 5$, since they cover all possible triples (k, a, δ); see [Nik83, Theorem 4.3.1] and also [AN06, § 2.3].

Now, suppose that the intersection form on $\mathrm{Cl}(X)$ is $U \oplus A_1^{k-2}$. Then it is known that there is an elliptic fibration $p\colon X \to \mathbb{P}^1$ with a section E and $k-2$ reducible fibers; see [Kon89, Lemma 3.1]. In fact, if e, f is the natural basis of U and $v_1, \ldots, v_{k-2}$ is an orthogonal basis of A_1^{k-2}, we can assume that the class of E is $f - e$ and v_i are represented by components of the reducible fibers not intersecting E.

By [Nik83, Theorem 4.2.2], the ramification divisor of σ is the disjoint union of a smooth irreducible curve of genus $12-k$ and a smooth irreducible rational curve. This implies that C is transverse to the fibers of p; hence, any fiber is preserved by σ and the section E is the rational curve in the ramification divisor.

A basis of $\mathrm{Cl}(X)$ is given by $e, f-e, v_1, \ldots, v_{k-2}$. It follows from Lemma 6.2(ii) and (iii) that the Picard lattice of Y has intersection form

$$\begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix} \oplus (-1)^{k-2}.$$

Consequently, the classification of minimal rational surfaces yields that Y is the blowing up of the Hirzebruch surface $\mathbb{F}_4$ at $k-2$ points.

Now assume that the intersection form on $\mathrm{Cl}(X)$ is $U(2) \oplus A_1^{k-2}$. Then, by [Nik83, Theorem 4.2.2], the ramification divisor has only one connected component and this is a smooth irreducible curve of genus $11-k$. Thus, Proposition 6.1(ii) gives $\mathrm{Cl}(X) = \pi^*(\mathrm{Cl}(Y))$. It follows that the intersection form on $\mathrm{Cl}(Y)$ is $U \oplus (-1)^{k-2}$. Hence, as before, we can conclude that Y is the blowing up of $\mathbb{F}_0$ at $k-2$ points.

Similarly, if the intersection form on $\mathrm{Cl}(X)$ is $(2) \oplus A_1$, then we obtain that the intersection form on $\mathrm{Cl}(Y)$ is $(1) \oplus (-1)$, the ramification divisor is a smooth irreducible curve of genus 9 and conclude that Y is the blowing up of $\mathbb{P}^2$ at one point. □

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Proposition 6.4. *Let X be a generic K3 surface with a non-symplectic involution. Suppose that:*

- *the branch divisor of the associated double cover $\pi\colon X \to Y$ is of the form $B = C_1 + C_B$ with $C_1, C_B \subseteq Y$ irreducible and C_1 rational;*
- *the Cox ring of Y is a polynomial ring $S = S'[t_1]$ with the canonical section t_1 of C_1 and a finitely generated $\mathbb{C}$-algebra S'.*

Moreover, denote by $f \in S'$ the canonical section of C_B. Then the Cox ring of X is given as

$$R = \pi^*(S')[T_1, T_2]/\langle T_2^2 - \pi^*(f)\rangle,$$

with the $\mathrm{Cl}(X)$-grading defined by $\deg(\pi^*(g)) := \pi^*(\deg(g))$ for any homogeneous $g \in S'$ and

$$\deg(T_1) := \frac{\pi^*(w_1)}{2}, \quad \deg(T_2) := -\frac{\pi^*(2K_Y + w_1)}{2}.$$

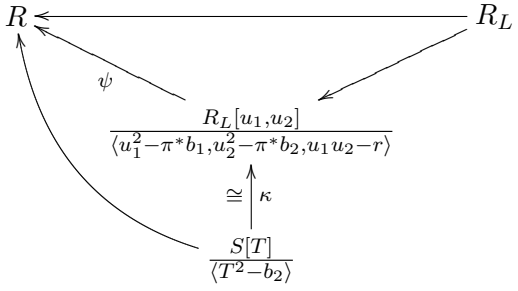
Moreover, the pull-back homomorphism $\pi^*\colon S \to R$ of graded rings is given on the grading groups by $\mathbb{Z}^r \to \mathbb{Z}^r$, $w \mapsto Aw$ and as a ring homomorphism by

$$t_1 \mapsto T_1^2, \quad g \mapsto \pi^*(g) \text{ for any homogeneous } g \in S'.$$

Proof. First note that by Proposition 4.6, the Cox ring R of X inherits finite generation from the Cox ring S of Y. Consider the pull-back group of $\mathrm{Cl}(Y)$ and the corresponding Veronese subalgebra

$$L := \pi^*(\mathrm{Cl}(Y)) \subseteq \mathrm{Cl}(X), \quad R_L := \bigoplus_{w \in L} R_w.$$

Write, for the moment, $B = B_1 + B_2$ and let r and b_i denote the canonical sections of $\pi^{-1}(B)$ and B_i, respectively. We claim that there is a commutative diagram of finite ring homomorphisms



where, denoting by r_1 and r_2 the canonical sections of the reduced divisors $\pi^{-1}(B_1)$ and $\pi^{-1}(B_2)$, respectively, the homomorphism ψ is induced by $u_i \mapsto r_i$.

In this claim, everything is straightforward except the definition of the isomorphism κ. By Proposition 4.3, we know that R_L is generated as a $\pi^*(S)$-module by 1 and a section $s \in R_L$ satisfying $s^2 = \pi^*(b)$, where b denotes the canonical section of B. According to Lemma 6.2, we may choose s to be the canonical section r of the ramification divisor $\pi^{-1}(B)$. Thus, we obtain isomorphisms

$$\begin{aligned} R_L[u_1, u_2]/\langle u_1^2 - b_1, u_2^2 - b_2, u_1u_2 - r\rangle &\cong \pi^*(S)[y, u_1, u_2]/\langle y^2 - b, u_1^2 - b_1, u_2^2 - b_2, u_1u_2 - y\rangle \\ &\cong \pi^*(S)[u_1, u_2]/\langle u_1^2 - b_1, u_2^2 - b_2\rangle. \end{aligned}$$

Now we use our assumption $S = S'[t_1]$. This enables us to define a ring homomorphism

$$\widetilde{\kappa}\colon S[T] \to \pi^*(S)[u_1, u_2], \quad S' \ni g \mapsto \pi^*(g) \in \pi^*(S'), \quad t_1 \mapsto u_1, \quad T \mapsto u_2.$$

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It sends t_1^2 to u_1^2, which defines the same element in $\pi^*(S)[u_1,u_2]/\langle u_1^2-b_1, u_2^2-b_2\rangle$ as $\pi^*(t_1)$. Consequently, $\widetilde{\kappa}$ induces the desired isomorphism

$$\kappa\colon S[T]/\langle T^2-b_2\rangle \to \pi^*(S)[u_1,u_2]/\langle u_1^2-b_1, u_2^2-b_2\rangle.$$

The next step is to show that the homomorphism ψ of the above diagram is an isomorphism. For this, it is enough to show that $S[T]/\langle T^2-b_2\rangle$ is a normal ring. Indeed, $R_L \to R$ is of degree two,

$$R_L \to R_L[u_1,u_2]/\langle u_1^2-b_1, u_2^2-b_2, u_1u_2-r\rangle$$

is of degree at least two and thus ψ is a finite morphism of degree one. If we know that $S[T]/\langle T^2-b_2\rangle$ is normal, we can conclude that ψ is an isomorphism.

In order to show that $S[T]/\langle T^2-b_2\rangle$ is normal, note that S can be made into a $\mathbb{Z}$-graded ring by assigning to each $\mathbb{Z}^r$-homogeneous element the w_1-component of its $\mathbb{Z}^r$-degree. In particular, then, $\deg(b_2)$ is odd. Moreover, $b_2 \in S$ is a prime element. Thus, we can apply the result [SS84, p. 45] and obtain that $S[T]/\langle T^2-b_2\rangle$ is even factorial. In particular, it is normal.

Having verified that ψ is an isomorphism, the commutative diagram tells us that the Cox ring R of X is isomorphic to $S[T]/\langle T^2-b_2\rangle$. Consequently, R is the polynomial ring $\pi^*(S')[T_1,T_2]$ divided by the relation $T_2^2-\pi^*(b_2)$, where $\pi^*(b_2)$ only depends on the first variable. The degrees of the generators T_i are easily computed using Lemma 6.2(ii). □

We are ready to compute the Cox rings of generic K3 surfaces X admitting a non-symplectic involution and satisfying $2 \leqslant \varrho(X) \leqslant 5$. We will work with the curves $D_1, D_2 \subseteq \mathbb{F}_0$ given by

$$D_1 := \{0\} \times \mathbb{P}_1, \quad D_2 := \mathbb{P}_1 \times \{0\},$$

the integral curves $C_1, C_2, C_3 \subseteq \mathbb{F}_4$ given by

$$C_1^2 = -4, \quad C_2 := q^{-1}(0), \quad C_3 := q^{-1}(\infty),$$

where $q\colon \mathbb{F}_4 \to \mathbb{P}_1$ is the bundle projection, and the curves $E_1, E_2 \subseteq \mathrm{Bl}_1(\mathbb{P}^2)$ with

$$E_1^2 = 1, \quad E_2^2 = -1.$$

Moreover, on blow ups of the surfaces $\mathbb{F}_0$ and $\mathbb{F}_4$, we denote the proper transforms of the curves D_i and C_j again by D_i and C_j.

Proposition 6.5. *Let X be a generic K3 surface admitting a non-symplectic involution, and let $\pi\colon X \to Y$ be the associated double cover. If $\varrho(X) = 2$ holds, then the following cases can occur.*

(i) *We have $Y = \mathbb{F}_0$. Then $\mathrm{Cl}(X) = \mathbb{Z}\cdot\pi^*(w_1) \oplus \mathbb{Z}\cdot\pi^*(w_2)$ holds, where $w_i \in \mathrm{Cl}(Y)$ is the class of $D_i \in \mathrm{WDiv}(Y)$. The Cox ring of X is*

$$\mathcal{R}(X) = \mathbb{C}[T_1,\ldots,T_5]/\langle T_5^2 - f\rangle$$

with a polynomial $f \in \mathbb{C}[T_1,\ldots,T_4]$ and the degree of T_i with respect to the above basis is the ith column of the matrix

$$Q = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix}.$$

(ii) *We have $Y = \mathbb{F}_4$. Then $\mathrm{Cl}(X) = \mathbb{Z}\cdot\pi^*(w_1)/2 \oplus \mathbb{Z}\cdot\pi^*(w_2)$ holds, where $w_i \in \mathrm{Cl}(Y)$ denotes the class of $C_i \in \mathrm{WDiv}(Y)$. The Cox ring of X is*

$$\mathcal{R}(X) = \mathbb{C}[T_1,\ldots,T_5]/\langle T_5^2 - f\rangle$$

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with a polynomial $f \in \mathbb{C}[T_1^2, T_2, T_3, T_4]$ *and, with respect to the above basis, the degree of* T_i *is the* i*th column of the matrix*

$$Q = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 4 & 1 & 6 \end{bmatrix}.$$

(iii) *We have* $Y = Bl_1(\mathbb{P}^2)$. *Then* $\mathrm{Cl}(X) = \mathbb{Z} \cdot \pi^*(w_1) \oplus \mathbb{Z} \cdot \pi^*(w_2)$ *holds, where* $w_i \in \mathrm{Cl}(Y)$ *denotes the class of* $E_i \in \mathrm{WDiv}(Y)$. *The Cox ring of* X *is*

$$\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_5]/\langle T_5^2 - f \rangle$$

with a polynomial $f \in \mathbb{C}[T_1, T_2, T_3, T_4]$ *and, with respect to the above basis, the degree of* T_i *is the* i*th column of the matrix*

$$Q = \begin{bmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 3 \end{bmatrix}.$$

Proof. First note that by Proposition 6.3, the surface Y is one of the three types listed in the assertion.

If $Y = \mathbb{P}_1 \times \mathbb{P}_1$ holds, then $\mathrm{Cl}(Y) \cong \mathbb{Z}^2$ is generated by the classes w_1, w_2 of D_1, D_2 and the Cox ring of Y is given by

$$\mathbb{C}[T_1, \ldots, T_4], \quad \deg(T_1) = \deg(T_3) = w_1, \quad \deg(T_2) = \deg(T_4) = w_2;$$

use e.g. Construction 5.1. Similarly, if $Y = \mathrm{Bl}_1(\mathbb{P}^2)$, then $\mathrm{Cl}(Y) \cong \mathbb{Z}^2$ is generated by the classes w_1, w_2 of E_1, E_2 and the Cox ring of Y is given by

$$\mathbb{C}[T_1, \ldots, T_4], \quad \deg(T_1) = w_1, \quad \deg(T_2) = \deg(T_3) = w_1 - w_2, \quad \deg(T_4) = w_2.$$

In both cases, Proposition 6.3 tells us that the branch divisor $B \subseteq Y$ is irreducible. Propositions 6.1(ii) and 4.3 thus show that the Cox ring is as claimed in (i) and (iii).

If $Y = \mathbb{F}_4$ holds, then $\mathrm{Cl}(Y) \cong \mathbb{Z}^2$ is generated by classes w_1, w_2 of C_1, C_2 and the Cox ring of Y is given by

$$\mathbb{C}[T_1, \ldots, T_4], \quad \deg(T_i) = w_1, \quad \deg(T_3) = w_1 + 4w_2, \quad \deg(T_2) = \deg(T_4) = w_2.$$

This time, Lemma 6.2 and Proposition 6.4 show that the Cox ring of X is as claimed in (ii). □

Proposition 6.6. *Let* X *be a generic K3 surface admitting a non-symplectic involution, and let* $\pi\colon X \to Y$ *be the associated double cover. If* $\varrho(X) = 3$ *holds, then the following cases can occur.*

(i) *The surface* Y *is the blow up of* $\mathbb{F}_0$ *at the point* $(0, 0)$. *If* $D_3 \subseteq Y$ *denotes the exceptional curve, then*

$$\mathrm{Cl}(X) = \mathbb{Z} \cdot \pi^*(w_1) \oplus \mathbb{Z} \cdot \pi^*(w_2) \oplus \mathbb{Z} \cdot \pi^*(w_3)$$

holds, where $w_i \in \mathrm{Cl}(Y)$ *denotes the class of* $D_i \in \mathrm{WDiv}(Y)$. *Moreover, the Cox ring of* X *is given by*

$$\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_6]/\langle T_6^2 - f \rangle$$

with a polynomial $f \in \mathbb{C}[T_1, T_2, \ldots, T_5]$ *and, with respect to the above basis, the degree of* T_i *is the* i*th column of the matrix*

$$Q = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{bmatrix}.$$

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(ii) *The surface Y is the blow up of $\mathbb{F}_4$ at the point in $C_1 \cap C_2$, where $S_0 \in \mathbb{F}_4$ is the zero section. Then the divisor class group of X is*

$$\mathrm{Cl}(X) = \mathbb{Z} \cdot \frac{\pi^*(w_1)}{2} \oplus \mathbb{Z} \cdot \pi^*(w_2) \oplus \mathbb{Z} \cdot \pi^*(w_3),$$

where $w_i \in \mathrm{Cl}(Y)$ denotes the class of $C_i \in \mathrm{WDiv}(Y)$. Moreover, the Cox ring of X is given by

$$\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_6]/\langle T_6^2 - f \rangle$$

with a polynomial $f \in \mathbb{C}[T_1^2, T_2, \ldots, T_5]$ and, with respect to the above basis, the degree of T_i is the ith column of the matrix

$$Q = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 & 1 & 5 \end{bmatrix}.$$

Proof. The fact that Y is either $\mathbb{P}_1 \times \mathbb{P}_1$ blown up at a point p or $\mathbb{F}_4$ blown up at a point p follows from Proposition 6.3. Moreover, the same proposition yields that the branch divisor has one component in the first case and two components in the second one. In both cases, applying a suitable automorphism, we may assume that the point p to be blown up is as in the assertion. Then, in both cases, the surface Y is toric and the computation of the Cox rings then goes in the same way as in the preceding proposition. □

Proposition 6.7. *Let X be a generic K3 surface admitting a non-symplectic involution, and let $\pi\colon X \to Y$ be the associated double cover. If $\varrho(X) = 4$ holds, then the following cases can occur.*

(i) *The surface Y is the blow up of $\mathbb{F}_0$ at the points $(0,0)$ and (∞,∞). If $D_3, D_4 \subseteq Y$ are exceptional curves corresponding to these points, then*

$$\mathrm{Cl}(X) = \mathbb{Z} \cdot \pi^*(w_1) \oplus \cdots \oplus \mathbb{Z} \cdot \pi^*(w_4)$$

holds, where $w_i \in \mathrm{Cl}(Y)$ denotes the class of $D_i \in \mathrm{WDiv}(Y)$. Moreover, the Cox ring of X is given by

$$\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_7]/\langle T_7^2 - f \rangle$$

with a polynomial $f \in \mathbb{C}[T_1, T_2, \ldots, T_6]$ and, with respect to the above basis, the degree of T_i is the ith column of the matrix

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 \end{bmatrix}.$$

(ii) *The surface Y is the blow up of $\mathbb{F}_4$ at the two points $p_1 \in C_1 \cap C_2$ and $p_2 \in C_1 \cap C_3$. We have*

$$\mathrm{Cl}(X) = \mathbb{Z} \cdot \frac{\pi^*(w_1)}{2} \oplus \mathbb{Z} \cdot \pi^*(w_2) \oplus \mathbb{Z} \cdot \pi^*(w_3) \oplus \mathbb{Z} \cdot \pi^*(w_4),$$

where $w_i \in \mathrm{Cl}(Y)$ is the class of $C_i \in \mathrm{WDiv}(Y)$ and $C_4 \subseteq Y$ is the exceptional curve over $p_1 \in \mathbb{F}_4$. The Cox ring of X is given by

$$\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_7]/\langle T_7^2 - f \rangle$$

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with a polynomial $f \in \mathbb{C}[T_1^2, T_2, \ldots, T_6]$ *and, with respect to the above basis, the degree of* T_i *is the* i*th column of the matrix*

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 & 3 & 1 & 5 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 4 \end{bmatrix}.$$

Proof. Again, Proposition 6.3 tells us that Y is either $\mathbb{P}_1 \times \mathbb{P}_1$ blown up at two points p, q or $\mathbb{F}_4$ blown up at two points p, q and that the branch divisor has one component in the first case and two components in the second one. In both cases, we may apply a suitable automorphism, and achieve that the points p, q to be blown up are as in the assertion. Thus, again, the surface Y is toric and the computation of the Cox rings proceeds as before. □

Proposition 6.8. *Let* X *be a generic K3 surface admitting a non-symplectic involution, and let* $X \to Y$ *be the associated double cover. If* $\varrho(X) = 5$ *holds, then the following cases can occur.*

(i) *The surface* Y *is the blow up of* $\mathbb{F}_0$ *at three general points. Then the Cox ring of* X *is*

$$\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_{11}]/\langle f_1, \ldots, f_5, T_{11}^2 - g \rangle,$$

where $f_1, \ldots, f_5$ *are the Plücker relations in the variables* $T_1, \ldots, T_{10}$*, i.e. we have*

$$f_1 = T_2T_5 - T_3T_6 + T_4T_7, \quad f_2 = T_1T_5 - T_3T_8 + T_4T_9,$$
$$f_3 = T_1T_6 - T_2T_8 + T_4T_{10}, \quad f_4 = T_1T_7 - T_2T_9 + T_3T_{10},$$
$$f_5 = T_5T_{10} - T_6T_9 + T_7T_8$$

and $g \in \mathbb{C}[T_1, \ldots, T_{10}]$ *is a prime polynomial. The degree of* $T_i \in \mathcal{R}(X)$ *is the* i*th column of*

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & -3 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 1 \end{bmatrix}.$$

(ii) *The surface* Y *is the blow up of* $\mathbb{F}_4$ *at three general points. Then the Cox ring of* X *is*

$$\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_9]/\langle T_2T_5 + T_4T_6 + T_7T_8, T_9^2 - f \rangle,$$

where $f \in \mathbb{C}[T_1, \ldots, T_8]$ *is a prime polynomial and the degree of* $T_i \in \mathcal{R}(X)$ *is the* i*th column of*

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & 3 & 4 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 1 \end{bmatrix}.$$

Proof. The facts that only (i) and (ii) are possible and that the branch divisor has one component in (i) and two in (ii) follow from Proposition 6.3.

If we are in the situation (i), then Y is the blow up of $\mathbb{P}_2$ at four general points and hence is a del Pezzo surface. Its Cox ring is the ring of (3×3)-minors of a generic (3×5)-matrix; see [BP04, Proposition 4.1]. Moreover, by Proposition 6.1(ii), the pull-back $\pi^*\colon \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$ is an isomorphism. Thus, taking the same basis of $\mathrm{Cl}(Y)$ as in the proof of [BP04, Proposition 4.1], the assertion follows from Propositions 4.3 and 6.1(ii).

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If we are in situation (ii), then the assertion is a direct consequence of Propositions 5.3 and 6.4 and Lemma 6.2. Note that the canonical class of Y can be determined according to [BH07, Proposition 8.5] as the degree of the relation minus the sum of the degrees of the generators of the Cox ring of Y. □

If X is a generic K3 surface with a non-symplectic involution such that the associated double cover has an irreducible branch divisor, then we can proceed with the computation of Cox rings as follows.

Proposition 6.9. *Let X be a generic K3 surface with a non-symplectic involution, associated double cover $\pi\colon X \to Y$ and intersection form $U(2) \oplus A_1^{k-2}$, where $5 \leqslant k \leqslant 9$. Then Y is a del Pezzo surface of Picard number k and:*

(i) *the Cox ring $\mathcal{R}(X)$ is generated by the pull-backs of the (-1)-curves of Y, the section T defining the ramification divisor and, for $k = 9$, the pull-back of an irreducible section of $H^0(Y, -K_Y)$;*

(ii) *the ideal of relations of $\mathcal{R}(X)$ is generated by quadratic relations of degree $\pi^*(D)$, where $D^2 = 0$ and $D \cdot K_Y = -2$, and the relation $T^2 - f$ in degree $-2\pi^*(K_Y)$, where f is the pull-back of the canonical section of the branch divisor.*

Proof. As in the proof of Proposition 6.3, we use [Nik83, Theorem 4.2.2] to see that the ramification divisor of $\pi\colon X \to Y$ is irreducible. Then Proposition 6.1(ii) yields $\mathrm{Cl}(X) = \pi^*(\mathrm{Cl}(Y))$. It follows that the intersection form on $\mathrm{Cl}(Y)$ is $U \oplus (-1)^{k-2}$. Consequently, Y is the blow up of $\mathbb{F}_0$ at $k-2$ general points and hence is a del Pezzo surface.

It is known that $\mathcal{R}(Y)$ is generated by all the (-1)-curves of Y plus, if $k = 9$, an irreducible section of $H^0(Y, -K_Y)$; see [BP04, Theorem 3.2]. The ideal of relations of $\mathcal{R}(Y)$ is generated by quadratic relations of degree D, where D is a conic bundle, i.e. we have $D^2 = 0$ and $D \cdot K_Y = -2$; see [LV09, Lemmas 2.2, 2.3 and 2.4]. Thus, the statement follows from Proposition 4.3. □

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