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THE TRANSLATIONAL HULL OF A SEMILATTICE OF WEAKLY REDUCTIVE SEMIGROUPS

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1. Introduction and summary. The translational hull is of central importance in the construction of ideal extensions and the study of densely embedded ideals particularly for weakly reductive semigroups (see [**4**, Chapter III]). The translational hull of semigroups belonging to a few special classes is known in an explicit form, and for some other classes of semigroups, certain properties of their translational hulls have been established (see [**4**, Chapter V]). We have generalized in [**5**] the concept of an inverse limit of groups in order to give a construction of the translational hull of a semigroup which is a semilattice of groups. The purpose of this paper is further to generalize the construction in [**5**] in order to construct the translational hull of any semilattice of weakly reductive semigroups. Based on this construction, we are able to consider a variety of special cases providing extra information peculiar to these cases.

We list in §2 most of the needed definitions and notation; those not explicitly stated can be found in [**4**]. In §3, we present the main construction and establish the needed characterization of the translational hull of a semilattice of weakly reductive semigroups. The special case of a strong semilattice of weakly reductive semigroups is treated in §4. A discussion of the further special case of a sturdy semilattice of weakly reductive semigroups is the content of §5. The translational hull of a semigroup which is a subdirect product of a semilattice and a cancellative semigroup is constructed in §6. Finally, in §7, we provide certain information about the translational hull of a spined product of semigroups satisfying certain restrictions. We deduce a number of corollaries concerning semilattices of cancellative and some other special kinds of semigroups.

2. Preliminaries. Let S be any semigroup and let x and y stand for arbitrary elements of S. A function λ (resp. ρ), written on the left (resp. right) mapping S into itself is a *left* (resp. *right*) *translation* if $\lambda(xy) = (\lambda x)y$ (resp. $(xy)\rho = x(y\rho)$); in addition, the pair (λ, ρ) is a *bitranslation* if also $x(\lambda y) = (x\rho)y$. The set of all bitranslations of S under the operation $(\lambda, \rho)(\lambda', \rho') = (\lambda\lambda', \rho\rho')$, where $(\lambda\lambda')x = \lambda(\lambda' x)$ and $x(\rho\rho') = (x\rho)\rho'$, is a semigroup, the *translational hull* of S, to be denoted by $\Omega(S)$. We will denote the pair (λ, ρ) by a single letter ω and consider it as a *bioperator* on S with $\omega x = \lambda x$, $x\omega = x\rho$. For any $s \in S$,

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the function λ_s (resp. ρ_s) defined by $\lambda_s x = sx$ (resp. $x\rho_s = xs$) is the *inner left* (resp. *right*) *translation* induced by s. We write $\pi_s = (\lambda_s, \rho_s)$ and note that $\Pi(S) = \{\pi_s | s \in S\}$ is an ideal of $\Omega(S)$, called the *inner part* of $\Omega(S)$. The mapping $\pi : s \rightarrow \pi_s$ is the canonical homomorphism of S onto $\Pi(S)$, and is one-to-one if and only if S is weakly reductive.

If I is an ideal of S, then S is an (*ideal*) *extension* of I. If also the equality relation on S is the only congruence on S whose restriction to I is the equality relation on I, then S is a *dense extension* of I; if S is (under inclusion) a maximal dense extension of I, then I is a *densely embedded ideal* of S. For a subsemigroup A of S, the idealizer of A in S is the largest subsemigroup of S having A as an ideal, and is given by

$$i_S(A) = \{s \in S | sa, as \in A \text{ for all } a \in A\}.$$

An embedding χ of a semigroup T into S is *dense* if $T\chi$ is a densely embedded ideal of its idealizer in S.

An ideal I of S is a *retract ideal*, and S is a *retract extension* of I, if there exists a homomorphism of S onto I which leaves the elements of I fixed. If Y is a semilattice, its ideals form a semigroup $\mathscr{I}_Y$ under intersection; its retract ideals $\mathscr{R}_Y$ form a subsemigroup of $\mathscr{I}_Y$. The principal ideal generated by an element α of a semilattice Y will be denoted by (α). The elements I of $\mathscr{R}_Y$ are characterized by the property that $I \cap (\alpha)$ is a principal ideal for every $\alpha \in Y$.

Let σ be a semilattice congruence on S (i.e., S/σ is a semilattice); then S is a *semilattice Y of semigroups S_α* where $Y = S/\sigma$ and S_α are the σ-classes. Such an S can be constructed from the semigroups S_α if these are weakly reductive as follows.

Let $\{S_\alpha\}_{\alpha \in Y}$ be a family of pairwise disjoint weakly reductive semigroups indexed by a semilattice Y. For each pair $\alpha \geqq \beta$, let a function $\psi_{\alpha,\beta} : S_\alpha \rightarrow \Omega(S_\beta)$ be given, $\psi_{\alpha,\beta} : a \rightarrow a\psi_{\alpha,\beta}$, and assume that:

(i) $\psi_{\alpha,\alpha}$ is the canonical isomorphism $S_\alpha \rightarrow \Pi(S_\alpha)$;

(ii) $(a\psi_{\alpha,\alpha\beta})(b\psi_{\beta,\alpha\beta}) \in \Pi(S_{\alpha\beta})$ for all $a \in S_\alpha$, $b \in S_\beta$;

(iii) if $\alpha > \beta\gamma$, then for all $a \in S_\alpha$, $b \in S_\beta$,

$$(1) \quad [(a\psi_{\alpha,\alpha\beta})(b\psi_{\beta,\alpha\beta})]\psi_{\alpha\beta,\alpha\beta}^{-1}\psi_{\alpha\beta,\gamma} = (a\psi_{\alpha,\gamma})(b\psi_{\beta,\gamma}).$$

On $S = \bigcup_{\alpha \in Y} S_\alpha$ define an operation $*$ by

$$(2) \quad a*b = [(a\psi_{\alpha,\alpha\beta})(b\psi_{\beta,\alpha\beta})]\psi_{\alpha\beta,\alpha\beta}^{-1} \qquad (a \in S_\alpha,\ b \in S_\beta).$$

Then S is a semilattice Y of semigroups S_α, in notation $S = (Y; S_\alpha, \psi_{\alpha,\beta})$. Conversely, every semilattice Y of semigroups S_α can be so constructed.

A special case of particular interest is obtained by taking Y and S_α as above and a system of homomorphisms $\varphi_{\alpha,\beta} : S_\alpha \rightarrow S_\beta$ for all pairs $\alpha \geqq \beta$, with $\varphi_{\alpha,\alpha}$ the identity mapping on S_α, satisfying the transitivity condition: if $\alpha > \beta > \gamma$, then $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$ (functions written on the right), with an operation $*$ on S defined by

$$a*b = (a\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta}) \qquad (a \in S_\alpha,\ b \in S_\beta).$$

Then S is a *strong semilattice Y of semigroups S_α*, in notation $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$. If all $\varphi_{\alpha,\beta}$ are one-to-one, S is a *sturdy semilattice Y of semigroups S_α*, in notation $S = \langle Y; S_\alpha, \varphi_{\alpha,\beta}\rangle$. We will be mainly interested in semilattices of semigroups belonging to a class $\mathscr{C}$ rather than in a special semilattice Y of fixed semigroups S_α.

As usual, E_S denotes the (partially ordered) set of idempotents of S.

For undefined concepts as well as for a full discussion of the notions listed above, see [**4**]. Further information concerning the translational hull is summarized in [**1**].

3. The main construction theorem. The result in question is preceded by some auxiliary statements, notation and constructions. These are of basic importance for a large part of the paper.

LEMMA 1. *Let S be a semigroup, σ be a congruence on S such that S/σ is reductive and $\omega \in \Omega(S)$. If $a\ \sigma\ b$, then $\omega a\ \sigma\ \omega b$ and $a\omega\ \sigma\ b\omega$.*

Proof. Assume that $a\ \sigma\ b$. Then for any $c \in S$, we have $(c\omega)a\ \sigma\ (c\omega)b$ and thus also $c(\omega a)\ \sigma\ c(\omega b)$. Let $x \to \bar{x}$ be the canonical homomorphism of S onto S/σ. Then $\bar{c}\ \overline{\omega a} = \bar{c}\ \overline{\omega b}$ for all $\bar{c} \in S/\sigma$. Since S/σ is reductive, it follows that $\overline{\omega a} = \overline{\omega b}$. Consequently $\omega a\ \sigma\ \omega b$; the relation $a\omega\ \sigma\ b\omega$ is proved similarly.

LEMMA 2. *Let $S = (Y; S_\alpha, \psi_{\alpha,\beta})$. We define a mapping ϵ by*

$$\epsilon : \omega \to \bar{\omega} \qquad (\omega \in \Omega(S))$$

where $\bar{\omega}$ is defined on Y by

$$\bar{\omega}\alpha = \alpha\bar{\omega} = \beta \quad \text{if } a \in S_\alpha,\ \omega a \in S_\beta.$$

Then ϵ is a homomorphism of $\Omega(S)$ into $\Omega(Y)$. Moreover, if $\omega a \in S_\beta$, then $a\omega \in S_\beta$.

Proof. By Lemma 1, $\bar{\omega}$ is well-defined. If $a \in S_\alpha$, $b \in S_\beta$, then

$$\omega(ab) \in S_{\bar{\omega}(\alpha\beta)}, \qquad (\omega a)b \in S_{\bar{\omega}\alpha}S_\beta \subseteq S_{(\bar{\omega}\alpha)\beta}$$

and thus $\bar{\omega}(\alpha\beta) = (\bar{\omega}\alpha)\beta$. Hence $\bar{\omega} \in \Omega(Y)$ since Y is a semilattice. If $\omega, \theta \in \Omega(S)$, $a \in S_\alpha$, then

$$(\omega\theta)a \in S_{\overline{\omega\theta}\alpha}, \qquad \omega(\theta a) \in \omega S_{\bar{\theta}\alpha} \subseteq S_{(\bar{\omega}\bar{\theta})\alpha}$$

which proves that $\overline{\omega\theta} = \bar{\omega}\bar{\theta}$. Consequently ϵ is a homomorphism.

In order to prove the last statement, let $a \in S_\alpha$, $\omega a \in S_\beta$, $a\omega \in S_\gamma$. Then $\omega a^2 \in S_\beta$ by Lemma 1, so that $(\omega a)a = \omega a^2$ implies $\beta\alpha = \beta$. Hence $\beta \leqq \alpha$ and similarly $\gamma \leqq \alpha$. Further, $(a\omega)a = a(\omega a)$ implies $\gamma\alpha = \alpha\beta$ which finally implies $\gamma = \beta$ since $\beta, \gamma \leqq \alpha$.

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Recall from [**4**, V. 6.1] that for any semilattice Y, the mapping

$$\omega \to I_\omega = \omega Y$$

is an isomorphism of $\Omega(Y)$ onto $\mathscr{R}_Y$; for more information on semilattices see [**2**]. Returning to the situation in Lemma 2, if we let

$$I_\omega = \{\alpha \in Y | \omega S \cap S_\alpha \neq \phi\}, \qquad I_{\bar{\omega}} = \bar{\omega} Y,$$

we see from Lemma 2 that $I_\omega = I_{\bar{\omega}}$ and

LEMMA 3. *The mapping*

$$\omega \to I_\omega$$

is a homomorphism of $\Omega(S)$ *into* $\mathscr{R}_Y$.

Let $S = (Y; S_\alpha, \psi_{\alpha,\beta})$ where each S_α is weakly reductive. We will consistently use the following notation.

$$\mathscr{F}(Y; S_\alpha, \psi_{\alpha,\beta}) = \bigcup_{I \in \mathscr{I}_Y} \prod_{\alpha \in I} \Omega(S_\alpha)$$

with multiplication

$$(\omega_\alpha)_{\alpha \in I} \cdot (\theta_\alpha)_{\alpha \in J} = (\omega_\alpha \theta_\alpha)_{\alpha \in I \cap J}.$$

We will usually write $\mathscr{F}$ instead of $\mathscr{F}(Y; S_\alpha, \psi_{\alpha,\beta})$. It is easy to see that $\mathscr{F}$ is a semigroup, in fact,

$$\mathscr{F} \cong [\mathscr{I}_Y; \Omega_I, \psi_{I,J}]$$

where $\Omega_I = \Pi_{\alpha \in I} \Omega(S_\alpha)$ for any $I \in \mathscr{I}_Y$ and

$$\Psi_{I,J} : (\omega_\alpha)_{\alpha \in I} \to (\omega_\alpha)_{\alpha \in J} \qquad (I, J \in \mathscr{I}_Y, I \supseteq J).$$

Next let $\mathscr{B} = \mathscr{B}(Y; S_\alpha, \psi_{\alpha,\beta})$ be the set of all $(\omega_\alpha)_{\alpha \in I}$ in $\mathscr{F}$ satisfying:

(C1) $I \in \mathscr{R}_Y$, write $(\alpha) \cap I = (\bar{\alpha})$,

(C2) for every $a \in S_\alpha$ there exist $a', a'' \in S_\alpha$ such that

$$a' \psi_{\bar{\alpha},\beta} = \omega_\beta (a \psi_{\alpha,\beta}), \qquad a'' \psi_{\bar{\alpha},\beta} = (a \psi_{\alpha,\beta}) \omega_\beta \qquad (\beta \leqq \bar{\alpha}).$$

Finally, let $\mathscr{C} = \mathscr{C}(Y; S_\alpha, \psi_{\alpha,\beta})$ be the set of all $(\omega_\alpha)_{\alpha \leqq \gamma}$ in $\mathscr{F}$ satisfying:

(C3) there exists $c \in S_\gamma$ such that

$$\omega_\alpha = c \psi_{\gamma,\alpha} \qquad (\alpha \leqq \gamma).$$

Both $\mathscr{B}$ and $\mathscr{C}$ inherit the multiplication from $\mathscr{F}$. It will follow from the theorem below that they are both semigroups. We will adhere to this notation as well as to $\bar{\omega}$, I_ω, $I_{\bar{\omega}}$ introduced above.

THEOREM 1. *Let* $S = (Y; S_\alpha, \psi_{\alpha,\beta})$, *where each* S_α *is weakly reductive. The mapping* χ *defined by*

$$\chi : \omega \to (\omega|_{S_\alpha})_{\alpha \in I} \qquad (\omega \in \Omega(S))$$

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is an isomorphism of $\Omega(S)$ *into* $\mathscr{F}$ *satisfying*

$$\Pi(S)\chi = \mathscr{C}, \qquad \Omega(S)\chi = \mathscr{B} = i_{\mathscr{F}}(\mathscr{C}).$$

Proof. Let $\omega \in \Omega(S)$. Then $\bar{\omega} \in \Omega(Y)$ by Lemma 2 and hence $\bar{\omega}$ leaves every element of $I_{\bar{\omega}}$ fixed. It follows from Lemma 2 that ω maps S_α into itself for every $\alpha \in I_\omega = I_{\bar{\omega}}$. Consequently $\omega|_{S_\alpha} \in \Omega(S_\alpha)$ for all $\alpha \in I_\omega$. Hence χ maps $\Omega(S)$ into $\mathscr{F}$.

Let $\omega, \theta \in \Omega(S)$. Using Lemma 3, we deduce

$$\begin{aligned}(\omega\theta)\chi &= ((\omega\theta)|_{S_\alpha})_{\alpha\in I_{\omega\theta}} = ((\omega\theta)|_{S_\alpha})_{\beta\in I_\omega\cap I_\theta} \\ &= (\omega|_{S_\alpha})_{\alpha\in I_\omega}\cdot(\theta|_{S_\alpha})_{\alpha\in I_\theta} = (\omega\chi)(\theta\chi),\end{aligned}$$

that is, χ is a homomorphism.

Let $\omega \in \Omega(S)$ and $\alpha \in Y$. Then $\bar{\omega}\alpha \leqq \alpha$ and $\bar{\omega}\alpha \in I_\omega$ so that $(\bar{\omega}\alpha) \subseteq (\alpha) \cap I_\omega$. Let $\beta \in (\alpha) \cap I_\omega$. Then $\beta \leqq \alpha$ and $\beta \in I_\omega$ which implies that $\bar{\omega}\beta = \beta$. Consequently

$$(\bar{\omega}\alpha)\beta = \bar{\omega}(\alpha\beta) = \bar{\omega}(\beta\alpha) = (\bar{\omega}\beta)\alpha = \beta\alpha = \beta$$

which shows that $\beta \leqq \bar{\omega}\alpha$, and thus $(\alpha) \cap I_\omega \subseteq (\bar{\omega}\alpha)$. Hence $(\alpha) \cap I_\omega = (\bar{\omega}\alpha)$ proving that $I_\omega \in \mathscr{R}_Y$. This establishes condition (C1) above. We write $(\alpha) \cap I_\omega = (\bar{\alpha})$ and will prove next condition (C2). Let $a \in S_\alpha$, $a' = \omega a$, $a'' = a\omega$ so that $a', a'' \in S_{\bar{\alpha}}$. Writing $\omega_\beta = \omega|_{S_\beta}$ for any $\beta \in I_\omega$, we obtain for any $b \in S_\beta$, $\beta \leqq \bar{\alpha}$,

$$\begin{aligned}(a'\psi_{\bar{\alpha},\beta})\, b &= a'b = (\omega a)b = \omega(ab) = \omega_\beta(a\psi_{\alpha,\beta})b \\ b(a'\psi_{\bar{\alpha},\beta}) &= ba' = b(\omega a) = (b\omega)a = b\omega_\beta(a\psi_{\alpha,\beta})\end{aligned}$$

which proves that $a'\psi_{\bar{\alpha},\beta} = \omega_\beta(a\psi_{\alpha,\beta})$. This establishes the first formula in (C2); the second formula is proved similarly. Therefore χ maps $\Omega(S)$ into $\mathscr{B}$.

Now let $\omega, \theta \in \Omega(S)$ and suppose that $\omega\chi = \theta\chi$. Then $I_\omega = I_\theta$ and $\omega|_{S_\alpha} = \theta|_{S_\alpha}$ for all $\alpha \in I_\omega$. For $\beta = \bar{\alpha}$ in (C2), we obtain

$$(\omega a)\psi_{\bar{\alpha},\bar{\alpha}} = \omega_{\bar{\alpha}}(a\psi_{\alpha,\bar{\alpha}}) = \theta_{\bar{\alpha}}(a\psi_{\alpha,\bar{\alpha}}) = (\theta a)\psi_{\bar{\alpha},\bar{\alpha}}$$

which by weak reductivity of $S_{\bar{\alpha}}$ yields $\omega a = \theta a$. One shows similarly that $a\omega = a\theta$. Consequently $\omega = \theta$ and hence χ is one-to-one.

Let $(\omega_\alpha)_{\alpha\in I} \in \mathscr{B}$. Using the notation introduced in conditions (C1) and (C2), we let

$$\omega a = a', \quad a\omega = a'' \qquad (a \in S).$$

We will now prove that $\omega \in \Omega(S)$ and that $\omega\chi = (\omega_\alpha)_{\alpha\in I}$. Let $a \in S_\alpha$, $b \in S_\beta$. Using (1), (2) of §2 and (C2), we obtain

$$\begin{aligned}(ab)'\psi_{\overline{\alpha\beta},\overline{\alpha\beta}} &= \omega_{\alpha,\overline{\alpha\beta}}[(ab)\psi_{\alpha\beta,\overline{\alpha\beta}}] = \omega_{\overline{\alpha\beta}}(a\psi_{\alpha,\overline{\alpha\beta}})(b\psi_{\beta,\overline{\alpha\beta}}) \\ &= [\omega_{\overline{\alpha\beta}}(a\psi_{\alpha,\overline{\alpha\beta}})](b\psi_{\beta,\overline{\alpha\beta}}) = (a'\psi_{\bar{\alpha},\overline{\alpha\beta}})(b\psi_{\beta,\overline{\alpha\beta}}) = (a'b)\psi_{\overline{\alpha\beta},\overline{\alpha\beta}}\end{aligned}$$

which by weak reductivity in $S_{\overline{\alpha\beta}}$ yields $\omega(ab) = (\omega a)b$. One proves similarly

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that $(ab)\omega = a(b\omega)$. Further,

$$(ab')\psi_{\overline{\alpha\beta},\overline{\alpha\beta}} = (a\psi_{\alpha,\overline{\alpha\beta}})(b'\psi_{\overline{\beta},\overline{\alpha\beta}}) = (a\psi_{\alpha,\overline{\alpha\beta}})[\omega_{\overline{\alpha\beta}}(b\psi_{\beta,\overline{\alpha\beta}})]$$
$$= [(a\psi_{\alpha,\overline{\alpha\beta}})\omega_{\overline{\alpha\beta}}](b\psi_{\beta,\overline{\alpha\beta}}) = (a''\psi_{\overline{\alpha},\overline{\alpha\beta}})(b\psi_{\beta,\overline{\alpha\beta}}) = (a''b)\psi_{\overline{\alpha\beta},\overline{\alpha\beta}}$$

and thus $a(\omega b) = (a\omega)b$. Consequently $\omega \in \Omega(S)$. It is clear that $I = I_\omega$. For $\alpha \in I$ and $a \in S_\alpha$, we have

$$a'\psi_{\alpha,\alpha} = \omega_\alpha(a\psi_{\alpha,\alpha}) = (\omega_\alpha a)\psi_{\alpha,\alpha}$$

so that $\omega a = \omega_\alpha a$. One shows similarly that $a\omega = a\omega_\alpha$. Hence $\omega|_{S_\alpha} = \omega_\alpha$ which implies that $\omega\chi = (\omega_\alpha)_{\alpha\in I}$. Therefore χ maps $\Omega(S)$ onto $\mathscr{B}$.

Let $c \in S_\gamma$, $(\omega_\alpha)_{\alpha\leqq\gamma} = \pi_c\chi$. Then for any $a \in S_\alpha$, $\alpha \leqq \gamma$, we have

$$\omega_\alpha a = ca = (c\psi_{\gamma,\alpha})a, \qquad a\omega_\alpha = ac = a(c\psi_{\gamma,\alpha})$$

and thus $\omega_\alpha = c\psi_{\gamma,\alpha}$. Hence (C3) holds and thus $\pi_c\chi \in \mathscr{C}$. Conversely, if $\omega_\alpha = c\psi_{\gamma,\alpha}$ for some $c \in S_\gamma$ and all $\alpha \leqq \gamma$, then

$$\pi_c\chi = (c\psi_{\gamma,\alpha})_{\alpha\leqq\gamma} = (\omega_\alpha)_{\alpha\leqq\gamma}.$$

Consequently χ maps $\Pi(S)$ onto $\mathscr{C}$.

It remains to show that $\mathscr{B} = i_{\mathscr{F}}(\mathscr{C})$. Since $\Pi(S)$ is an ideal of $\Omega(S)$, we have that $\mathscr{C}$ is an ideal of $\mathscr{B}$ because of the isomorphism χ. Hence $\mathscr{B} \subseteq i_{\mathscr{F}}(\mathscr{C})$. In order to prove the opposite inclusion, we let $(\omega_\alpha)_{\alpha\in I} \in i_{\mathscr{F}}(\mathscr{C})$. By virtue of the isomorphism χ, for every $c \in S_\gamma$ there exists $c' \in S_{\bar{\gamma}}$ such that

$$(\omega_\alpha)_{\alpha\in I} \cdot (c\psi_{\gamma,\alpha})_{\alpha\leqq\gamma} = (c'\psi_{\bar{\gamma},\alpha})_{\alpha\leqq\bar{\gamma}}.$$

Consequently $I \cap (\gamma) = (\bar{\gamma})$ and $\omega_\alpha(c\psi_{\gamma,\alpha}) = c'\psi_{\bar{\gamma},\alpha}$ for all $\alpha \leqq \bar{\gamma}$. It follows that (C1) and the first formula in (C2) are satisfied; the second formula in (C2) is proved similarly. Hence $(\omega_\alpha)_{\alpha\in I} \in \mathscr{B}$ which proves the inclusion $i_{\mathscr{F}}(\mathscr{C}) \subseteq \mathscr{B}$.

COROLLARY 1. *Let S be as in Theorem 1. Then the function ζ defined on S by*

$$\zeta : a \to (a\psi_{\alpha,\beta})_{\beta\leqq\alpha} \quad \text{if } a \in S_\alpha$$

is a dense embedding of S into $\mathscr{F}$.

Proof. It follows easily from the proof of Theorem 1 that ζ is the composition of the canonical isomorphism $a \to \pi_a$ and χ. Recall from [**4**, III.5.9] that $\Pi(S)$ is a densely embedded ideal of $\Omega(S)$. Theorem 1 implies that $\mathscr{C}$ is a densely embedded ideal of $\mathscr{B}$ because of the isomorphism χ. Finally, by Theorem 1 we conclude that $\mathscr{B} = i_{\mathscr{F}}(S\zeta)$, and therefore ζ is a dense embedding.

Recall that a semigroup S is *separative* if for any $x, y \in S$, $xy = x^2$, $yx = y^2$ implies $x = y$ and $xy = y^2$, $yx = x^2$ implies $x = y$.

COROLLARY 2. *Every (commutative) separative semigroup can be densely embedded into a strong semilattice of (commutative) cancellative monoids.*

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Proof. Let S be a (commutative) separative semigroup. By [**4**, II.6.4], $S = (Y; S_\alpha, \psi_{\alpha,\beta})$ where each S_α is a (commutative) cancellative semigroup. By Corollary 1, S can be densely embedded into $\mathscr{F}(Y; S_\alpha, \psi_{\alpha,\beta})$; the latter is isomorphic to $[\mathscr{I}_Y; \Omega_I, \Psi_{I,J}]$ as noted before Theorem 1. Since S_α is (commutative) cancellative, so is $\Omega(S_\alpha)$ by [**4**, III.5.9, 5.14, 5.16] and hence also $\Omega_I = \Pi_{\alpha \in I}\Omega(S_\alpha)$.

In view of [**4**, II.6.4], Corollary 2 implies that if S is (commutative) separative, so is $\Omega(S)$. This can be proved directly using the definition of a separative semigroup. It is easy to see that if T is a semilattice of cancellative monoids, then E_T is a subsemigroup of T, and thus the semilattice composition is strong. Hence "strong" in Corollary 2 is automatic. Corollary 1 can be applied to any semilattice of monoids having a property preserved by direct products to yield a result similar to Corollary 2.

4. Strong compositions. If the composition in Theorem 1 is strong, we can make much more precise statements about $\Omega(S)$ as follows.

THEOREM 2. *Let $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$, where each S_α is weakly reductive. Then the following statements hold.*

(i) *$\mathscr{B}$ consists of all $(\omega_\alpha)_{\alpha \in I}$ in $\mathscr{F}$ satisfying* (C1) *and*
(C2′) *for every $a \in S_\alpha$, $\alpha \geqq \beta$, $\alpha \in I$,*

$$(\omega_\alpha a)\varphi_{\alpha,\beta} = \omega_\beta(a\varphi_{\alpha,\beta}), \quad (a\omega_\alpha)\varphi_{\alpha,\beta} = (a\varphi_{\alpha,\beta})\omega_\beta.$$

(ii) *$\mathscr{B} \cong [\mathscr{R}_Y, \mathscr{B} \cap \Omega_I, \Phi_{I,J}]$ where $\Phi_{I,J} = \Psi_{I,J}|_{\mathscr{B}\cap\Omega_I}$.*
(iii) *$\mathscr{C} = \{(\pi_{a_\alpha})_{\alpha \leqq \gamma} \in \mathscr{F} \mid a_\beta\varphi_{\beta,\alpha} = a_\alpha$ if $b \in S_\beta$, $\alpha \leqq \beta \leqq \gamma\}$.*

Proof. The hypothesis that the composition is strong implies

$$a\psi_{\alpha,\beta} = (a\varphi_{\alpha,\beta})\psi_{\beta,\beta} \qquad (a \in S_\alpha,\ \alpha \geqq \beta).$$

(i) Assume first that (C1) and (C2) hold. For $a \in S_\alpha$, we obtain

$$a'\psi_{\bar\alpha,\bar\alpha} = \omega_{\bar\alpha}(a\psi_{\alpha,\bar\alpha}) = \omega_{\bar\alpha}[(a\varphi_{\alpha,\bar\alpha})\psi_{\bar\alpha,\bar\alpha}] = [\omega_{\bar\alpha}(a\varphi_{\alpha,\bar\alpha})]\psi_{\bar\alpha,\bar\alpha}$$

which implies $a' = \omega_{\bar\alpha}(a\varphi_{\alpha,\bar\alpha})$. One shows similarly that $a'' = (a\varphi_{\alpha,\bar\alpha})\omega_{\bar\alpha}$. Next let $a \in S_\alpha$, $\alpha \geqq \beta$, $\alpha \in I$. Then

$$[(\omega_\alpha a)\varphi_{\alpha,\beta}]\psi_{\beta,\beta} = (\omega_\alpha a)\psi_{\alpha,\beta} = \omega_\beta(a\psi_{\alpha,\beta}) = [\omega_\beta(a\varphi_{\alpha,\beta})]\psi_{\beta,\beta}$$

and thus $(\omega_\alpha a)\varphi_{\alpha,\beta} = \omega_\beta(a\varphi_{\alpha,\beta})$. The second formula in (C2′) is proved analogously.

Now assume that (C1) and (C2′) hold. For $a \in S_\alpha$, let $a' = \omega_{\bar\alpha}(a\varphi_{\alpha,\bar\alpha})$, $a'' = (a\varphi_{\alpha,\bar\alpha})\omega_{\bar\alpha}$. If $\beta \leqq \bar\alpha$, we obtain

$$a'\psi_{\bar\alpha,\beta} = (a'\varphi_{\bar\alpha,\beta})\psi_{\beta,\beta} = \{[\omega_{\bar\alpha}(a\varphi_{\alpha,\bar\alpha})]\varphi_{\bar\alpha,\beta}\}\psi_{\beta,\beta} = [(\omega_\alpha a)\varphi_{\alpha,\bar\alpha}\varphi_{\bar\alpha,\beta}]\psi_{\beta,\beta}$$
$$= [(\omega_\alpha a)\varphi_{\alpha,\beta}]\psi_{\beta,\beta} = [\omega_\beta(a\varphi_{\alpha,\beta})]\psi_{\beta,\beta} = \omega_\beta[(a\varphi_{\alpha,\beta})\psi_{\beta,\beta}] = \omega_\beta(a\psi_{\alpha,\beta})$$

giving the first formula in (C2). The second formula in (C2) is proved similarly.

(ii) We note first that for any $I \in \mathscr{R}_Y$, $\mathscr{B} \cap \Omega_I$ contains $(\omega_\alpha)_{\alpha \in I}$ with $\omega_\alpha = (\iota_{S_\alpha}, \iota_{S_\alpha})$, the identity bitranslation. Consequently $\mathscr{B} \cap \Omega_I \neq \phi$. For $I, J \in \mathscr{R}_Y$, $I \supseteq J$, $(\omega_\alpha)_{\alpha \in I} \in \mathscr{B} \cap \Omega_I$, condition (C2′) for $(\omega_\alpha)_{\alpha \in J}$ is the restriction of condition (C2′) for $(\omega_\alpha)_{\alpha \in I}$. Consequently $(\omega_\alpha)_{\alpha \in J} \in \mathscr{B} \cap \Omega_J$, and $\Phi_{I,J}$ maps $\mathscr{B} \cap \Omega_I$ into $\mathscr{B} \cap \Omega_J$. The assertion now follows from the multiplication in $\mathscr{F}$.

(iii) The remark at the beginning of the proof implies that

$$\mathscr{C} = \{((c\varphi_{\gamma,\alpha})\psi_{\alpha,\alpha})_{\alpha \leqq \gamma} \text{ for some } c \in S_\gamma,\ \gamma \in Y\}.$$

It is easy to see that this set coincides with the set in item (iii).

Condition (C2′) can be schematically represented as the commutativity of the following diagram:

$$\begin{array}{ccc} S_\alpha & \xrightarrow{\omega_\alpha} & S_\alpha \\ {\scriptstyle \varphi_{\alpha,\beta}}\downarrow & & \downarrow{\scriptstyle \varphi_{\alpha,\beta}} \\ S_\beta & \xrightarrow{\omega_\beta} & S_\beta \end{array}$$

COROLLARY 1. *If S is a strong semilattice of (commutative) cancellative semigroups, so is $\Omega(S)$.*

In [**5**] we have modified the notion of an inverse limit of groups to describe the translational hull of a semilattice of groups. We now offer the following variant of this concept.

For a given system $[Y; S_\alpha, \varphi_{\alpha,\beta}]$, we let

$$\text{Inv lim}_{\mathscr{R}}\{S_\alpha\}_{\alpha \in Y} = \{(a_\alpha)_{\alpha \in I} | I \in \mathscr{R}_Y,\ a_\alpha \in S_\alpha,\ a_\alpha\varphi_{\alpha,\beta} = a_\beta \text{ if } \alpha > \beta\}$$

with multiplication

$$(a_\alpha)_{\alpha \in I}(b_\alpha)_{\alpha \in J} = (a_\alpha b_\alpha)_{\alpha \in I \cap J}.$$

COROLLARY 2. *Let $S = (Y; S_\alpha, \psi_{\alpha,\beta})$, where each S_α has an identity e_α and the set $E = \{e_\alpha | \alpha \in Y\}$ is a subsemigroup of S. Then* $\mathbf{\Omega(S)} \cong \text{Inv lim}_{\mathscr{R}}\{S_\alpha\}_{\alpha \in Y}$.

Proof. Since each S_α has an identity, for any $\alpha > \beta$, the semigroup $S_\alpha \cup S_\beta$ is an extension of S_β determined by the homomorphism $\varphi_{\alpha,\beta} : a \rightarrow ae_\beta = e_\beta a$ by [**4**, III.4.5]. The hypothesis that E is a subsemigroup easily implies that $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$ if $\alpha > \beta > \gamma$. Consequently the composition is strong.

Let $(\omega_\alpha)_{\alpha \in I} \in \mathscr{B}$. Since S_α has an identity, we must have $\Omega(S_\alpha) = \Pi(S_\alpha)$ by [**4**, V.1.4]. Hence $\omega_\alpha = a_\alpha \psi_{\alpha,\alpha}$ for some $a_\alpha \in S$, $\alpha \in I$. If now $\alpha \in I$, $\alpha > \beta$, we obtain by (C2′),

$$a_\alpha\varphi_{\alpha,\beta} = [(a_\alpha\psi_{\alpha,\alpha})e_\alpha]\varphi_{\alpha,\beta} = (a_\beta\psi_{\beta,\beta})(e_\alpha\varphi_{\alpha,\beta}) = a_\beta e_\beta = a_\beta,$$

which shows that $(a_\alpha)_{\alpha \in I} \in \text{Inv lim}_{\mathscr{R}}\{S_\alpha\}_{\alpha \in Y}$. Conversely, if $(a_\alpha)_{\alpha \in I} \in \text{Inv lim}_{\mathscr{R}}$-$\{S_\alpha\}_{\alpha \in Y}$, then it is clear that $(a_\alpha\psi_{\alpha,\alpha})_{\alpha \in I} \in \mathscr{B}$.

For a given system $[Y; S_\alpha, \varphi_{\alpha,\beta}]$, we can also define

$$\text{inv lim}\{S_\alpha\}_{\alpha\in Y} = \{(a_\alpha) \in \Pi_{\alpha\in Y} S_\alpha | a_\alpha\varphi_{\alpha,\beta} = a_\beta \text{ if } \alpha > \beta\}$$

with multiplication inhereted from the direct product. It is easy to see that

$$\text{Inv lim}_{\mathscr{R}}\{S_\alpha\}_{\alpha\in Y} \cong [\mathscr{R}_Y, \text{inv lim}\{S_\alpha\}_{\alpha\in I}, \bar{\varphi}_{I,J}]$$

where

$$\bar{\varphi}_{I,J} : (a_\alpha)_{\alpha\in I} \to (a_\alpha)_{\alpha\in J} \qquad (I, J \in \mathscr{R}_Y, I \supseteq J).$$

Most of the results in [**5**, §3] can be obtained by specializing some of the statements of this section to a semigroup which is a semilattice of groups.

5. Sturdy compositions. In this section we fix a sturdy composition

$$S = \langle Y; S_\alpha, \varphi_{\alpha,\beta} \rangle$$

of weakly reductive semigroups S_α.

LEMMA 4. *The functions $\Phi_{I,J}$ defined in Theorem 2 are one-to-one.*

Proof. Let $(\omega_\alpha)_{\alpha\in I}, (\theta_\alpha)_{\alpha\in I} \in \mathscr{B}$, $I \supseteq J$, $I, J \in \mathscr{R}_Y$ and assume that

$$(\omega_\alpha)_{\alpha\in I}\Phi_{I,J} = (\theta_\alpha)_{\alpha\in I}\Phi_{I,J}.$$

Then $\omega_\alpha = \theta_\alpha$ for all $\alpha \in J$. Let $a \in S_\alpha$, $\alpha \geqq \beta$, $\alpha \in I$, $\beta \in J$. Then

$$(\omega_\alpha a)\varphi_{\alpha,\beta} = \omega_\beta(a\varphi_{\alpha,\beta}) = \theta_\beta(a\varphi_{\alpha,\beta}) = (\theta_\alpha a)\varphi_{\alpha,\beta},$$

which by hypothesis on $\varphi_{\alpha,\beta}$ implies $\omega_\alpha a = \theta_\alpha a$. One shows analogously that $a\omega_\alpha = a\theta_\alpha$, so that $\omega_\alpha = \theta_\alpha$. Consequently $(\omega_\alpha)_{\alpha\in I} = (\theta_\alpha)_{\alpha\in I}$.

On any sturdy composition $T = \langle Z; T_\alpha, \zeta_{\alpha,\beta} \rangle$ define a relation σ by

$$a \ \sigma \ b \text{ if } a\zeta_{\alpha,\alpha\beta} = b\zeta_{\beta,\alpha\beta} \qquad (a \in S_\alpha, b \in S_\beta).$$

It is proved in [**4**, III.7.11] that σ is a congruence. We will use the notation $\bar{T} = T/\sigma$. Caution: σ depends on the way T is decomposed into a semilattice of subsemigroups. The class of σ containing an element $a \in T$ will be denoted by $[a]$ in any semigroup. We now let

$$\mathscr{B} = \langle \mathscr{R}_Y; \mathscr{B} \cap \Omega_I, \Phi_{I,J} \rangle$$

so that, by Theorem 2 and Lemma 4, we have $\mathscr{B} \cong B$. Let $\bar{S} = S/\sigma$ and $\bar{B} = B/\sigma$ where both σ's are defined relative to the particular semilattice decompositions expressed by the above notation. From Lemma 4 and [**4**, III.7.11], we immediately obtain

COROLLARY. *The mapping*

$$\zeta : (\omega_\alpha)_{\alpha\in I} \to (I, [(\omega_\alpha)_{\alpha\in I}]) \qquad ((\omega_\alpha)_{\alpha\in I} \in \mathscr{B})$$

is an isomorphism of $\mathscr{B}$ onto a subdirect product of $\mathscr{R}_Y$ and $\bar{B}$.

THEOREM 3. *On $\bar{B}$ define a mapping η by*

$$\eta : [(\omega_\alpha)_{\alpha \in I}] \rightarrow \omega$$

where for $a \in S_\alpha$, $(\alpha) \cap I = (\bar{\alpha})$,

$$\omega[a] = [\omega_{\bar{\alpha}}(a\varphi_{\alpha,\bar{\alpha}})], \qquad [a]\omega = [(a\varphi_{\alpha,\bar{\alpha}})\omega_{\bar{\alpha}}].$$

Then η is an embedding of $\bar{B}$ into $\Omega(\bar{S})$.

Proof. Let $(\omega_\alpha)_{\alpha \in I} \in \mathscr{B}$ and $\omega = [(\omega_\alpha)_{\alpha \in I}]\eta$. In order to show that ω is well-defined, we let $a \in S_\alpha$, $b \in S_\beta$, $[a] = [b]$, $(\gamma) \cap I = (\bar{\gamma})$ for all $\gamma \in Y$. Then $a\varphi_{\alpha,\alpha\beta} = b\varphi_{\beta,\alpha\beta}$ and thus

$$\begin{aligned}(\omega_{\bar{\alpha}}(a\varphi_{\alpha,\bar{\alpha}}))\varphi_{\bar{\alpha},\overline{\alpha\beta}} &= \omega_{\overline{\alpha\beta}}(a\varphi_{\alpha,\bar{\alpha}}\varphi_{\bar{\alpha},\overline{\alpha\beta}}) = \omega_{\overline{\alpha\beta}}(a\varphi_{\alpha,\overline{\alpha\beta}}) \\ &= \omega_{\overline{\alpha\beta}}(a\varphi_{\alpha,\alpha\beta}\varphi_{\alpha\beta,\overline{\alpha\beta}}) = \omega_{\overline{\alpha\beta}}(b\varphi_{\beta,\alpha\beta}\varphi_{\alpha\beta,\overline{\alpha\beta}}) \\ &= \omega_{\overline{\alpha\beta}}(b\varphi_{\beta,\overline{\alpha\beta}}) = \omega_{\overline{\alpha\beta}}(b\varphi_{\beta,\bar{\beta}}\varphi_{\bar{\beta},\overline{\alpha\beta}}) = (\omega_{\bar{\beta}}(b\varphi_{\beta,\bar{\beta}}))\varphi_{\bar{\beta},\overline{\alpha\beta}}\end{aligned}$$

so that $\omega[a] = \omega[b]$. A similar argument shows that also $[a]\omega = [b]\omega$.

We show next that $\omega \in \Omega(\bar{S})$. With the same notation, we obtain

$$\begin{aligned}(\omega[a])[b] &= [\omega_{\bar{\alpha}}(a\varphi_{\alpha,\bar{\alpha}})][b] = [(\omega_{\bar{\alpha}}(a\varphi_{\alpha,\bar{\alpha}}))\varphi_{\bar{\alpha},\overline{\alpha\beta}}(b\varphi_{\beta,\overline{\alpha\beta}})] \\ &= [(\omega_{\overline{\alpha\beta}}(a\varphi_{\alpha,\overline{\alpha\beta}}))(b\varphi_{\beta,\overline{\alpha\beta}})] = [\omega_{\overline{\alpha\beta}}((a\varphi_{\alpha,\overline{\alpha\beta}})(b\varphi_{\beta,\overline{\alpha\beta}}))] \\ &= [\omega_{\overline{\alpha\beta}}(((a\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta}))\varphi_{\alpha\beta,\overline{\alpha\beta}})] = [\omega_{\overline{\alpha\beta}}((ab)\varphi_{\alpha\beta,\overline{\alpha\beta}})] \\ &= \omega([ab]) = \omega([a][b]),\end{aligned}$$

and similarly

$$[a]([b]\omega) = ([a][b])\omega.$$

Further,

$$\begin{aligned}([a]\omega)[b] &= [(a\varphi_{\alpha,\bar{\alpha}})\omega_{\bar{\alpha}}][b] = [((a\varphi_{\alpha,\bar{\alpha}})\omega_{\bar{\alpha}})\varphi_{\bar{\alpha},\overline{\alpha\beta}}(b\varphi_{\alpha,\overline{\alpha\beta}})] \\ &= [((a\varphi_{\alpha,\overline{\alpha\beta}})\omega_{\overline{\alpha\beta}})(b\varphi_{\beta,\overline{\alpha\beta}})] = [(a\varphi_{\alpha,\overline{\alpha\beta}})(\omega_{\overline{\alpha\beta}}(b\varphi_{\beta,\overline{\alpha\beta}}))] = [a](\omega[b]).\end{aligned}$$

Consequently $\omega \in \Omega(\bar{S})$.

Let $(\omega_\alpha)_{\alpha \in I}$, $(\theta_\alpha)_{\alpha \in J} \in \mathscr{B}$. For any $\alpha \in Y$, let

$$(\alpha) \cap I = (\bar{\alpha}), \qquad (\alpha) \cap J = (\hat{\alpha}), \qquad \alpha^* = \bar{\hat{\alpha}}.$$

Then

$$(\alpha^*) = (\hat{\alpha}) \cap I = ((\alpha) \cap J) \cap I = (\alpha) \cap (I \cap J),$$

and thus, for any $a \in S_\alpha$, we have

$$\begin{aligned}[(\omega_\alpha)_{\alpha \in I}]\eta[(\theta_\alpha)_{\alpha \in J}]\eta[a] &= [(\omega_\alpha)_{\alpha \in I}]\eta[\theta_{\hat{\alpha}}(a\varphi_{\alpha,\hat{\alpha}})] \\ &= [\omega_{\alpha^*}((\theta_{\hat{\alpha}}(a\varphi_{\alpha,\hat{\alpha}}))\varphi_{\hat{\alpha},\alpha^*}] = [\omega_{\alpha^*}\theta_{\alpha^*}(a\varphi_{\alpha,\alpha^*})] \\ &= [(\omega_\alpha\theta_\alpha)_{\alpha \in I}]\eta[a] = ([(\omega_\alpha)_{\alpha \in I}][(\theta_\alpha)_{\alpha \in J}])\eta[a].\end{aligned}$$

The formula with $[a]$ on the left is proved analogously. Hence η is a homomorphism.

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With the same notation, assume that

$$[(\omega_\alpha)_{\alpha \in I}]\eta = [(\theta_\alpha)_{\alpha \in J}]\eta.$$

Then for any $a \in S_\alpha$, $\alpha \in I \cap J$, we have $[\omega_\alpha a] = [\theta_\alpha a]$ which evidently implies $\omega_\alpha a = \theta_\alpha a$; analogously $a\omega_\alpha = a\theta_\alpha$. Consequently $[(\omega_\alpha)_{\alpha \in I}] = [(\theta_\alpha)_{\alpha \in J}]$ which proves that η is one-to-one.

COROLLARY. *$\Omega(S)$ can be embedded into $\Omega(Y) \times \Omega(\bar{S})$.*

Proof. By Theorem 1, $\Omega(S) \cong \mathscr{B}$; by [**4**, V.6.1], $\Omega(Y) \cong \mathscr{R}_Y$. It remains to apply Theorem 3 and the corollary preceding it.

This corollary establishes a connection between the translational hull of S and the translational hulls of two of its homomorphic images. For example, if each S_α is (commutative) cancellative, then by [**4**, III.7.11], $\bar{S}$ is also, and the corollary implies that if S is a subdirect product of a semilattice and a (commutative) cancellative semigroup, then so is $\Omega(S)$, for the corresponding statement holds both for semilattices and (commutative) cancellative semigroups by [**4**, III.5.14, 5.17]. In the next section, we will prove this statement directly and obtain some additional information.

6. Subdirect product of a semilattice and a cancellative semigroup.

For these semigroups we establish here precise statements concerning their translational hulls.

THEOREM 4. *Let S be a subdirect product of a semilattice Y and a cancellative semigroup C. For any $\omega \in \Omega(S)$, there exist unique $\omega' \in \Omega(Y)$ and $\omega'' \in \Omega(C)$ such that*

$$(1) \quad \omega(\alpha, a) = (\omega'\alpha, \omega''a), \qquad (\alpha, a)\omega = (\alpha\omega', a\omega'') \qquad ((\alpha, a) \in S).$$

The mapping

$$\epsilon : \omega \longrightarrow (\omega', \omega'') \qquad (\omega \in \Omega(S))$$

is an isomorphism of $\Omega(S)$ onto $i_{\Omega(Y)\times\Omega(C)}(\Pi(S)\epsilon)$.

Proof. We may suppose that S is a subsemigroup of $Y \times C$. Let $\omega \in \Omega(S)$ and define bioperators σ and τ on S by the following formulae

$$\omega(\alpha, a) = (\sigma(\alpha, a), \tau(\alpha, a)),$$
$$(\alpha, a)\omega = ((\alpha, a)\sigma, (\alpha, a)\tau).$$

For any (α, a), $(\beta, b) \in S$, using the properties of ω, we easily derive

$$[\sigma(\alpha, a)]\beta = \sigma(\alpha\beta, ab) \qquad [\tau(\alpha, a)]b = \tau(\alpha\beta, ab)$$
$$\alpha[(\beta, b)\sigma] = (\alpha\beta, ab)\sigma \qquad a[(\beta, b)\tau] = (\alpha\beta, ab)\tau$$
$$[(\alpha, a)\sigma]\beta = \alpha[\sigma(\beta, b)] \qquad [(\alpha, a)\tau]b = a[\tau(\beta, b)].$$

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The relation η defined on S by

$$(\alpha, a)\ \eta\ (\beta, b) \quad \text{if } \alpha = \beta$$

is evidently a semilattice congruence. Now let $(\alpha, a), (\alpha, b) \in S$. Then $(\alpha, a)\eta$-(α, b) which by Lemma 1 implies $\omega(\alpha, a)\ \eta\ \omega(\alpha, b)$. This means that $\sigma(\alpha, a) = \sigma(\alpha, b)$. Hence we may write $\sigma\alpha = \sigma(\alpha, a)$ and consider σ defined on Y. The same type of argument is valid for $(\alpha, a)\omega$, and hence writing $\alpha\sigma = (\alpha, a)$, we have a bioperator mapping Y into itself. The properties of σ stated above imply at once that $\sigma \in \Omega(Y)$.

Next let $(\alpha, a), (\beta, a) \in S$. Then

$$\tau(\alpha, a)a = \tau(\alpha\beta, a^2) = \tau(\beta\alpha, a^2) = \tau(\beta, a)a$$

which by cancellation in C implies $\tau(\alpha, a) = \tau(\beta, a)$. Hence we may write $\tau a = \tau(\alpha, a)$. The same type of argument is valid for $(\alpha, a)\tau$ and we may write $a\tau = (\alpha, a)\tau$. Consequently τ is a bioperator mapping C into itself. The properties of τ stated above yield $\tau \in \Omega(C)$.

The uniqueness of σ and τ follows immediately from the hypothesis that S is a subdirect product of Y and C. Letting $\omega' = \sigma$ and $\omega'' = \tau$, we obtain formulae (1). It is very easy to see that ϵ is an isomorphism of $\Omega(S)$ into $\Omega(Y) \times \Omega(C)$. Since $\Pi(S)$ is an ideal of $\Omega(S)$, it follows that $\Pi(S)\epsilon$ is an ideal of $\Omega(S)\epsilon$ and thus

(2) $\Omega(S)\epsilon \subseteq i_{\Omega(Y)\times\Omega(C)}(\Pi(S)\epsilon)$.

It is easy to see that

(3) $\epsilon : \pi_{(\alpha,a)} \to (\pi_\alpha, \pi_a) \qquad ((\alpha, a) \in S)$.

In order to establish the opposite inclusion in (2), we let $(\sigma, \tau) \in i_{\Omega(Y)\times\Omega(C)}(\Pi$-$(S)\epsilon)$. Let $(\alpha, a) \in S$. In view of (3), there exist unique $(\alpha, a)', (\alpha, a)'' \in S$ such that

(4) $(\sigma, \tau)(\pi_{(\alpha,a)}\epsilon) = \pi_{(\alpha,a)'}\epsilon, \qquad (\pi_{(\alpha,a)}\epsilon)(\sigma, \tau) = \pi_{(\alpha,a)''}\epsilon.$

Now writing $(\alpha, a)' = (\lambda(\alpha, a), \rho(\alpha, a))$, by (3) and the first formula in (4), we have

$$(\sigma, \tau)(\pi_\alpha, \pi_a) = (\pi_{\lambda(\alpha,a)}, \pi_{\rho(\alpha,a)})$$

and thus

$$\pi_{\sigma\alpha} = \sigma\pi_\alpha = \pi_{\lambda(\alpha,a)}, \qquad \pi_{\tau a} = \tau\pi_a = \pi_{\rho(\alpha,a)}$$

so that $\sigma\alpha = \lambda(\alpha, a)$ and $\tau a = \rho(\alpha, a)$. Consequently $(\sigma\alpha, \tau a) = (\alpha, a)' \in S$. A similar argument, using the second formula in (4), can be used to prove that $(\alpha\sigma, a\tau) = (\alpha, a)'' \in S$. Now letting

$$\omega(\alpha, a) = (\sigma\alpha, \tau a), \qquad (\alpha, a)\omega = (\alpha\sigma, a\tau) \qquad ((\alpha, a) \in S)$$

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we evidently have $\omega \in \Omega(S)$ and $\omega\epsilon = (\sigma, \tau)$. Consequently $(\sigma, \tau) \in \Omega(S)\epsilon$, as required.

COROLLARY 1. *Let the notation be as in Theorem 4. Then S can be densely embedded into $\Omega(Y) \times \Omega(C)$ and also into $\mathscr{I}_Y \times \Omega(C)$.*

Proof. For the first embedding it suffices to take $\pi\epsilon$, where $\pi : S \to \Pi(S)$ is the canonical isomorphism, and apply Theorem 4. By [**4**, V.6.1], $\Omega(Y) \cong \mathscr{R}_Y$. Hence S can be densely embedded into $\mathscr{R}_Y \times \Omega(C)$ in view of the first embedding. It is easy to verify that the image of S in $\mathscr{R}_Y \times \Omega(C)$ has the same idealizer in $\mathscr{R}_Y \times \Omega(C)$ as in $\mathscr{I}_Y \times \Omega(C)$. Consequently S can also be densely embedded into $\mathscr{I}_Y \times \Omega(C)$.

COROLLARY 2. *If S is a subdirect product of a semilattice and a* (*commutative*) *cancellative semigroup, so is $\Omega(S)$.*

Proof. In the notation of Theorem 4, $\Omega(Y)$ is a semilattice by [**4**, V.6.2]; $\Omega(C)$ is cancellative by [**4**, III.5.14], $\Omega(C)$ is also commutative if C is by [**4**, III.5.16]. The assertion now follows from the fact that $\Omega(S)\epsilon$ is a subdirect product of its projections in $\Omega(Y)$ and $\Omega(C)$.

COROLLARY 3. *Let Y be a semilattice and C be a cancellative semigroup. Then $\Omega(Y \times C) \cong \Omega(Y) \times \Omega(C)$.*

Proof. This follows easily from the proof of Theorem 4.

The principal results of [**5**, §4] follow from the statements of this section by specialization.

7. Spined products. These represent a special case of a subdirect product, the pertinent definitions are given below.

Let $S_1, S_2, \ldots, S_n$ be semigroups. If $\sigma_i \in \Omega(S_i)$ for $i = 1, 2, \ldots, n$, it is easy to see that the bioperator ω defined on the direct product $S_1 \times S_2 \times \ldots \times S_n$ by

$$\omega(s_1, s_2, \ldots, s_n) = (\sigma_1 s_1, \sigma_2 s_2, \ldots, \sigma_n s_n),$$
$$(s_1, s_2, \ldots, s_n)\omega = (s_1\sigma_1, s_2\sigma_2, \ldots, s_n\sigma_n)$$

is a bitranslation of $S_1 \times S_2 \times \ldots \times S_n$; we write $\omega = (\sigma_1, \sigma_2, \ldots, \sigma_n)$.

Definition 1. The *bitranslations of* $S_1 \times S_2 \times \ldots \times S_n$ *split* if every $\omega \in \Omega(S_1 \times S_2 \times \ldots \times S_n)$ is of the form $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ for some $\sigma_i \in \Omega(S_i)$, $i = 1, 2, \ldots, n$.

Note that $\sigma_1, \sigma_2, \ldots, \sigma_n$ are unique and that this property implies that

$$\Omega(S_1 \times S_2 \times \ldots \times S_n) \cong \Omega(S_1) \times \Omega(S_2) \times \ldots \times \Omega(S_n).$$

For example, this is the case when S_1 is a semilattice, S_2 a left zero semigroup, S_3 a group, S_4 a right zero semigroup according to [**4**, V.6.8, Exercise 2]. It follows from Theorem 4 that this is the case also when S_1 is a semilattice and S_2 is a cancellative semigroup.

We next modify the notion of a spined product as follows:

Definition 2. For $i = 1, 2, \ldots, n$, let S_i be a semilattice Y of semigroups S_i^α. Then the subsemigroup

$$\bigcup_{\alpha \in Y} (S_1^\alpha \times S_2^\alpha \times \ldots \times S_n^\alpha)$$

of the direct product $S_1 \times S_2 \times \ldots \times S_n$ is the *spined product of* $S_1, S_2, \ldots, S_n$ *over* Y. The phrase "over Y" will be omitted if Y is the greatest semilattice decomposition of each S_i.

It should be remarked that Y is common to all S_i and that the decomposition of S_i induced by Y need not be the greatest semilattice decomposition. For examples of spined products see [**4**, IV.4.6, 4.7]; we will encounter some below. The desired theorem can now be stated. For simplicity, we consider only the case $n = 2$; the general case then follows by induction, or by an obvious modification of the proof below.

THEOREM 5. *Let S be a spined product of $T = (Y; T_\alpha, \varphi_{\alpha,\beta})$ and $V = (Y; V_\alpha, \psi_{\alpha,\beta})$ over Y, and assume that for every $\alpha \in Y$, both T_α and V_α are weakly reductive and the bitranslations of $T_\alpha \times V_\alpha$ split. Then $\Omega(S)$ is a spined product of $\Omega(T)$ and $\Omega(V)$ over $\mathscr{R}_Y$.*

Proof. In light of Definition 2, we can write

$$S = (Y; S_\alpha, \chi_{\alpha,\beta})$$

where $S_\alpha = T_\alpha \times V_\alpha$ and

$$(1) \quad (t, v)\chi_{\alpha,\beta} = (t\varphi_{\alpha,\beta}, v\psi_{\alpha,\beta})$$

for all $(t, v) \in S_\alpha$, $\alpha \geqq \beta$. Each S_α is weakly reductive, so by Theorem 1, we can consider $\mathscr{B} = \mathscr{B}(Y; S_\alpha, \chi_{\alpha,\beta})$ instead of $\Omega(S)$.

Let $(\omega_\alpha)_{\alpha \in I} \in \mathscr{B}$. For each $\alpha \in I$, we have $\omega_\alpha \in \Omega(T_\alpha \times V_\alpha)$, which by the hypothesis of splitting implies that $\omega_\alpha = (\tau_\alpha, \nu_\alpha)$ for some $\tau_\alpha \in \Omega(T_\alpha)$ and $\nu_\alpha \in \Omega(V_\alpha)$. Now let $(t, v) \in T_\alpha \times V_\alpha$; $(\alpha) \cap I = (\bar{\alpha})$ by condition (C1). In view of condition (C2), there exist $(t, v)', (t, v)'' \in T_{\bar{\alpha}} \times V_{\bar{\alpha}}$ such that

$$(2) \quad (t,v)'\chi_{\bar{\alpha},\beta} = \omega_\beta[(t,v)\chi_{\alpha,\beta}], \qquad (t,v)''\chi_{\bar{\alpha},\beta} = [(t,v)\chi_{\alpha,\beta}]\omega_\beta \qquad (\beta \leqq \bar{\alpha}).$$

We now write $(t, v)' = ((t, v)\xi, (t, v)\eta) \in T_{\bar{\alpha}} \times V_{\bar{\alpha}}$, so that the first formula in (2) by virtue of (1) becomes

$$([(t, v)\xi]\varphi_{\bar{\alpha},\beta}, [(t, v)\eta]\psi_{\bar{\alpha},\beta}) = (\tau_\beta, \nu_\beta)(t\varphi_{\alpha,\beta}, t\psi_{\alpha,\beta}).$$

Writing this expression by coordinates gives

$$(3) \quad [(t, v)\xi]\varphi_{\bar{\alpha},\beta} = \tau_\beta(t\varphi_{\alpha,\beta}), \qquad [(t, v)\eta]\nu_{\bar{\alpha},\beta} = \nu_\beta(v\psi_{\alpha,\beta}).$$

The right hand side of the first formula in (3) does not contain v, so that the left hand side is independent of v. Consequently we can write t' instead of $(t, v)\xi$. Similarly, in the second formula in (3) we can write v' instead of $(t, v)\eta$. Hence (3) takes on the form

$$(4) \quad t'\varphi_{\bar{\alpha},\beta} = \tau_\beta(t\varphi_{\alpha,\beta}), \qquad v'\psi_{\bar{\alpha},\beta} = \nu_\beta(v\psi_{\alpha,\beta}).$$

An analogous argument shows that (t'', v'') can be found in $T_{\bar{\alpha}} \times V_{\bar{\alpha}}$ satisfying

$$(5) \quad t''\varphi_{\bar{\alpha},\beta} = (t\varphi_{\alpha,\beta})\tau_\beta, \qquad v''\psi_{\bar{\alpha},\beta} = (v\psi_{\alpha,\beta})\nu_\beta.$$

Formulae (4) and (5) are valid for all $\alpha \in Y$ which in view of Theorem 1 implies that $(\tau_\alpha)_{\alpha\in I} \in \mathscr{B}_1$, $(\nu_\alpha)_{\alpha\in I} \in \mathscr{B}_2$ where $\mathscr{B}_1 = \mathscr{B}(Y; T_\alpha, \varphi_{\alpha,\beta})$, $\mathscr{B}_2 = \mathscr{B}(Y; V_\alpha, \psi_{\alpha,\beta})$.

Since for each $\alpha \in I$, ω_α uniquely determines τ_α and ν_α, we have that

$$\zeta : (\omega_\alpha)_{\alpha\in I} \rightarrow ((\tau_\alpha)_{\alpha\in I}, (\nu_\alpha)_{\alpha\in I})$$

maps $\mathscr{B}(Y; S_\alpha, \chi_{\alpha,\beta})$ into the spined product of $\mathscr{B}_1$ and $\mathscr{B}_2$ over $\mathscr{R}_Y$. It is now clear that if we start with $((\tau_\alpha)_{\alpha\in I}, (\nu_\alpha)_{\alpha\in I}) \in \mathscr{B}_1 \times \mathscr{B}_2$, then $(\tau_\alpha, \nu_\alpha)_{\alpha\in I}$ is the unique element $(\omega_\alpha)_{\alpha\in I}$ of $\mathscr{B}$ such that

$$(\omega_\alpha)_{\alpha\in I}\eta = ((\tau_\alpha)_{\alpha\in I}, (\nu_\alpha)_{\alpha\in I}).$$

An easy verification shows that ζ is also a homomorphism. Therefore ζ is the required isomorphism of $\mathscr{B}$ onto a spined product of $\mathscr{B}_1$ and $\mathscr{B}_2$. The assertion of the theorem now follows from Theorem 1.

Semigroups which are orthodox bands of groups have been characterized in [**6**, Theorem 3.2] as spined products of bands and semilattices of groups. It follows from [**4**, V.3.12] that the bitranslations of $B \times G$ split, where B is a rectangular band and G is a group. Hence the theorem yields

COROLLARY 1. *If S is a spined product of $T = (Y; T_\alpha, \varphi_{\alpha,\beta})$, where each T_α is a rectangular band, and $V = (Y; V_\alpha, \psi_{\alpha,\beta})$, where each V_α is a group, then $\Omega(S)$ is a spined product of $\Omega(T)$ and $\Omega(V)$ over $\mathscr{R}_Y$.*

It follows from [**7**, Theorem 5 and Corollary 2 to Theorem 7] that completely regular orthodox semigroups in which both Green's relations $\mathscr{L}$ and $\mathscr{R}$ are congruences can be characterized as spined products of a left regular band, a semilattice of groups and a right regular band. Since by [**4**, V.3.12], the bitranslations of $L \times G \times R$ split, where L is a left zero semigroup, G is a group and R is a right zero semigroup, the theorem implies

COROLLARY 2. *If S is a spined product of $L = (Y; L_\alpha, \varphi_{\alpha,\beta})$, $G = (Y; G_\alpha, \omega_{\alpha,\beta})$, $R = (Y; R_\alpha, \psi_{\alpha,\beta})$, where each L_α is a left zero semigroup, G_α is a group and R_α is a right zero semigroup, then $\Omega(S)$ is a spined product of $\Omega(L)$, $\Omega(G)$ and $\Omega(R)$ over $\mathscr{R}_Y$.*

Completely regular semigroups whose idempotents form a normal band have been characterized in [**3**, Construction 4.4] as spined products of a left normal band, a semilattice of groups and a right normal band. All three of these semigroups are strong compositions of their $\mathscr{N}$-classes. Hence we are dealing with a semigroup S which is a spined product of

$$[Y; L_\alpha, \varphi_{\alpha,\beta}], \quad [Y; G_\alpha, \omega_{\alpha,\beta}], \quad [Y; R_\alpha, \psi_{\alpha,\beta}].$$

Corollary 2 is applicable in this case. We will give a more precise description of the translational hull of each of these three semigroups. First note that the translational hull of a left (resp. right) zero semigroup A can be identified with the semigroup of all transformations on A written on the left (resp. right), see [**4**, V.3.12], and that all bitranslations of a group are inner. We now introduce some convenient notation.

Let $[Y; T_\alpha, \varphi_{\alpha,\beta}; F_\alpha]$ stand for the following: $[Y; T_\alpha, \varphi_{\alpha,\beta}]$ is a system as defined previously, F_α is a nonempty set of functions on T_α written on the left or right. Let

$$\lim[Y; T_\alpha, \varphi_{\alpha,\beta}; F_\alpha]$$

be the set of all $(\psi_\alpha)_{\alpha \in Y} \in \Pi_{\alpha \in Y} F_\alpha$ for which the diagram

$$\begin{array}{ccc} T_\alpha & \xrightarrow{\psi_\alpha} & T_\alpha \\ {\scriptstyle \varphi_{\alpha,\beta}} \downarrow & & \downarrow {\scriptstyle \varphi_{\alpha,\beta}} \\ T_\beta & \xrightarrow{\psi_\beta} & T_\beta \end{array}$$

is commutative whenever $\alpha > \beta$, with the multiplication inherited from the direct product. Next let

$$\operatorname{Lim}\,[Y; T_\alpha, \varphi_{\alpha,\beta}; F_\alpha] = \bigcup_{I \in \mathscr{R}_Y} \lim\,[I; T_\alpha, \varphi_{\alpha,\beta}; F_\alpha]$$

with the multiplication

$$(\psi_\alpha)_{\alpha \in I} \cdot (\delta_\alpha)_{\alpha \in J} = (\psi_\alpha \delta_\alpha)_{\alpha \in I \cap J}.$$

For example, if all T_α are groups, F_α are right translations, then it is easy to see that $(\rho_{a_\alpha})_{\alpha \in Y} \in \lim[Y; T_\alpha, \varphi_{\alpha,\beta}; F_\alpha]$ if and only if $(a_\alpha)_{\alpha \in Y} \in \operatorname{inv}\lim\{T_\alpha\}_{\alpha \in Y}$. Consequently $(\rho_{a_\alpha})_{\alpha \in I} \in \operatorname{Lim}[Y; T_\alpha, \varphi_{\alpha,\beta}; F_\alpha]$ if and only if $(a_\alpha)_{\alpha \in I} \in \operatorname{Inv} \lim_{\mathscr{R}}\{T_\alpha\}_{\alpha \in Y}$. Hence the above concept can be considered as a generalization of the inverse limit of groups.

In view of the above discussion, the theorem yields

COROLLARY 3. *If S is a spined product of*

$$[Y; L_\alpha, \varphi_{\alpha,\beta}], \quad [Y; G_\alpha, \omega_{\alpha,\beta}], \quad [Y; R_\alpha, \psi_{\alpha,\beta}],$$

where $L_\alpha \times G_\alpha \times R_\alpha$ is a rectangular group, then $\Omega(S)$ is a spined product of

$$\operatorname{Lim}[Y; L_\alpha, \varphi_{\alpha,\beta}; \mathscr{T}(L_\alpha)], \quad \operatorname{Inv}\lim\{G_\alpha\}_{\alpha \in Y}, \quad \operatorname{Lim}[Y; R_\alpha, \psi_{\alpha,\beta}; \mathscr{T}'(R_\alpha)]$$

over $\mathscr{R}_Y$.

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