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ON SEMI-ARTINIAN MODULES AND INJECTIVITY CONDITIONS

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It is well known that a module M has finite length if and only if it is semi-artinian and Noetherian or, equivalently, semi-noetherian and artinian. Our main result shows that finite length is often achieved by just assuming that M is semi-artinian, semi-noetherian and has finitely generated socle.

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Introduction

Throughout R is a ring with identity and all modules are unital right R-modules. The category of all such modules is denoted by mod-R. A module M is called *semi-artinian* (or a *Loewy module*) if every non-zero homomorphic image of M contains a simple submodule or, equivalently, if every non-zero homomorphic image of M has essential socle. Dually, a module M is *semi-noetherian* if every non-zero submodule contains a maximal submodule. It is well known (see, for example, [**1**, §11]) that a module M has finite length if and only if it is semi-artinian and Noetherian or, equivalently, it is semi-noetherian and Artinian. The main result of this note shows that we can often get finite length by just assuming that M is semi-artinian, semi-noetherian and has finitely generated socle. As a consequence of this we can obtain a recent characterization of quasi-Frobenius rings in [**9**].

The ring R is called *right semi-artinian* (respectively *right semi-noetherian*) if the right R-module R is semi-artinian (semi-noetherian). It is well known (see, for example, [**10**, Proposition 22.32]) that if R is right semi-artinian then each non-zero M in mod-R is also semi-artinian. Chapter 22 of [**10**] contains further information on semi-artinian rings and rings for which every module is semi-noetherian, therein called *socular* and *B*-rings respectively. The relationship between these two classes of rings is considered in [**24**]. In particular [**24**, Théorème 3.1] shows that any commutative semi-artinian ring is semi-noetherian.

For any module M, $E(M)$ will denote its injective hull. The socle of M will be denoted by $\mathrm{Soc}(M)$. The *second socle* $\mathrm{Soc}_2(M)$ is the submodule of M containing $\mathrm{Soc}(M)$ such that $\mathrm{Soc}_2(M)/\mathrm{Soc}(M)=\mathrm{Soc}(M/\mathrm{Soc}(M))$.

Results

Our first lemma features in [**6**, Remarks (2), (3)] and in [**4**, Proposition 4.4 and Corollary 4.5]. (An analogue also appears in [**3**, Lemmas 2.1–2.3] with the descending chain condition on essential right ideals instead of the semi-artinian condition.)

Lemma 1. (1) *Let* $\{R_\lambda: \lambda \in \Lambda\}$ *be a (non-empty) collection of right semi-artinian rings and let* $R = \prod_\Lambda R_\lambda$. *Then R is right semi-artinian if and only if* Λ *is finite.*

(2) *Let R be a right semi-artinian right or left self-injective von Neumann regular ring. Then R is semiprime Artinian.*

In the seminal paper [**5**], Bass shows in his Theorem P that every left perfect ring is right semi-artinian (see also [**1**, Theorem 28.4]). We now note that the converse is true in the presence of self-injectivity.

Proposition 2. *Let R be a right or left self-injective ring. Then R is left perfect if and only if R is right semi-artinian.*

Proof. Suppose that R is right semi-artinian and let J denote the Jacobson radical of R. Then J is left T-nilpotent [**1**, Remark 28.5] and R/J is right or left self-injective von Neumann regular [**10**, Theorem 19.27]. By Lemma 1(2) and [**1**, Theorem 28.4], R is left perfect. □

Corollary 3. *Let R be a right or left self-injective ring. Suppose that R is right semi-artinian. Then R is left semi-noetherian.*

Proof. By [**1**, Theorem 28.4]. □

The converse to Corollary 3 is not true in general. To see this, let K be any field and $R = \prod_{n=1}^{\infty} K_n$, where $K_n = K$ for each $n \geqq 1$. Then R is a commutative self-injective von Neumann regular ring, so that R is semi-noetherian (being a V-ring), by [**20**, Theorem 2.1]. However, R is not semi-artinian by Lemma 1(1).

A well known open question in ring theory asks whether a right and left perfect right self-injective ring R must be quasi-Frobenius. In [**9**] this was answered affirmatively under the additional assumption that the second right socle $\mathrm{Soc}_2(R)$ is finitely generated. Our next corollary extends this result.

Corollary 4. *Let R be a right and left semi-artinian right self-injective ring such that* $\mathrm{Soc}_2(R_R)$ *is a finitely generated right ideal. Then R is a quasi-Frobenius ring.*

Proof. By Proposition 2 and [**9**, Theorem]. □

We aim to generalise Corollary 4. First we prove the main result of this note. Recall that a module is *finitely cogenerated* if its socle is essential and finitely generated.

Theorem 5. *Let R be a ring satisfying the property:*

$$(*) \qquad \mathrm{Soc}_2(\mathrm{E}(U)) \text{ is finitely generated for each simple right } R\text{-module } U.$$

Then a right R-module M has finite length if and only if M is semi-artinian, semi-noetherian and $\mathrm{Soc}(M)$ *is finitely generated.*

Proof. The necessity is clear. Conversely, suppose that M is semi-artinian, semi-noetherian and $\mathrm{Soc}(M)$ is finitely generated. Suppose that M is not Artinian. By Zorn's Lemma there exists a submodule P of M minimal in the collection of submodules L of M such that M/L is not finitely cogenerated (see [**1**, Proposition 10.10]).

Because M is semi-artinian, M/P has essential socle. Thus $\mathrm{Soc}(M/P)$ is not finitely generated. Note that $P \neq 0$ and, because M is semi-noetherian, P contains a maximal submodule Q. By the choice of P, M/Q is finitely cogenerated and hence $\mathrm{Soc}(M/Q)$ is finitely generated. Thus, without loss of generality we can suppose that $Q=0$. In this case, $\mathrm{Soc}_2(M)$ is not finitely generated, because $\mathrm{Soc}(M/P)$ is not finitely generated.

Let $S_1 = \mathrm{Soc}(M)$. There exists a positive integer n and simple submodules U_i for $1 \leqq i \leqq n$ of M such that $S_1 = U_1 \oplus \cdots \oplus U_n$. Then, without loss of generality, M is a submodule of $X = \mathrm{E}(U_1) \oplus \cdots \oplus \mathrm{E}(U_n)$ and so M/S_1 is a submodule of $X/S_1 \simeq [\mathrm{E}(U_1)/U_1] \oplus \cdots \oplus [\mathrm{E}(U_n)/U_n]$. By property $(*)$, $\mathrm{Soc}_2(X)$ is finitely generated and hence $\mathrm{Soc}_2(M)/S_1$ is also finitely generated. It follows that $\mathrm{Soc}_2(M)$ is finitely generated, a contradiction. Thus M is Artinian and so has finite length. □

Our theorem has the following corollaries.

Corollary 6. *Let R be a ring for which there exists an injective cogenerator X for mod-R such that* $\mathrm{Soc}_2(X)$ *is finitely generated. Then a right R-module M has finite length if and only if M is semi-artinian, semi-noetherian and* $\mathrm{Soc}(M)$ *is finitely generated.*

Proof. If X is an injective cogenerator for mod-R such that the socle of $X/\mathrm{Soc}(X)$ is finitely generated then $\mathrm{Soc}_2(\mathrm{E}(U))$ is finitely generated for each simple right R-module U. □

Corollary 7. *Let R be a ring and M be an injective cogenerator for mod-R. Then M has finite length if and only if M is semi-artinian, semi-noetherian and* $\mathrm{Soc}_2(M)$ *is finitely generated.*

Proof. The necessity is clear. Conversely, suppose that M is semi-artinian, semi-noetherian and $\mathrm{Soc}_2(M)$ is finitely generated. Clearly $M/\mathrm{Soc}(M)$ has finite Goldie dimension n (say). Suppose that $\mathrm{Soc}(M)$ is not finitely generated. Then

$$\mathrm{Soc}(M) = S_1 \oplus \cdots \oplus S_{n+1}$$

for some non-finitely generated submodules $S_i (1 \leqq i \leqq n+1)$. Thus

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$$M = E(S_1) \oplus \cdots \oplus E(S_{n+1})$$

and hence

$$M/\mathrm{Soc}(M) \simeq \bigoplus_{i=1}^{n+1} E(S_i)/S_i.$$

It follows that there exists $1 \leqq j \leqq n+1$ such that $E(S_j) = S_j \leqq \mathrm{Soc}_2(M)$, which is finitely generated. Being a direct summand of $\mathrm{Soc}_2(M)$, S_j is finitely generated, a contradiction. It follows that $\mathrm{Soc}(M)$ is finitely generated. By the theorem, M has finite length. □

We note that Corollary 4 above is now a consequence of Corollary 7 (taking $M = R$), Corollary 3 and the well-known fact that any right self-injective right Artinian ring is quasi-Frobenius.

Remarks and examples

(**1**). Let $R = \mathbb{Z}$, the ring of integers. Then

(i) the R-module R is Noetherian (whence semi-noetherian) with zero socle but is not Artinian,

(ii) for any prime p, the Prüfer p-group is an Artinian (whence semi-artinian) R-module with simple socle but is not Noetherian, and

(iii) any non-finitely generated semisimple R-module is semi-artinian and semi-noetherian but does not have finite length.

Thus is is not clear how the theorem can be improved.

(**2**). Following J. P. Jans [**16**], a ring R is called *right co-Noetherian* if every factor module of every finite cogenerated right R-module is again finitely cogenerated or, equivalently, every finitely cogenerated right R-module is Artinian. As a consequence of results of P. Vámos [**26**], R is right co-Noetherian if and only if each simple right R-module has an Artinian injective hull. Thus any right co-Noetherian ring R satisfies property $(*)$ of Theorem 5. In what follows we give some indication of the ubiquity of co-Noetherian rings.

Firstly, Vámos [*loc. cit*] has shown that a commutative ring is co-Noetherian if and only if each localization R_M is Noetherian for all maximal ideals M of R. Also, trivially, every right V-ring is right co-Noetherian.

Theorem 2 of Jategaonkar [**17**] states that the injective hull of a simple module over a Noetherian P. I. ring is an Artinian module. Consequently any Noetherian P. I. ring is co-Noetherian. On the other hand, Example 7.14 of Chatters and Hajarnavis [**7**] shows that there are Artinian rings R which do not satisfy property $(*)$ of Theorem 5. More specifically, let D be a division ring with a subdivision ring K such that D is finite-dimensional as a left vector space over K but not as a right vector space over K. Let

$$R = \begin{bmatrix} K & D \\ 0 & D \end{bmatrix}.$$

Then R is left and right Artinian. Moreover $M = De_{11} + De_{12}$ is a right R-module which is an essential extension of the simple right R-module $N = De_{12}$ and the R-submodules of M/N correspond to the right K-subspaces of D so that M/N is not Artinian. It follows that the socle of $\mathrm{E}(N)/N$ is not finitely generated.

Gupta and Varadarajan have shown [**15**, Proposition 2.14] that a ring R is right co-Noetherian if and only if there is a cogenerator for mod-R which is a direct sum of Artinian modules. They also consider when the endomorphism ring of a finitely generated quasi-projective module is left co-Noetherian. Some of their arguments have been generalised by García Hernández and Gómez Pardo [**14**].

From Theorem 4 of S. Singh [**25**] it follows that any hereditary Noetherian prime ring is also co-Noetherian.

Theorem A of I. Musson [**22**] implies that if R is the group ring $S[G]$ where G is a polycyclic-by-finite group and the coefficient ring S is either $\mathbb{Z}$ or an absolute field then R is co-Noetherian. Also the Main Theorem of [**23**] shows that if K is a non-absolute field and G is a polycyclic-by-finite group then $K[G]$ is co-Noetherian if and only if G is abelian-by-finite. In fact, Theorem 3.1 of [**23**] provides another example of a Noetherian ring which does not satisfy Theorem 5's property $(*)$ by proving that if K is a non-absolute field and G is a nilpotent-by-finite group which is not abelian-by-finite then there is a simple $K[G]$-module V such that $\mathrm{Soc}(\mathrm{E}(V)/V)$ is not finitely generated.

(3). Following Vámos [**27**], a ring R is defined to be (right) *classical* if $\mathrm{E}(V)$ is linearly compact for each simple right R-module V. An account of linearly compact modules can be found in the recent monograph by Xue [**28**]. In particular, Proposition 3.4 there shows that linearly compact modules have finite Goldie dimension. Consequently, classical rings satisfy property $(*)$ of Theorem 5. Moreover, by Lemma 3.1 of [**28**], every Artinian module is linearly compact and so right co-Noetherian rings are classical. In view of this, we now give a brief discussion of classical rings.

From Vámos [*loc. cit*] it follows, using results of Matlis [**18**] and Müller [**21**] respectively, that almost maximal valuation rings are classical and any commutative ring with a Morita duality is also classical. Indeed, Phạm Ngọc Ánh [2] has recently characterized commutative classical rings as being those rings R for which the completion of the localization R_M of R at each maximal ideal M has a Morita duality.

Vámos [*loc. cit*] defines an R-module M to be *subdirectly irreducible* if $\mathrm{E}(M) \simeq \mathrm{E}(V)$ for some simple R-module V or, equivalently, if M has a simple essential socle. He also defines a commutative ring R to be a *SISI* ring if, for each ideal I or R, the factor ring R/I is self-injective if the R-module R/I is subdirectly irreducible. He proves [**27**, Proposition 3.2] that a commutative classical ring is SISI but gives the following example R to show that the converse is false. (A similar example appears in Section 1.11 of C. Menini and A. Orsatti [**19**].)

Let F be a field, let $P = F[x_1, x_2, x_3, \ldots]$ be the polynomial ring over F in a countable number of indeterminates, let I be the ideal of P generated by the set of products $\{x_i x_j\colon i \geqq 1,\ j \geqq 1\}$, and let $R = P/I$. Then R is a local ring and its maximal ideal M is a countable direct sum of copies of $R/M \simeq F$. Moreover $\bar{E} = \mathrm{E}(R/M)/(R/M)$ is an infinite direct sum of copies of R/M (see [**27**] for details). Thus R is not classical and does not satisfy Theorem 5's property $(*)$.

For further information about SISI rings see [11], [12].

(4). We now give an example of a commutative ring R which satisfies property $(*)$ but is not classical. The ring in question featured in [8] (for other purposes) and we follow its presentation given there.

Let A be a discrete valuation ring with maximal ideal $I=At$ and quotient field K. Moreover assume that A is countable, so that A is not complete in the I-adic topology. Let M denote the A-module K/A and, for each positive integer n, let M_n denote the A-submodule At^{-n}/A of M where

$$At^{-n}=\{at^{-n}\colon a\in A\}=\{ut^k\colon u \text{ is a unit in } A,\, k\in\mathbb{Z},\, k\geqq -n\}.$$

Then M is the direct union $\bigcup_{n=1}^{\infty}M_n$ and any proper nonzero. A-submodule of M is M_n for some $n\geqq 1$.

Now let $R=A\oplus M$ be the trivial extension of the ring A by its module M. Then the nonzero ideals of the ring R properly contained in M are precisely the A-submodules M_n for each n while the other nonzero ideals of R each contain M and, apart from M and R, are of the form $Rt^n=At^n\oplus M$, with $M=\bigcap_{n=1}^{\infty}Rt^n$. In fact the ideals of R are linearly ordered, forming the following chain:

$$0\subset M_1\subset M_2\subset\cdots\subset M_n\subset\cdots\subset M\subset\cdots\subset Rt^n\subset\cdots\subset Rt\subset R.$$

Thus M_1 is, up to isomorphism, the unique simple R-module and $\mathrm{E}(M_1)\simeq\mathrm{E}(R)$.

Since M is a faithful injective A-module, it follows from the discussion on page 22 of [13] that $\mathrm{E}(R)\simeq\mathrm{End}_A(M)\oplus M$. Moreover, by arguments in [8], $\mathrm{End}_A(M)\simeq\hat{A}$, where $\hat{A}$ is the I-adic completion of A. Since A is countable and $\bigcap_{n=1}^{\infty}I^n=0$, we can regard A as properly embedded in $\hat{A}$. It follows that R is not self-injective and has $\hat{A}\oplus M$ as injective hull. Since $(\hat{A}\oplus M)/M_1\simeq\hat{A}\oplus M$ as R-modules and the latter has M_1 as a unique minimal submodule, it follows that $\mathrm{E}(M_1)/M_1$ has finitely generated socle. Hence R satisfies property $(*)$.

It remains to see that R is not classical. For this, consider the countable set of congruences in R given by

$$\left\{x\equiv\sum_{k=1}^{n}t^k(\mathrm{mod}\,Rt^{n+1})\colon n\in\mathbb{N}\right\}.$$

Then, for any fixed n, setting $x=\sum_{k=1}^{n}t^k$ gives a simultaneous solution to the first n of these congruences. However there is no simultaneous solution to the complete set of congruences. Hence R is not linearly compact. Thus $\mathrm{E}(R)=\mathrm{E}(M_1)$ is not a linearly compact R-module and so, since M_1 is the only simple R-module up to isomorphism, it follows that R is not classical.

In fact, R is not SISI. To see this, note that the R-module R is subdirectly irreducible yet R is not a self-injective ring, and so, with I as the zero ideal, R fails to satisfy the definition of an SISI ring.

(5). From the above remarks we have the following strict implications for any ring R:

$$R \text{ is co-Noetherian} \Rightarrow R \text{ is classical} \Rightarrow R \text{ has property } (*).$$

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