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q-SERIES IDENTITIES AND REDUCIBILITY OF BASIC DOUBLE HYPERGEOMETRIC FUNCTIONS

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1. Introduction, notations and preliminaries. For real or complex q, $|q| < 1$, let

$$(1.1) \qquad (\lambda; q)_\mu = \prod_{j=0}^{\infty} \left(\frac{1 - \lambda q^j}{1 - \lambda q^{\mu+j}} \right)$$

for arbitrary λ and μ, so that

$$(1.2) \qquad (\lambda; q)_n = \begin{cases} 1, \text{ if } n = 0, \\ (1 - \lambda)(1 - \lambda q) \dots (1 - \lambda q^{n-1}), \\ \qquad \forall n \in \{1, 2, 3, \dots \}, \end{cases}$$

and

$$(1.3) \qquad (\lambda; q)_\infty = \prod_{j=0}^{\infty} (1 - \lambda q^j).$$

Define, as usual, a generalized q-hypergeometric function by (cf. [**26**, Chapter 3]; see also [**18**])

$$(1.4) \qquad {}_{p+1}\Phi_{p+r} \left[\begin{matrix} \alpha_1, \dots, \alpha_{p+1}; \\ \beta_1, \dots, \beta_{p+r}; \end{matrix} \quad q, z \right] = \sum_{n=0}^{\infty} (-1)^{rn} q^{\frac{1}{2}rn(n-1)} \frac{(\alpha_1; q)_n \dots (\alpha_{p+1}; q)_n}{(\beta_1; q)_n \dots (\beta_{p+r}; q)_n} \frac{z^n}{(q; q)_n},$$

where, for convergence, $|q| < 1$ and $|z| < \infty$ when r is a positive integer, or $|z| < 1$ when $r = 0$, provided that no zeros appear in the denominator.

Various q-extensions of the Appell functions (and of their confluent cases) in two variables (cf. [**6**]) were first considered by Jackson ([**20**], [**21**]) in his systematic presentation of the expansion theory of these basic double hypergeometric functions along the lines of Burchnall and Chaundy ([**10**], [**11**]). All of Jackson's extensions are contained in the basic double hypergeometric series defined by (cf. [**17**], [**22**], [**29**], and [**31**]; see also [**30**] for the multivariable case)

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$$(1.5)\quad \Phi^{p:h;u}_{l:m;n}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p:\beta_1,\ldots,\beta_h;\ \gamma_1,\ldots,\gamma_u;\ q;\ x,y\\ \lambda_1,\ldots,\lambda_l:\mu_1,\ldots,\mu_m;\ \nu_1,\ldots,\nu_n;\ i,j,k\end{matrix}\right]$$

$$= \sum_{r,s=0}^{\infty} q^{\frac{1}{2}ir(r-1)+\frac{1}{2}js(s-1)+krs}$$

$$\times \frac{\prod_{\tau=1}^{p}(\alpha_\tau;q)_{r+s}\prod_{\tau=1}^{h}(\beta_\tau;q)_r\prod_{\tau=1}^{u}(\gamma_\tau;q)_s}{\prod_{\tau=1}^{l}(\lambda_\tau;q)_{r+s}\prod_{\tau=1}^{m}(\mu_\tau;q)_r\prod_{\tau=1}^{n}(\nu_\tau;q)_s}$$

$$\times \frac{x^r}{(q;q)_r}\frac{y^s}{(q;q)_s},$$

for special values of i, j, k and of the nonnegative integers p, h, u, l, m, n. When $i = j = k = 0$, the first member of (1.5) is written simply as (see [**17**] and [**29**])

$$(1.6)\quad \Phi^{p:h;u}_{l:m;n}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p:\beta_1,\ldots,\beta_h;\ \gamma_1,\ldots,\gamma_u;\\ \lambda_1,\ldots,\lambda_l:\mu_1,\ldots,\mu_m;\ \nu_1,\ldots,\nu_n;\end{matrix}\ q;\ x,y\right].$$

A great deal of literature is available on ordinary hypergeometric functions of two and more variables (cf., e.g., [**6**], [**26**, Chapter 8], and [**16**]). However, the literature on basic multiple hypergeometric functions seems to be a lot less extensive. Apart from the aforementioned work on basic Appell series by Jackson ([**20**], [**21**]), Agarwal ([**1**], [**2**]), Slater ([**25**]; see also [**26**, Chapter 9]), Andrews ([**3**], [**4**]), and others ([**17**], [**22**], [**29**], [**30**], [**31**], [**32**], [**33**]) have developed various interesting properties of basic Appell series (and of their generalizations in two and more variables). In the present paper we prove a number of general q-series identities and transformations which are shown to be applicable in the derivation of reduction and transformation formulas for various classes of basic double hypergeometric functions. We also consider several other interesting consequences of some of the results presented here.

2. q-series identities. For every bounded sequence $\{A_n\}$ of complex numbers, we first prove the following q-series identities:

$$(2.1)\quad \sum_{m,n=0}^{\infty}(-1)^n A_{m+n}\frac{(a;q)_m(a;q)_n(-b/a;q)_m(-b/a;q)_n}{(b;q)_m(b;q)_n}$$

$$\times \frac{x^{m+n}}{(q;q)_m(q;q)_n}$$

$$= \sum_{n=0}^{\infty} A_{2n} \frac{(a^2; q^2)_n (b^2/a^2; q^2)_n (-b; q)_{2n}}{(b^2; q^2)_n (b; q)_{2n}} \frac{x^{2n}}{(q^2; q^2)_n},$$

(2.2) $$\sum_{m,n=0}^{\infty} (-1)^n A_{m+n} \frac{(a^2; q^2)_m (b^2; q^2)_n}{(a^2; q)_m (b^2; q)_n} \frac{x^{m+n}}{(q; q)_m (q; q)_n}$$

$$= \sum_{n=0}^{\infty} A_{2n} \frac{(a^2 b^2; q^2)_{2n}}{(a^2 q; q^2)_n (b^2 q; q^2)_n (a^2 b^2; q^2)_n} \frac{x^{2n}}{(q^2; q^2)_n},$$

(2.3) $$\sum_{m,n=0}^{\infty} A_{m+n} \frac{(a^2; q^2)_m (b^2; q^2)_n (a^2 q; q^2)_m (b^2 q; q^2)_n}{(a^4; q^2)_m (b^4 q^2; q^2)_n}$$

$$\times \frac{x^m}{(q^2; q^2)_m} \frac{(qx)^n}{(q^2; q^2)_n}$$

$$= \sum_{n=0}^{\infty} A_n \frac{(a^2 b^2; q)_{2n}}{(a^2 b^2; q)_n (-a^2; q)_n (-b^2 q; q)_n} \frac{x^n}{(q; q)_n},$$

(2.4) $$\sum_{m,n=0}^{\infty} A_{m+n} \frac{(a; q^2)_m (a; q^2)_n (b/aq; q^2)_m (b/aq; q^2)_n}{(b; q^2)_m (b; q^2)_n}$$

$$\times \frac{x^m}{(q^2; q^2)_m} \frac{(qx)^n}{(q^2; q^2)_n}$$

$$= \sum_{n=0}^{\infty} A_n \frac{(a; q)_n (b/q; q^2)_n (b/aq; q)_n}{(b; q^2)_n (b/q; q)_n} \frac{x^n}{(q; q)_n},$$

(2.5) $$\sum_{m,n=0}^{\infty} A_{m+n} \frac{(a; q)_m (b; q)_m (c/ab; q)_n}{(c; q)_m} \frac{(cx/ab)^m}{(q; q)_m} \frac{x^n}{(q; q)_n}$$

$$= \sum_{n=0}^{\infty} A_n \frac{(c/a; q)_n (c/b; q)_n}{(c; q)_n} \frac{x^n}{(q; q)_n},$$

and

(2.6) $$\sum_{m,n=0}^{\infty} q^{\frac{1}{2}m(m-1)} A_{m+n} \frac{(a; q)_m (bc/aq; q)_{m+2n}}{(b; q)_m (c; q)_n} \frac{(-bx/a)^m}{(q; q)_m} \frac{x^n}{(q; q)_n}$$

$$= \sum_{n=0}^{\infty} A_n \frac{(b/a; q)_n (bc/aq; q)_n (bcq^{n-1}; q)_n}{(b; q)_n (c; q)_n} \frac{x^n}{(q; q)_n}$$

provided that the series involved are absolutely convergent.

Proof of Formula (2.1). For convenience, let Ω_1 denote the left-hand side of the q-series identity (2.1). Then, using the definitions (1.1), (1.2) and (1.4), it is easily seen that

$$\text{(2.7)} \quad \Omega_1 = \sum_{n=0}^{\infty} A_n \frac{(a;\, q)_n(-b/a;\, q)_n}{(b;\, q)_n} \frac{(-x)^n}{(q;\, q)_n} \times {}_4\Phi_3 \left[\begin{matrix} q^{-n},\, q^{1-n}/b, & a, & -b/a; \\ & b,\, q^{1-n}/a, & -aq^{1-n}/b; \end{matrix} \; q,\, q \right].$$

The well-poised ${}_4\Phi_3$ series can be summed by a known q-analogue of Dixon's theorem (cf. [**35**, p. 1023, Eq. (2.3)]), and we are led to the q-series identity (2.1).

Formula (2.1) provides a q-extension of a series identity due to Buschman and Srivastava [**12**, p. 437, Eq. (2.7)]. For $b \to 0$, (2.1) yields a q-extension of a result of Srivastava [**27**, p. 297, Eq. (17)]. On the other hand, if in (2.1) we replace x by x/a and then let $a \to \infty$, we shall obtain a q-extension of another series identity due to Srivastava [**27**, p. 295, Eq. (5)]. Finally, our q-series identity (2.1) with b replaced by $-ab$ would yield a q-extension of a result of Buschman and Srivastava [**12**, p. 438, Eq. (2.8)].

Proof of Formula (2.2). If we denote the left-hand side of (2.2) by Ω_2, then

$$\text{(2.8)} \quad \Omega_2 = \sum_{n=0}^{\infty} A_n \frac{(a^2;\, q^2)_n}{(a^2;\, q)_n} \frac{x^n}{(q;\, q)_n} \times {}_4\Phi_3 \left[\begin{matrix} q^{-n},\, q^{1-n}/a^2,\, b,\, -b; \\ q^{1-n}/a,\, -q^{1-n}/a,\, b^2; \end{matrix} \; q,\, q \right],$$

and, upon summing the ${}_4\Phi_3$ series by a q-analogue of Watson's theorem (due to Andrews [**5**, p. 333, Theorem 1]), we obtain the q-series identity (2.2).

Formula (2.2) provides a q-extension of another series identity due to Buschman and Srivastava [**12**, p. 440, Eq. (3.10)].

Proof of Formula (2.3). Denoting the left-hand side of (2.3) by Ω_3, we have

$$\text{(2.9)} \quad \Omega_3 = \sum_{n=0}^{\infty} A_n \frac{(a^2;\, q^2)_n(a^2q;\, q^2)_n}{(a^4;\, q^2)_n} \frac{x^n}{(q^2;\, q^2)_n} \times {}_4\Phi_3 \left[\begin{matrix} q^{-2n},\, q^{2-2n}/a^4,\, b^2,\, b^2q; \\ q^{1-2n}/a^2,\, q^{2-2n}/a^2,\, b^4q^2; \end{matrix} \; q^2,\, q^2 \right].$$

Now apply a known result [**36**, p. 265, Eq. (5.3)] to sum the ${}_4\Phi_3$ series, and the q-series identity (2.3) follows readily.

Proof of Formula (2.4). Let Ω_4 denote the left-hand side of (2.4), so that

$$(2.10)\quad \Omega_4 = \sum_{n=0}^{\infty} A_n \frac{(a;\, q^2)_n (b/aq;\, q^2)_n}{(b;\, q^2)_n} \frac{x^n}{(q^2;\, q^2)_n} \times \left[{}_4\Phi_3 \begin{matrix} q^{-2n}, q^{2-2n}/b, & a, & b/aq; \\ b, q^{2-2n}/a, aq^{3-2n}/b; & & \end{matrix} \; q^2, q^4 \right].$$

If we sum the well-poised ${}_4\Phi_3$ series by applying Carlitz's formula [**13**]:

$$(2.11)\quad \left[{}_4\Phi_3 \begin{matrix} q^{-n}, & a, & b, q^{1/2-n}/ab; \\ q^{1-n}/a, q^{1-n}/b, & & abq^{1/2}; \end{matrix} \; q, q^2 \right] = \frac{(ab;\, q)_n (a;\, q^{1/2})_n (b;\, q^{1/2})_n (-q^{1/2};\, q^{1/2})_n}{(ab;\, q^{1/2})_n (a;\, q)_n (b;\, q)_n},\ n = 0, 1, 2, \dots,$$

we shall obtain the q-series identity (2.4).

Proof of Formulas (2.5) *and* (2.6). Denoting the left-hand sides of (2.5) and (2.6) by Ω_5 and Ω_6, respectively, we find that

$$(2.12)\quad \Omega_5 = \sum_{n=0}^{\infty} A_n (c/ab;\, q)_n \frac{x^n}{(q;\, q)_n}\, {}_3\Phi_2 \left[\begin{matrix} a, b, q^{-n}; \\ c, abq^{1-n}/c; \end{matrix} \; q, q \right]$$

and

$$(2.13)\quad \Omega_6 = \sum_{n=0}^{\infty} A_n \frac{(bc/aq;\, q)_{2n}}{(c;\, q)_n} \frac{x^n}{(q;\, q)_n}\, {}_3\Phi_2 \left[\begin{matrix} a, q^{1-n}/c, q^{-n}; \\ b, aq^{2-2n}/bc; \end{matrix} \; q, q \right].$$

Each of the ${}_3\Phi_2$ series, occurring in (2.12) and (2.13), is summable by the q-extension of Saalschütz's theorem [**26**, p. 97, Eq. (3.3.2.2)], and we readily arrive at the q-series identities (2.5) and (2.6).

Formula (2.5) with c replaced by abc and $b \to 0$ yields a q-extension of Srivastava's identity [**27**, p. 297, Eq. (16)]. On the other hand, our q-series identity (2.6) reduces, when $a \to \infty$, to a q-extension of another result due to Srivastava [**27**, p. 295, Eq. (4)].

3. Reduction formulas for basic double hypergeometric functions. Making use of the notations (1.5) and (1.6), we can readily specialize the coefficients A_n in our q-series identities (2.1) to (2.5) in order to derive the following cases of reducibility of the basic double hypergeometric function (1.6) with $y = -x$, $y = qx$, et cetera.

$$(3.1)\quad \Phi^{p:2;2}_{l:1;1} \left[\begin{matrix} \alpha_1, \dots, \alpha_p\colon a, -b/a;\ a, -b/a; \\ \beta_1, \dots, \beta_l\colon \quad b; \quad b; \end{matrix} \; q;\, x, -x \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1;\, q)_{2n} \dots (\alpha_p;\, q)_{2n} (a^2;\, q^2)_n (b^2/a^2;\, q^2)_n (-b;\, q)_{2n}}{(\beta_1;\, q)_{2n} \dots (\beta_l;\, q)_{2n} (b^2;\, q^2)_n (b;\, q)_{2n}}$$

$$\times \frac{x^{2n}}{(q^2;\, q^2)_n}.$$

(3.2)
$$F^{p:2;2}_{l:1;1}\left[\begin{matrix}\alpha_1, \ldots, \alpha_p\colon a,\ -a;\ b,\ -b;\\ \beta_1, \ldots, \beta_l\colon \quad a^2; \quad b^2;\end{matrix}\ q;\ x,\ -x\right]$$
$$= \sum_{n=0}^{\infty} \frac{(\alpha_1;\, q)_{2n} \ldots (\alpha_p;\, q)_{2n}(a^2b^2;\, q^2)_{2n}}{(\beta_1;\, q)_{2n} \ldots (\beta_l;\, q)_{2n}(a^2q;\, q^2)_n(b^2q;\, q^2)_n(a^2b^2;\, q^2)_n}$$
$$\times \frac{x^{2n}}{(q^2;\, q^2)_n}.$$

(3.3)
$$\Phi^{p:2;2}_{l:1;1}\left[\begin{matrix}\alpha_1, \ldots, \alpha_p\colon a^2,\ a^2q;\ b^2,\ b^2q;\\ \beta_1, \ldots, \beta_l\colon \quad a^4; \quad b^4q^2;\end{matrix}\ q^2;\ x,\ qx\right]$$
$$= \sum_{n=0}^{\infty} \frac{(\alpha_1;\, q^2)_n \ldots (\alpha_p;\, q^2)_n(a^2b^2;\, q)_{2n}}{(\beta_1;\, q^2)_n \ldots (\beta_l;\, q^2)_n(a^2b^2;\, q)_n(-a^2;\, q)_n(-b^2q;\, q)_n}$$
$$\times \frac{x^n}{(q;\, q)_n}.$$

(3.4)
$$\Phi^{p:2;2}_{l:1;1}\left[\begin{matrix}\alpha_1, \ldots, \alpha_p\colon a,\ b/aq;\ a,\ b/aq;\\ \beta_1, \ldots, \beta_l\colon \quad b; \quad b;\end{matrix}\ q^2;\ x,\ qx\right]$$
$$= \sum_{n=0}^{\infty} \frac{(\alpha_1;\, q^2)_n \ldots (\alpha_p;\, q^2)_n(a;\, q)_n(b/q;\, q^2)_n(b/aq;\, q)_n}{(\beta_1;\, q^2)_n \ldots (\beta_l;\, q^2)_n(b;\, q^2)_n(b/q;\, q)_n} \frac{x^n}{(q;\, q)_n}.$$

(3.5)
$$\Phi^{p:2;1}_{l:1;0}\left[\begin{matrix}\alpha_1, \ldots, \alpha_p\colon a,\ b;\ c/ab;\\ \beta_1, \ldots, \beta_l\colon \quad c;\ \text{——};\end{matrix}\ q;\ cx/ab,\ x\right]$$
$$= \sum_{n=0}^{\infty} \frac{(\alpha_1;\, q)_n \ldots (\alpha_p;\, q)_n(c/a;\, q)_n(c/b;\, q)_n}{(\beta_1;\, q)_n \ldots (\beta_l;\, q)_n(c;\, q)_n} \frac{x^n}{(q;\, q)_n}.$$

Upon similarly specializing the coefficients A_n, our q-series identity (2.6) yields

(3.6)
$$\sum_{m,n=0}^{\infty} q^{\frac{1}{2}m(m-1)} \frac{(\alpha_1;\, q)_{m+n} \ldots (\alpha_p;\, q)_{m+n}(a;\, q)_m(bc/aq;\, q)_{m+2n}}{(\beta_1;\, q)_{m+n} \ldots (\beta_l;\, q)_{m+n}(b;\, q)_m(c;\, q)_n}$$
$$\times \frac{(-bx/a)^m}{(q;\, q)_m} \frac{x^n}{(q;\, q)_n}$$
$$= \sum_{n=0}^{\infty} \frac{(\alpha_1;\, q)_n \ldots (\alpha_p;\, q)_n(b/a;\, q)_n(bc/aq;\, q)_n(bc/q;\, q)_{2n}}{(\beta_1;\, q)_n \ldots (\beta_l;\, q)_n(b;\, q)_n(c;\, q)_n(bc/q;\, q)_n}$$
$$\times \frac{x^n}{(q;\, q)_n}.$$

From the definition (1.2) it is easily verified that

$$(3.7) \quad (\lambda; q)_{2n} = (\lambda; q^2)_n(\lambda q; q^2)_n, \ (\lambda; q^2)_n = (\sqrt{\lambda}; q)_n(-\sqrt{\lambda}; q)_n, \quad n = 0, 1, 2, \ldots .$$

Thus the right-hand side of each of the reduction formulas (3.1) to (3.6) can easily be rewritten in terms of a basic hypergeometric function of the type (1.4).

4. Applications of the reduction formulas (3.1) **to** (3.6). Formula (3.1) provides a q-extension of the known result [**12**, p. 439, Eq. (3.4)]. For $p = l = 1$, it yields the reduction formula

$$(4.1) \quad \Phi^{1:2;2}_{1:1;1}\left[\begin{matrix}\lambda: a, & -b/a; & a, & -b/a; \\ \mu: & b; & & b;\end{matrix} \; q; x, -x\right] = {}_6\Phi_5\left[\begin{matrix}\lambda, \lambda q, a^2, b^2/a^2, -b, -bq; \\ \mu, \mu q, b^2, \quad b, \quad bq;\end{matrix} \; q^2, x^2\right],$$

which is due essentially to Verma and Jain [**35**, p. 1030, Eq. (2.33)].

If in the reduction formula (3.1) we let $b \to 0$, we shall obtain a q-extension of the known result [**12**, p. 439, Eq. (3.3)]. On the other hand, if in (3.1) we replace x by x/a and then let $a \to \infty$, we shall arrive at a q-extension of another known result [**12**, p. 439, Eq. (3.7)]. Formula (3.1), with b replaced by $-ab$, would readily yield a q-extension of yet another known result [**12**, p. 439, Eq. (3.6)].

Next we consider the special cases of (3.1) when $p - 1 = l = 0$ and $p = l - 1 = 0$. In the former case we are led to

$$(4.2) \quad \Phi^{1:2;2}_{0:1;1}\left[\begin{matrix}\lambda: a, & -b/a; & a, & -b/a; \\ -: & b; & & b;\end{matrix} \; q; x, -x\right] = {}_6\Phi_5\left[\begin{matrix}\lambda, \lambda q, a^2, b^2/a^2, -b, -bq; \\ b^2, b, bq, 0, 0;\end{matrix} \; q^2, x^2\right],$$

which is a q-extension of a result due to Bailey [**8**, p. 239, Eq. (4.5)] on the reducibility of Appell's function F_2. In the latter case of (3.1) (with $p = l - 1 = 0$), if we replace b by $-ab$, we obtain

$$(4.3) \quad \Phi^{0:2;2}_{1:1;1}\left[\begin{matrix}-: & a, b; & a, b; \\ \mu: & -ab; & -ab;\end{matrix} \; q; x, -x\right] = {}_6\Phi_5\left[\begin{matrix}a^2, b^2, ab, abq, 0, 0; \\ \mu, \mu q, a^2b^2, -ab, -abq;\end{matrix} \; q^2, x^2\right],$$

which provides a q-extension of another result of Bailey [**8**, p. 239, Eq. (4.3)] on the reducibility of Appell's function F_3.

Lastly, in our reduction formula (3.1) we set $p - 2 = l = 0$, replace x by x/a, and then let $a \to \infty$; we thus find that

$$(4.4)\quad \Phi^{2:0;0}_{0:1;1}\left[\begin{matrix}\lambda,\ \mu: & -; & -; & q;\ x,\ -x\\ -: & b; & b; & 1,\ 1,\ 0\end{matrix}\right]$$

$$= {}_6\Phi_6\left[\begin{matrix}\lambda,\ \lambda q,\ \mu,\ \mu q,\ -b,\ -bq; \\ b^2,\ b,\ bq,\ 0,\ 0,\ 0;\end{matrix}\ q^2,\ x^2\right],$$

which is a q-extension of a known result [**27**, p. 296, Eq. (9)] on the reducibility of Appell's function F_4.

For $p - 1 = l = 0$, the reduction formula (3.2) immediately yields (cf. [**22**, p. 359, Eq. (4.18)])

$$(4.5)\quad \Phi^{1:2;2}_{0:1;1}\left[\begin{matrix}\lambda: & a,\ -a; & b,\ -b; \\ -: & a^2; & b^2;\end{matrix}\ q;\ x,\ -x\right]$$

$$= {}_6\Phi_5\left[\begin{matrix}\lambda,\ \lambda q,\ ab,\ -ab,\ abq,\ -abq; \\ a^2q,\ b^2q,\ a^2b^2,\ 0,\ 0;\end{matrix}\ q^2,\ x^2\right],$$

which provides a q-extension of yet another result of Bailey [**8**, p. 239, Eq. (4.4)] involving the Appell function F_2.

Letting $b \to 0$ in (4.5), we obtain the quadratic transformation

$$(4.6)\quad {}_3\Phi_2\left[\begin{matrix}\lambda,\ a,\ -a; \\ a^2,\ -\lambda x;\end{matrix}\ q,\ x\right] = \frac{(-x;\ q)_\infty}{(-\lambda x;\ q)_\infty}\ {}_2\Phi_1\left[\begin{matrix}\lambda,\ \lambda q; \\ a^2q;\end{matrix}\ q^2,\ x^2\right],$$

which is a q-extension of the familiar result [**24**, p. 65, Theorem 24]. (See also Jain [**23**, p. 957, Eq. (3.1)] for an alternative proof of (4.6).)

Now, in our reduction formula (3.2) we set $p = l = 1$, $\alpha_1 = \lambda$, $\beta_1 = \lambda x$, and let $b \to \infty$; we thus find that

$$(4.7)\quad {}_3\Phi_2\left[\begin{matrix}\lambda,\ a,\ -a; \\ a^2,\ 0;\end{matrix}\ q,\ x\right] = \frac{(\lambda x;\ q)_\infty}{(x;\ q)_\infty}\ {}_2\Phi_3\left[\begin{matrix}\lambda,\ \lambda q; \\ \lambda x,\ \lambda qx,\ a^2q;\end{matrix}\ q^2,\ a^2x^2q\right],$$

which is another q-extension of the aforementioned result [**24**, p. 65, Theorem 24]. Use is made here of the following limiting case of the q-extension of Gauss's summation theorem [**26**, p. 97, Eq. (3.3.2.5)]:

$$(4.8)\quad {}_1\Phi_1\left[\begin{matrix}a; \\ c;\end{matrix}\ q,\ c/a\right] = \frac{(c/a;\ q)_\infty}{(c;\ q)_\infty}.$$

Lastly, for $p = l = 0$, (3.2) reduces to the following q-extension of Bailey's result [7, p. 246, Eq. (2.11)]:

$$(4.9)\quad {}_2\Phi_1\left[\begin{matrix}a,\ -a; \\ a^2;\end{matrix}\ q,\ x\right]{}_2\Phi_1\left[\begin{matrix}b,\ -b; \\ b^2;\end{matrix}\ q,\ -x\right]$$

$$= {}_4\Phi_3\left[\begin{matrix}ab,\ -ab,\ abq,\ -abq; \\ a^2q,\ b^2q,\ a^2b^2;\end{matrix}\ q^2,\ x^2\right],$$

which, for $b \to 0$, yields a q-extension of Kummer's second formula [**24**, p. 125, Theorem 42] in the form:

$$(4.10)\quad {}_2\Phi_1\left[\begin{matrix} a, -a; \\ a^2; \end{matrix}\; q, x\right] = (-x; q)_\infty\, {}_2\Phi_1\left[\begin{matrix} 0, 0; \\ a^2q; \end{matrix}\; q^2, x^2\right].$$

For $p = l = 0$, (3.3) reduces immediately to the interesting identity

$$(4.11)\quad {}_2\Phi_1\left[\begin{matrix} a^2, a^2q; \\ a^4; \end{matrix}\; q^2, x\;\; {}_2\Phi_1\begin{matrix} b^2, b^2q; \\ b^4q^2; \end{matrix}\; q^2, qx\right]$$
$$= {}_4\Phi_3\left[\begin{matrix} ab, -ab, ab\sqrt{q}, -ab\sqrt{q}; \\ a^2b^2, -a^2, -b^2q; \end{matrix}\; q, x\right],$$

while (3.4) similarly yields a q-extension of Clausen's identity (cf., e.g., [**15**, p. 185, Eq. 4.3(1)]) in the form:

$$(4.12)\quad {}_2\Phi_1\left[\begin{matrix} a, b; \\ abq; \end{matrix}\; q^2, x\right] {}_2\Phi_1\left[\begin{matrix} a, b; \\ abq; \end{matrix}\; q^2, qx\right]$$
$$= {}_4\Phi_3\left[\begin{matrix} a, b, \sqrt{ab}, -\sqrt{ab}; \\ ab, \sqrt{abq}, -\sqrt{abq}; \end{matrix}\; q, x\right].$$

Several further consequences of the identity (4.11) are worthy of note. First let $b \to 0$ in (4.11), and we have

$$(4.13)\quad {}_2\Phi_1\left[\begin{matrix} a^2, a^2q; \\ a^4; \end{matrix}\; q^2, x\right] = (qx; q^2)_\infty\, {}_2\Phi_1\left[\begin{matrix} 0, 0; \\ -a^2; \end{matrix}\; q, x\right],$$

while (4.11) for $b \to \infty$ yields

$$(4.14)\quad {}_2\Phi_1\left[\begin{matrix} a^2, a^2q; \\ a^4; \end{matrix}\; q^2, x\right] = \frac{1}{(x; q^2)_\infty}\, {}_0\Phi_1\left[\begin{matrix} -; \\ -a^2; \end{matrix}\; q, -a^2x\right].$$

On the other hand, if in (4.11) we let $a \to 0$ or $a \to \infty$, we obtain the identities

$$(4.15)\quad {}_2\Phi_1\left[\begin{matrix} b^2, b^2q; \\ b^4q^2; \end{matrix}\; q^2, qx\right] = (x; q^2)_\infty\, {}_2\Phi_1\left[\begin{matrix} 0, 0; \\ -b^2q; \end{matrix}\; q, x\right]$$

and

$$(4.16)\quad {}_2\Phi_1\left[\begin{matrix} b^2, b^2q; \\ b^4q^2; \end{matrix}\; q^2, qx\right] = \frac{1}{(qx; q^2)_\infty}\, {}_0\Phi_1\left[\begin{matrix} -; \\ -b^2q; \end{matrix}\; q, -b^2qx\right],$$

respectively. Each of the q-identities (4.13) to (4.16) is analogous to an interesting reduction formula for the Gaussian hypergeometric function ${}_2F_1$ (cf., e.g., [**24**, p. 70, Exercise 10]).

Formula (3.5) with c replaced by abc and $b \to 0$ yields

$$(4.17)\quad \Phi^{p:1;1}_{l:0;0}\left[\begin{matrix} \alpha_1, \dots, \alpha_p: & a; & c; \\ \beta_1, \dots, \beta_l: & -; & -; \end{matrix}\; q; cx, x\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1; q)_n \ldots (\alpha_p; q)_n (ac; q)_n}{(\beta_1; q)_n \ldots (\beta_l; q)_n} \frac{x^n}{(q; q)_n},$$

which is a q-extension of the known result [**12**, p. 439, Eq. (3.1)]. For $p = l = 1$, (4.17) evidently becomes

$$(4.18)\quad \Phi^{1:1;1}_{1:0;0}\left[\begin{matrix}\lambda: & a; & c; \\ \mu: & -; & -;\end{matrix}\ q; cx, x\right] = {}_2\Phi_1\left[\begin{matrix}\lambda, ac; \\ \mu;\end{matrix}\ q, x\right],$$

which, in view of Andrews' formula [**3**, p. 618, Theorem 1], leads us at once to the familiar result

$$(4.19)\quad {}_2\Phi_1\left[\begin{matrix}\lambda, \rho; \\ \mu;\end{matrix}\ q, x\right] = \frac{(\lambda; q)_\infty (\rho x; q)_\infty}{(\mu; q)_\infty (x; q)_\infty} {}_2\Phi_1\left[\begin{matrix}\mu/\lambda, x; \\ \rho x;\end{matrix}\ q, \lambda\right].$$

For $p = l = 0$ and x replaced by abx/c, (3.5) yields another familiar result

$$(4.20)\quad {}_2\Phi_1\left[\begin{matrix}a, b; \\ c;\end{matrix}\ q, x\right] = \frac{(abx/c; q)_\infty}{(x; q)_\infty} {}_2\Phi_1\left[\begin{matrix}c/a, c/b; \\ c;\end{matrix}\ q, abx/c\right],$$

while (3.5) with $p - 1 = l = 0$ and $b \to \infty$ gives us the transformation

$$(4.21)\quad {}_2\Phi_1\left[\begin{matrix}\lambda, \rho; \\ c;\end{matrix}\ q, x\right] = \frac{(\lambda x; q)_\infty}{(x; q)_\infty} {}_2\Phi_2\left[\begin{matrix}\lambda, c/\rho; \\ c, \lambda x;\end{matrix}\ q, \rho x\right],$$

which is due to Jackson [**19**, p. 145, Eq. (4)].

Formula (3.6) with $a \to \infty$ yields

$$(4.22)\quad \sum_{m,n=0}^{\infty} q^{m(m-1)} \frac{(\alpha_1; q)_{m+n} \ldots (\alpha_p; q)_{m+n}}{(\beta_1; q)_{m+n} \ldots (\beta_l; q)_{m+n} (b; q)_m (c; q)_m} \times \frac{(bx)^m}{(q; q)_m} \frac{x^n}{(q; q)_n}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1; q)_n \ldots (\alpha_p; q)_n (bc/q; q)_{2n}}{(\beta_1; q)_n \ldots (\beta_l; q)_n (b; q)_n (c; q)_n (bc/q; q)_n} \frac{x^n}{(q; q)_n},$$

which is a q-extension of the known result [**12**, p. 439, Eq. (3.2)]. For $p - 2 = l = 0$, (4.22) evidently reduces to (cf. (1.5))

$$(4.23)\quad \Phi^{2:0;0}_{0:1;1}\left[\begin{matrix}\lambda, \mu: & -; & -; & q; bx, x \\ -: & b; & c; & 2, \ 0, 0\end{matrix}\right]$$

$$= {}_6\Phi_5\left[\begin{matrix}\lambda, \mu, \sqrt{\dfrac{bc}{q}}, -\sqrt{\dfrac{bc}{q}}, \sqrt{bc}, -\sqrt{bc}; \\ b, c, \dfrac{bc}{q}, 0, 0;\end{matrix}\ q, x\right],$$

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which provides a q-extension of Burchnall's formula [**9**, p. 101, Eq. (37)] on the reducibility of the Appell function F_4.

Finally, we set $\lambda = \mu = 0$ in (4.23), and we obtain the formula

$$(4.24)\quad {}_0\Phi_1\left[\begin{matrix} -; \\ b; \end{matrix}\ q,\ bx\right]{}_2\Phi_1\left[\begin{matrix} 0,\ 0; \\ c; \end{matrix}\ q,\ x\right]$$

$$= {}_4\Phi_3\left[\begin{matrix} \sqrt{\frac{bc}{q}},\ -\sqrt{\frac{bc}{q}},\ \sqrt{bc},\ -\sqrt{bc}; \\ b,\ c,\ \frac{bc}{q}; \end{matrix}\ q,\ x\right],$$

which is a q-extension of a well-known result (cf., e.g., [**15**, p. 185, Eq. 4.3(2)]).

5. General q-series transformations. Let $\{A_n\}$ be a bounded sequence of complex numbers. The general q-series transformations to be proved in this section are

$$(5.1)\quad \sum_{m,n=0}^{\infty} q^{\frac{1}{2}(m+n)(m+n-1)} A_{m+n}(a;\ q)_m(b;\ q)_n \frac{x^m}{(q;\ q)_m}\frac{y^n}{(q;\ q)_n}$$

$$= \sum_{m,n=0}^{\infty} (-1)^m q^{\frac{1}{2}n(n-1)} A_{m+n}$$

$$\times (a;\ q)_m(abq^m;\ q)_n(ax/y;\ q)_m \frac{(y/a)^{m+n}}{(q;\ q)_m(q;\ q)_n},$$

which is a q-extension of Srivastava's formula [**28**, p. 139, Eq. (3)];

$$(5.2)\quad \sum_{m,n=0}^{\infty} q^{\frac{1}{2}m(m-1)+\frac{1}{2}n(n-1)} \frac{A_{m+n}}{(a;\ q)_m(b;\ q)_n}\frac{x^m}{(q;\ q)_m}\frac{y^n}{(q;\ q)_n}$$

$$= \sum_{m,n=0}^{\infty} q^{\frac{1}{2}m(m-1)+\frac{1}{2}n(n-1)} A_{m+n}$$

$$\times \frac{(abq^{-1};\ q)_{m+2n}(ay/x;\ q)_{m+n}}{(a;\ q)_{m+n}(abq^{-1};\ q)_{m+n}(b;\ q)_n(ay/x;\ q)_n}$$

$$\times \frac{x^m}{(q;\ q)_m}\frac{y^n}{(q;\ q)_n},$$

which is a q-extension of a mild generalization of a series identity due to Buschman and Srivastava [**12**, p. 436, Eq. (1.9)];

$$(5.3)\quad \sum_{m,n=0}^{\infty} q^{2m^2} \frac{A_{m+n}}{(a^2;\, q^2)_m(a^2;\, q^2)_n} \frac{x^{2m}}{(q^2;\, q^2)_m} \frac{y^{2n}}{(q^2;\, q^2)_n}$$

$$= \sum_{m,n=0}^{\infty} q^{\frac{1}{2}n^2} A_{m+n} \frac{(q^{n+\frac{1}{2}}x/y;\, q)_{2m}(a^2/q;\, q^2)_n}{(a^2;\, q^2)_{m+n}(a^2/q;\, q)_n} \frac{y^{2m}}{(q^2;\, q^2)_m} \frac{(xy)^n}{(q;\, q)_n},$$

which is a q-extension of Srivastava's result [**28**, p. 142, Eq. (15)];

$$(5.4)\quad \sum_{m,n=0}^{\infty} q^{\frac{1}{2}(m+n)^2} A_{m+n}(a^2;\, q^2)_m(a^2;\, q^2)_n$$

$$\times \frac{x^{2m}}{(q^2;\, q^2)_m} \frac{y^{2n}}{(q^2;\, q^2)_n}$$

$$= \sum_{m,n=0}^{\infty} q^{\frac{1}{2}n(n-1)} A_{m+n} \frac{(a^2;\, q)_{2m+n}(xy^{-1}\sqrt{q};\, q)_m(x^{-1}y\sqrt{q};\, q)_m}{(a^2q;\, q^2)_m}$$

$$\times \frac{(-xy)^m}{(q^2;\, q^2)_m} \frac{(xy)^n}{(q;\, q)_n};$$

$$(5.5)\quad \sum_{m,n=0}^{\infty} q^{\frac{1}{2}m(m-1)} A_{m+n} \frac{(a;\, q)_m(a;\, q)_n}{(b;\, q)_m(b;\, q)_n} \frac{x^m}{(q;\, q)_m} \frac{y^n}{(q;\, q)_n}$$

$$= \sum_{m,n=0}^{\infty} q^{n(n-1)} A_{m+2n} \frac{(a;\, q)_{m+n}(-q^n x/y;\, q)_m(b/a;\, q)_n}{(b;\, q)_{m+2n}(b;\, q)_n}$$

$$\times \frac{y^m}{(q;\, q)_m} \frac{(-axy)^n}{(q;q)_n},$$

which provides a q-extension of a result due to Buschman and Srivastava [**12**, p. 437, Eq. (2.4)];

$$(5.6)\quad \sum_{m,n=0}^{\infty} q^{mn+\frac{1}{2}n(n-1)} A_{m+n}(a;\, q)_m(a;\, q)_n$$

$$\times (b;\, q)_m(b;\, q)_n \frac{x^m}{(q;\, q)_m} \frac{y^n}{(q;\, q)_n}$$

$$= \sum_{m,n=0}^{\infty} q^{n(n-1)} A_{m+2n}$$

$$\times \frac{(ab;\, q)_{m+2n}(a;\, q)_{m+n}(b;\, q)_{m+n}(-q^n y/x;\, q)_m}{(ab;\, q)_{m+n}}$$

$$\times \frac{x^m}{(q;q)_m} \frac{(-xy)^n}{(q;q)_n},$$

which is a q-extension of a series identity due to Buschman and Srivastava [**12**, p. 437, Eq. (2.5)];

$$(5.7) \quad \sum_{m,n=0}^{\infty} q^{m(m-1)+\frac{1}{2}n(n-3)} \frac{A_{m+n}}{(a;q)_m(a;q)_n(b;q)_m(b;q)_n} \times \frac{x^m}{(q;q)_m} \frac{y^n}{(q;q)_n}$$
$$= \sum_{m,n=0}^{\infty} q^{mn+\frac{1}{2}m(m-3)+\frac{1}{2}n(3n-5)} A_{m+2n} \times \frac{(abq^{m+2n-1};q)_n(-q^{n+1}x/y;q)_m}{(a;q)_{m+2n}(b;q)_{m+2n}(a;q)_n(b;q)_n} \frac{y^m}{(q;q)_m} \frac{(xy)^n}{(q;q)_n},$$

which provides a q-extension of yet another result of Buschman and Srivastava [**12**, p. 437, Eq. (2.6)]. It is assumed in every case that each of the double series involved converges absolutely.

Proofs. Denoting, for convenience, the first member of the q-series transformation (5.1) by Λ, and making use of the definition (1.2), we readily observe that

$$(5.8) \quad \Lambda = \sum_{N=0}^{\infty} q^{\frac{1}{2}N(N-1)} A_N(a;q)_N \frac{x^N}{(q;q)_N} \times {}_2\Phi_1\left[\begin{matrix} q^{-N}, b; \\ q^{1-N}/a; \end{matrix} \quad q, qy/ax\right],$$

where we have made the rearrangement $m + n = N$. Now we transform the ${}_2\Phi_1$ occurring in (5.8) by appealing to Jackson's formula (4.21), and we find that

$$(5.9) \quad \Lambda = \sum_{N=0}^{\infty} A_N(a;q)_N(ax/y;q)_N \frac{(-y/a)^N}{(q;q)_N} \times {}_2\Phi_2\left[\begin{matrix} q^{-N}, q^{1-N}/ab; \\ q^{1-N}/a, q^{1-N}y/ax; \end{matrix} \quad q, bqy/ax\right].$$

Since this ${}_2\Phi_2$ series is terminating, we can make a second rearrangement, and (upon simplifying the resulting double series) we arrive at the q-series transformation (5.1).

The proof of the q-series transformation (5.2) depends similarly upon Jackson's formula (4.21).

Formulas (5.3) and (5.4) can be proven in precisely the same manner as (5.1) and (5.2); in place of Jackson's formula (4.21), however, we shall require the quadratic transformation [**34**, p. 426, Eq. (2.3)]

$$(5.10)\quad {}_2\Phi_1\left[\begin{matrix} a^2, b^2; \\ a^2q^2/b^2; \end{matrix}\; q^2, z^2\right]$$

$$= \frac{(abz/\sqrt{q};\, q)_\infty}{(bz/a\sqrt{q};\, q)_\infty}\, {}_4\Phi_3\left[\begin{matrix} a, -a, a\sqrt{q}/b, -a\sqrt{q}/b; \\ a^2q/b^2, abz/\sqrt{q}, aq\sqrt{q}/bz; \end{matrix}\; q, q\right],$$

$$a = q^{-n}(n = 0, 1, 2, \dots),$$

which is a q-extension of the terminating version of a familiar result [**24**, p. 65, Theorem 23].

In order to prove the q-series transformation (5.5), we let Δ denote its left-hand side and make the rearrangement as in the proofs of (5.1) to (5.4). We thus find that

$$(5.11)\quad \Delta = \sum_{N=0}^{\infty} A_N \frac{(a;\, q)_N}{(b;\, q)_N} \frac{y^N}{(q;\, q)_N}$$

$$\times\, {}_3\Phi_2\left[\begin{matrix} q^{-N}, q^{1-N}/b, \quad a; \\ b, q^{1-N}/a; \end{matrix}\; q, -bq^Nx/ay\right].$$

Transforming this well-poised terminating ${}_3\Phi_2$ series by means of the known formula [**13**, p. 195, Eq. (2.4)]

$$(5.12)\quad {}_3\Phi_2\left[\begin{matrix} a, \quad b, \quad c; \\ aq/b, aq/c; \end{matrix}\; q, aqz/bc\right]$$

$$= \frac{(az;\, q)_\infty}{(z;\, q)_\infty}\, {}_5\Phi_4\left[\begin{matrix} \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, aq/bc; \\ aq/b, aq/c, az, q/z; \end{matrix}\; q, q\right],$$

$$a = q^{-n}(n = 0, 1, 2, \dots),$$

we obtain

$$(5.13)\quad \Delta = \sum_{N=0}^{\infty} A_N \frac{(a;\, q)_N(-x/y;\, q)_N}{(b;\, q)_N} \frac{y^N}{(q;\, q)_N}$$

$$\times\, {}_5\Phi_4\left[\begin{matrix} q^{-\frac{1}{2}N}, -q^{-\frac{1}{2}N}, q^{\frac{1}{2}(1-N)}, -q^{\frac{1}{2}(1-N)}, b/a; \\ b, q^{1-N}/a, -x/y, -q^{1-N}y/x; \end{matrix}\; q, q\right].$$

Now we make a second rearrangement, this time with $N = m + 2n$, and (on simplifying the resulting double series) the proof of (5.5) is thus completed.

Similar are the derivations of the q-series transformations (5.6) and (5.7). We skip the details involved.

6. Special cases of (5.1) **to** (5.7). For $y = ax$, our q-series transformation (5.1) reduces immediately to the interesting result

$$(6.1)\quad \sum_{m,n=0}^{\infty} A_{m+n}(a;\,q)_m(b;\,q)_n \frac{x^m}{(q;\,q)_m}\frac{(ax)^n}{(q;\,q)_n} = \sum_{n=0}^{\infty} A_n(ab;\,q)_n \frac{x^n}{(q;\,q)_n},$$

which is a q-extension of Srivastava's formula [**27**, p. 297, Eq. (16)]. Indeed, as we pointed out in Section 2, (6.1) would follow also if we replace c in (2.5) by abc and let $b \to 0$.

Formula (5.1) with

$$(6.2)\quad A_n = \frac{(\lambda;\,q)_n}{(\mu;\,q)_n},\quad n = 0, 1, 2, \dots,$$

provides a q-extension of Carlson's identity [**14**, p. 222, Eq. (4)]. On the other hand, if in (5.1) we set

$$(6.3)\quad A_n = \frac{(c;\,q)_n}{(ab;\,q)_n(-cy/a;\,q)_n},\quad n = 0, 1, 2, \dots,$$

and make use of (4.8), we obtain the reduction formula

$$(6.4)\quad \Phi^{1:1;1}_{2:0;0}\left[\begin{matrix} c\colon\ a;\ \ b;\, q;\, x, y \\ ab,\ -cy/a\colon\ -;\ -;\ 1,\ 1,\ 1\end{matrix}\right] = \frac{(-y/a;\,q)_\infty}{(-cy/a;\,q)_\infty}\,{}_3\Phi_2\left[\begin{matrix} c,\ a,\ ax/y; \\ ab,\ 0;\end{matrix}\ q,\ -y/a\right].$$

The first member of each of the q-series transformations (5.2), (5.3) and (5.5) reduces, when

$$(6.5)\quad A_n = 1,\ n = 0, 1, 2, \dots,$$

to a product of two simple Φ-series defined by (1.4). These obvious consequences of (5.2), (5.3) and (5.5) under the constraint (6.5) provide interesting instances when the double series on the right-hand sides of (5.2), (5.3) and (5.5) simplify considerably.

Next we set $y = x\sqrt{q}$ in (5.4), and we obtain

$$(6.6)\quad \sum_{m,n=0}^{\infty} q^n A_{m+n}(a^2;\,q^2)_m(a^2;\,q^2)_n \frac{x^{2m+2n}}{(q^2;\,q^2)_m(q^2;\,q^2)_n} = \sum_{n=0}^{\infty} A_n(a^2;\,q)_n \frac{x^{2n}}{(q;\,q)_n},$$

where we have replaced A_n by $q^{-\frac{1}{2}n^2}A_n$ on both sides. Formula (6.6) would follow also when we set $y = x/\sqrt{q}$ in (5.4).

A number of other q-series identities can be derived similarly from our q-series transformations (5.1) to (5.7).

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