ALMOST SURE CONVERGENCE OF QUADRATIC FORMS IN RANDOM VARIABLES

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Let $X_1, X_2, \cdots$ be a sequence of random variables and let $\{a_{jk}\}, j, k = 1, 2, \cdots$, be a matrix of real numbers. Write

$$S_n = \sum_{j,k=1}^{n} a_{jk} X_j X_k.$$

We establish the following result.

THEOREM. *Let $\{X_n, n \geqq 1\}$ be a sequence of random variables with*

$$\mathscr{E}\{X_n \mid X_1, \cdots, X_{n-1}\} = 0 \tag{1}$$

and

$$\mathscr{E}\{X_n^2 \mid X_1, \cdots, X_{n-1}\} = 1 \tag{2}$$

for $n = 2, 3, \cdots$. Let $\{a_{jk}\}$, $j, k = 1, 2, \cdots$ be a matrix of real numbers and let $S_n = \sum_{j,k=1}^{n} a_{jk} X_j X_k$. If $\sum_{j,k=1}^{\infty} a_{jk}^2 < \infty$ and $\sum_{k=1}^{\infty} |a_{kk}| < \infty$ then S_n converges almost surely.

REMARKS.

1. We emphasize that we do not assume the independence of random variables X_i. Nor do we assume that the random variables are identically distributed.

2. If, however, the random variables are independent with $\mathscr{E}X_n = 0$ and $\mathscr{E}X_n = 1$ for $n = 1, 2, \cdots$ then our theorem yields Theorem 1 and Corollaries 1 and 2 of Varberg [1].

PROOF. Following Varberg [1] we write $S_n = K_n + L_n + M_n$, where

$$K_n = \sum_{j=1}^{n} X_j \sum_{k=1}^{j-1} a_{jk} X_k, \quad L_n = \sum_{k=1}^{n} \sum_{j=1}^{k-1} a_{jk} X_j, \quad \text{and} \quad M_n = \sum_{k=1}^{n} a_{kk} X_k^2.$$

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Now note that for integers i, j, l, m with $l < i$, $m < i$, $l < j$, $m < j$, $i \neq j$ we have, because of (1), $\mathscr{E}\{X_i X_j X_l X_m\} = 0$. It follows therefore that

$$\begin{aligned} \mathscr{E}\{K_n^2\} &= \mathscr{E}\left\{\sum_{i=1}^{n} X_i^2 \left(\sum_{j=1}^{i-1} a_{ij} X_j\right)^2\right\} + \mathscr{E}\left\{\sum_{i \neq j} \sum_{l=1}^{i-1} \sum_{m=1}^{j-1} a_{il} a_{jm} X_i X_j X_l X_m\right\} \\ &= \sum_{j=1}^{n} \mathscr{E}\left\{\left(\sum_{k=1}^{j-1} a_{jk} X_k\right)^2\right\} \\ &= \sum_{j=1}^{n} \mathscr{E}\left\{\sum_{k=1}^{j-1} a_{jk}^2 X_k^2 + \sum_{k \neq l} a_{jk} a_{jl} X_k X_l\right\} \\ &= \sum_{j=1}^{n} \sum_{k=1}^{j-1} a_{jk}^2 < \infty. \end{aligned}$$

Since K_n is a martingale with respect to the σ-field generated by $X_1, X_2, \cdots X_n$ it follows by the martingale convergence theorem that K_n (and similarly L_n) converges almost surely. Finally we write

$$M_n = \sum_{1}^{n} a_{kk}(X_k^2 - 1) + \sum_{1}^{n} a_{kk} = P_n + \sum_{1}^{n} a_{kk}$$

and note that P_n is a martingale satisfying

$$\mathscr{E}|P_n| \leqq \sum_{k=1}^{n} |a_{kk}|\, \mathscr{E}\{|X_k^2 - 1|\} \leqq 2 \sum_{1}^{n} |a_{kk}| < \infty.$$

It follows therefore that P_n and, hence M_n, converges almost surely.

COROLLARY 1. If $\sum_{j,k=1}^{\infty} |a_{jk}| < \infty$, *then* S_n *converges almost surely.*

COROLLARY 2. *If* $a_{jk} = \sum_{i=1}^{\infty} b_{ji} c_{ik}$ *where* $\sum b_{ji}^2 < \infty$ *and* $\sum c_{ik}^2 < \infty$, *then* S_n *converges almost surely.*

Reference

[1] D. E. Varberg, 'Almost sure convergence of quadratic forms in independent random variables', *Ann. Math. Statist.* 39 (1968), 1502–1506.

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