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On the importance of slow ions in the kinetic Bohm criterion

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Between a plasma and a solid target lies a positively charged sheath of several Debye lengths λ_D width, typically much smaller than the characteristic length scale L of the main plasma. This scale separation implies that the asymptotic limit $\epsilon = \lambda_D/L \to 0$ is useful to solve for the plasma-sheath system. In this limit, the Bohm criterion must be satisfied at the sheath entrance. A new derivation of the kinetic criterion, admitting a general ion velocity distribution, is presented. It is proven that, for $\epsilon \to 0$, the distribution of the velocity component normal to the target, v_x, and its first derivative must vanish for $|v_x| \to 0$ at the sheath entrance. These two conditions can be subsumed into a third integral one after it is integrated by parts twice. A subsequent interchange of the limits $\epsilon \to 0$ and $|v_x| \to 0$ is invalid, leading to a divergence which underlies the misconception that the criterion gives undue importance to slow ions.

Key words: plasma sheaths

1. Introduction

For a quasineutral plasma to exist next to a wall or a solid target, the most mobile charged species, typically electrons due to their smaller mass, must be reflected by the wall to achieve no net charge loss (and thus preserve quasineutrality). The wall is thus negatively charged and the region where electron reflection occurs is called the 'sheath' (Langmuir 1923, 1929; Tonks & Langmuir 1929). The sheath is positively charged and shields the quasineutral plasma from the negative charge on the wall. Its characteristic length scale is the Debye length, defined as $\lambda_D = \sqrt{\epsilon_0 T_e/n_e e^2}$, where n_e is the electron density, e is the proton charge, T_e is the electron temperature (in units of energy) and ϵ_0 is the permittivity of free space. The characteristic length scale L of unmagnetised plasmas, or magnetised plasmas where the magnetic field is perpendicular to the target, is defined to be the smallest one between: the collisional mean free path of ions, the ionisation mean free path of neutrals and the target curvature (Riemann 1991). It is usually much larger than the Debye length, such that the sheath is thin compared with the plasma

$$\epsilon = \frac{\lambda_D}{L} \ll 1. \tag{1.1}$$

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In magnetised plasmas with an oblique magnetic field impinging on the target, L can also be the ion sound Larmor radius $\rho_S = \sqrt{m_i(ZT_e + T_i)}/(ZeB)$, where T_i is the ion temperature, B is the magnetic field strength, m_i is the ion mass and Z is the ionic charge state (Chodura 1982).[1] Once they reach the target, ions and electrons are absorbed, eventually recombine and are subsequently re-emitted as neutrals that penetrate back through the sheath all the way into the plasma before re-ionising.

The physical principle treated in this paper states that the ions must reach the sheath at a minimum velocity, as first formalised by Bohm (1949) assuming mono-energetic (cold) ions

$$-u_x \geqslant v_B \equiv \sqrt{ZT_e/m_i}. \tag{1.2}$$

Here, $u_x = \boldsymbol{u} \cdot \boldsymbol{e}_x$, where $\boldsymbol{u}$ is the fluid velocity, $\hat{\boldsymbol{e}}_x$ is a unit vector normal to the target plane and pointing away from it, $v_B = \sqrt{ZT_e/m_i}$ is the Bohm speed (or cold-ion sound speed) and m_i is the ion mass. The negative sign in the velocity component is related to the choice of placing the target at $x = 0$ and the plasma at $x > 0$. The inequality (1.2) came to be known as the 'Bohm criterion'. It holds at the 'sheath entrance' (or 'sheath edge'), a position at this stage loosely defined as the distance from the wall below which collisions, ionisation, target curvature and curvature of ion Larmor orbits are negligible, and above which the space charge is negligible. It ensures an increasingly positive space charge in the sheath as the wall is approached, consistent with an increasing electric field directed towards the wall to reflect electrons.

Since mono-energetic ions are a significant idealisation, a kinetic formulation of the Bohm criterion, valid for arbitrary ion velocity distributions, is desirable. This was first given by Harrison & Thompson (1959)

$$v_B^2 \int_{-\infty}^{0} \frac{f(v_x)}{v_x^2}\, \mathrm{d}v_x \leqslant \frac{n_e}{Z}. \tag{1.3}$$

Here, f is the ion distribution function of the velocity component normal to the target, $v_x = \boldsymbol{v} \cdot \hat{\boldsymbol{e}}_x$ with velocity $\boldsymbol{v}$, at the sheath entrance. By plasma quasineutrality at the sheath entrance, the electron density n_e satisfies

$$n_e = Z \int_{-\infty}^{0} f(v_x)\, \mathrm{d}v_x. \tag{1.4}$$

The distribution f is, in practice, the full three-dimensional distribution function which has been integrated over the velocity components tangential to the target. Note that the fluid criterion (1.2) is always fulfilled by the flow velocity u_x defined by

$$u_x \int_{-\infty}^{0} f(v_x)\, \mathrm{d}v_x = \int_{-\infty}^{0} f(v_x) v_x\, \mathrm{d}v_x, \tag{1.5}$$

provided that f satisfies the kinetic criterion (1.3), as can be shown by two applications of Schwarz's inequality (Harrison & Thompson 1959; Riemann 1991). In deriving (1.3), it has been assumed that no ions travel away from the target at the sheath entrance, such that $f(v_x) = 0$ for $v_x > 0$.

Although Harrison & Thompson's kinetic criterion (1.3) stood the test of time, Hall (1962) promptly criticised their derivation for implicitly assuming the conditions

[1]Strongly magnetised plasmas may also be solved in the limit $\rho_S/L \to 0$. If the magnetic field is obliquely incident on the target, the 'magnetised plasma sheath' comprises two regions: the charged 'Debye sheath' of width $\sim \lambda_D$ and a (usually larger) quasineutral 'magnetic presheath' of width $\sim\rho_S$ (Geraldini, Brunner & Parra 2024).

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$f(0^-) \equiv \lim_{v_x \to 0^-} f(v_x) = 0$ and $f'(0^-) = 0$ without justification. The derivative of any function $G(\zeta)$ of a single variable ζ is denoted $G'(\zeta) \equiv \mathrm{d}G(\zeta)/\mathrm{d}\zeta$. Performing a calculation that will be shown here to be flawed, Hall refutes $f'(0^-) = 0$ and concludes that (1.3) ascribes 'undue importance' to the slow ions with $v_x \approx 0$.

In parallel, Caruso & Cavaliere (1962) recognised the asymptotic framework of the plasma-sheath transition as a singular perturbation theory (see § 7.2 of Bender & Orszag 1999), following work by Bertotti (1961). In the limit $\epsilon \to 0$ the equations on the plasma and sheath scales are distinct, and thus the two regions can be treated separately. The sheath entrance is defined in this limit by $x/L \to 0$ and $x/\lambda_D \to \infty$ (Bertotti & Cavaliere 1965). In practice, any point $x_\epsilon = \epsilon^q L$ on an intermediate length scale with $q \in (0, 1)$ satisfies $\lim_{\epsilon \to 0} x_\epsilon/\lambda_D \to \infty$ and $\lim_{\epsilon \to 0} x_\epsilon/L \to 0$ and is thus a valid reference choice for the sheath entrance.[2] The actual distribution function at this point is denoted $f_\epsilon(v_x)$, and the asymptotic sheath entrance distribution function is defined by

$$f(v_x) \equiv \lim_{\epsilon \to 0} f_\epsilon(v_x). \tag{1.6}$$

The Bohm criterion at the sheath entrance arises as a necessary condition for an electron-repelling sheath solution to exist in the limit $\epsilon \to 0$.

In § 3.2 of his seminal review, Riemann (1991) derived the kinetic Bohm criterion (1.3) in the limit $\epsilon \to 0$, including ion reflection and re-emission from the wall. For comparison with this work, his derivation is considered for a perfectly absorbing wall. There, it is proven in an intermediate step that $f(0^-) = 0$, while the condition $f'(0^-) = 0$ emerges from the assumption that f can be represented by a Taylor expansion in kinetic energy $v_x^2/2$ near $v_x = 0$. This assumption was criticised (Fernsler, Slinker & Joyce 2005; Baalrud & Hegna 2011, 2012; Baalrud *et al.* 2015) for restricting the class of possible ion distribution functions at the sheath entrance. Another objection to (1.3) emphasises the divergence of its left-hand side when applied to the actual distribution function

$$\lim_{\epsilon \to 0} v_B^2 \int_{-\infty}^{0} \frac{f_\epsilon(v_x)}{v_x^2}\, \mathrm{d}v_x = \lim_{\epsilon \to 0} \infty, \tag{1.7}$$

caused by $f_\epsilon(0) \neq 0$ and $f_\epsilon'(0) \neq 0$ for finite ϵ. Echoing (Hall 1962) around 50 years later, Baalrud & Hegna (2011, 2012) claim that the kinetic criterion places 'undue importance' on slow ions. While this assertion appears reasonable in light of (1.7), it is motivated by the misguided expectation that (1.3) should still apply as it is to f_ϵ for small but finite ϵ (Riemann 2012).

In the remainder of this article, the derivation of the kinetic Bohm criterion (1.3) is generalised to remove the assumption, and prove the physical legitimacy of, $f'(0^-) = 0$. Condition (1.3) is shown to emerge from

$$\lim_{v_x \to 0^-} \lim_{\epsilon \to 0} f_\epsilon(v_x) \equiv f(0^-) = 0, \tag{1.8}$$

$$\lim_{v_x \to 0^-} \lim_{\epsilon \to 0} f_\epsilon'(v_x) \equiv f'(0^-) = 0, \tag{1.9}$$

$$\lim_{v_x \to 0^-} \lim_{\epsilon \to 0} \int_{-\infty}^{v_x} f_\epsilon''(v_x') \ln(-1/v_x')\, \mathrm{d}v_x' \equiv \int_{-\infty}^{0} f''(v_x) \ln(-1/v_x)\, \mathrm{d}v_x \leqslant \frac{n_e}{Z v_B^2} \equiv \lim_{\epsilon \to 0} \frac{n_{e,\epsilon}}{Z v_B^2}, \tag{1.10}$$

[2] For small but finite ϵ, the plasma and sheath solutions can be asymptotically matched to the solution obtained in a transition region on a suitably chosen intermediate length scale between plasma and sheath (Franklin & Ockendon 1970; Riemann 1997, 2003, 2006; Slemrod 2002; Riemann *et al.* 2005); one may therefore argue that this transition region is the most accurate manifestation of the concept of a sheath entrance.

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by integrating (1.10) by parts and imposing (1.8) and (1.9). In (1.10), the electron density at x_ϵ was defined as $n_{e,\epsilon}$. To conclude, (1.8)–(1.10) are combined to obtain the appropriate reformulation of (1.3) in terms of f_ϵ. It is explained that the divergence (1.7) follows from an incautious interchange of the limits $\epsilon \to 0$ and $v_x \to 0^-$.

2. Derivation of the kinetic Bohm criterion

In this section, the kinetic Bohm criterion is derived by imposing the requirement of an electron-repelling potential solution in the sheath near the sheath entrance. In § 2.1, the collisionless ion kinetic equation in the sheath is solved and Poisson's equation is thus derived (assuming adiabatic electrons). Then, in § 2.2 the ion density is expanded near the sheath entrance using a matched asymptotic expansion (Bender & Orszag 1999). Here, the only assumption on the form of the ion distribution function at the sheath entrance for $|v_x| \ll 1$, $v_x < 0$ is that it has an asymptotic expansion in (possibly fractional) powers of v_x. In § 2.3, the charge density is evaluated for all possible values of the smallest exponent of v_x, associated with the largest term, in this expansion. Imposing an electron-repelling potential near the sheath entrance is shown in § 2.4 to lead to conditions (1.8) and (1.9) and to the inequality form of (1.10) and (1.3). Finally, in § 2.5 we verify that the equality form of the kinetic Bohm criterion (1.3) is consistent with an electron-repelling sheath solution.

2.1. *Poisson's equation in the sheath*

The following dimensionless variables are introduced: the sheath-scale position $\xi = x/\lambda_D$, the (negative of the) electrostatic potential relative to the sheath entrance $\chi = \lim_{\epsilon\to 0} e(\phi(x_\epsilon) - \phi(x))/T_e$, the ion velocity component directed towards the wall $w = -v_x/v_B$ and the ion distribution function $g(w) = Zf(v_x)v_B/n_e$ satisfying $\int_0^\infty g(w)\,\mathrm{d}w = 1$ from (1.4).

For $\epsilon \to 0$, in the sheath $x \leqslant x_\epsilon$, corresponding to $\xi \in [0, \infty]$, the ion distribution function $g_\xi(\xi, w)$ satisfies the kinetic equation

$$w\frac{\partial g_\xi}{\partial \xi} + \chi'\frac{\partial g_\xi}{\partial w} = 0. \tag{2.1}$$

Equation (2.1) results from the normalised full kinetic equation which has been integrated in the other two velocity components, after all the terms small in ϵ have vanished in the limit $\epsilon \to 0$. The ϵ-dependent terms are related to the target curvature, collisions and ionisation in the plasma, the magnetic force on the ions and explicit time dependence on the plasma scale $v_B/L = \epsilon v_B/\lambda_D$. By imposing the boundary condition at the sheath entrance $g_\xi(\infty, w) = g(w)$, assuming a perfectly absorbing wall, $g_\xi(0, w < 0) = 0$, and assuming no ion reflection within the sheath, $g_\xi(\xi, w < 0) = 0$, one obtains $g_\xi(\xi, w) = g(\sqrt{w^2 - 2\chi})$ from (2.1). In order for no ions to be reflected, $\chi(\xi) \geqslant 0$ is required. For the electrons, a Boltzmann distribution is assumed, which is justified when the sheath reflects most electrons back into the bulk such that $\mathrm{e}^{-\chi(0)} \ll 1$. Hence, Poisson's equation in the sheath is

$$\chi''(\xi) = \int_{\sqrt{2\chi}}^{\infty} g(\sqrt{w^2 - 2\chi})\,\mathrm{d}w - \mathrm{e}^{-\chi}. \tag{2.2}$$

2.2. *Matched asymptotic expansion of the ion density near the sheath entrance*

In order to solve (2.2) for $\chi(\xi)$ locally near the sheath entrance (where $\chi = 0$), the electron density may be expanded as a Taylor series in $\chi \ll 1$, $e^{-\chi} = 1 - \chi + \frac{1}{2}\chi^2 + O(\chi^3)$. The ion density integral for $\chi \ll 1$ is calculated via a matched asymptotic expansion (see § 7.4

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in Bender & Orszag 1999) hinging on the observation that the function $g(\sqrt{w^2-2\chi})$ can be represented using two different approximations with a common range of validity. For $\bar{w}=\sqrt{w^2-2\chi}\ll 1$ (slow ions), the function $g(\bar{w})$ is expanded as an asymptotic series in (possibly fractional) powers of $\bar{w}$,

$$g(\bar{w})=\sum_{\nu=1}^{N_{3+}-1} g_{p_\nu}\bar{w}^{p_\nu}+O(w^{p_{N_{3+}}}) \quad \text{for } \bar{w}\ll 1, \tag{2.3}$$

where $p\equiv p_1>-1$ (the density should remain finite) is defined to be the smallest power in the expansion of $g(\bar{w})$ near $\bar{w}=0$, $g_p\equiv g_{p_1}>0$ is the corresponding positive constant coefficient ($g(w)>0$), $p_\nu>p_{\nu-1}$ is the νth power in the expansion for $\nu>1$ and N_{3+} is the index corresponding to the smallest exponent in the expansion satisfying $p_{N_{3+}}>3$. By convention, it is considered that $g_{p'}=0$ for p' such that $p_\nu\neq p'$ for all possible values of the index ν. The expansion in (2.3) is truncated such that terms $\ll\bar{w}^3$ are not retained, as these terms will be negligible throughout the subsequent analysis. Note that the form of the distribution function $g(w)$ near $w=0$ assumed here is much more general than in previous derivations of the kinetic Bohm criterion. For $\bar{w}\sim w\gg\sqrt{2\chi}$, $g(\sqrt{w^2-2\chi})$ can be Taylor expanded near $g(w)$

$$\begin{aligned} g(\sqrt{w^2-2\chi}) &= g(w)-\chi\frac{g'(w)}{w}-\frac{1}{2}\chi^2\frac{g'(w)}{w^3}+\frac{1}{2}\chi^2\frac{g''(w)}{w^2} \\ &\quad +O\left(\frac{\chi^3 g'(w)}{w^5},\frac{\chi^3 g''(w)}{w^4},\frac{\chi^3 g'''(w)}{w^3}\right). \end{aligned} \tag{2.4}$$

The two approximations (2.3) and (2.4) have a common range of validity $\sqrt{2\chi}\ll w\ll 1$. Therefore, (2.2) can be re-expressed by choosing a cutoff velocity parameter w_c, generally satisfying $\sqrt{2\chi}\ll w_c\ll 1$, and using the approximation (2.3) for $w\leqslant w_c$ and (2.4) for $w\geqslant w_c$. This results in

$$\chi''=Q_0+Q_1+Q_2+\sum_{\nu=1}^{N_{3+}-1}\bar{Q}_{(p_\nu+1)/2}+O(\chi^3 w_c^{p-5},\chi^3,\chi w_c^{p_{N_{3+}}-1})=O(\chi^{(p+1)/2},\chi), \tag{2.5}$$

with the zeroth-order, first-order, second-order and slow ion charge densities, respectively, defined by

$$Q_0=\int_0^\infty g(w)\,\mathrm{d}w-1=0, \tag{2.6}$$

$$Q_1=\chi\left[1-\int_{w_c}^\infty\frac{g'(w)}{w}\,\mathrm{d}w\right]=O(\chi,\chi w_c^{p-1}), \tag{2.7}$$

$$Q_2=\frac{1}{2}\chi^2\left[\int_{w_c}^\infty\frac{g''(w)}{w^2}\,\mathrm{d}w-\int_{w_c}^\infty\frac{g'(w)}{w^3}\,\mathrm{d}w-1\right]=O(\chi^2,\chi^2 w_c^{p-3}), \tag{2.8}$$

$$\begin{aligned} \bar{Q}_{(p_\nu+1)/2} &= g_{p_\nu}\left[\int_{\sqrt{2\chi}}^{w_c}(w^2-2\chi)^{p_\nu/2}\,\mathrm{d}w-\int_0^{w_c}w^{p_\nu}\,\mathrm{d}w\right] \\ &= g_{p_\nu}\left[-\frac{(2\chi)^{(p_\nu+1)/2}}{p_\nu+1}+\sum_{n=1}^{\infty}\frac{(2\chi)^n}{n!}\prod_{m=0}^{n-1}\left(m-\frac{p_\nu}{2}\right)\int_{\sqrt{2\chi}}^{w_c}\frac{\mathrm{d}w}{w^{2n-p_\nu}}\right] \\ &= O(\chi^{(p_\nu+1)/2},\chi w_c^{p_\nu-1}). \end{aligned} \tag{2.9}$$

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The quasineutrality condition at the sheath entrance (where $\chi = 0$ and $\chi'' = 0$) was used in (2.6). The errors $O(\chi^3 w_c^{p-5}, \chi^3)$ in (2.5) result from integrating the errors in (2.4) in the interval $w \in [w_c, \infty]$ and from another $O(\chi^3)$ term in the expansion of the electron density. The error $O(\chi w_c^{p_{N_{3+}}-1})$ comes from the term $\bar{Q}_{(p_{N_{3+}}+1)/2}$ which has been neglected in (2.5). The terms Q_2 and $\bar{Q}_{(p_\nu+1)/2}$ for $1 < p_\nu \leqslant 3$ ($N_{1+} \leqslant \nu < N_{3+} - 1$) are retained in (2.5), despite being $\ll \chi$, in anticipation of a higher-order analysis that will be necessary for $p > 1$ (see § 2.5). Hall (1962) uses a Taylor expansion instead of the more general expansion in (2.3), thus considering $p = 0$ (and $p_\nu = \nu - 1$) or $p = 1$ (and $p_\nu = \nu$) while keeping terms up to order $\sim \chi$, but incorrectly takes $w_c \sim \sqrt{2\chi}$.[3] Even the more general expansion (2.5) is invalid for $w_c \sim \sqrt{2\chi}$ and $p \leqslant 1$, since some of the neglected terms are of order $O(\chi^3 w_c^{p-5})$, which would be $O(\chi^{(p+1)/2})$ just like the dominant terms in (2.5). For (2.5) to have an error $\ll \chi$ when $p \leqslant 1$, it is required that w_c lie within the asymptotic region of overlap $\chi^{1/2} \lesssim \chi^{2/(5-p)} \ll w_c \ll 1$. In §§ 2.4 and 2.5, it will be seen that the asymptotic region of overlap changes with the value of p and with the order in χ of the expansion. However, provided that such a region exists and w_c lies within it, the expansion up to the desired order is independent of the value of w_c.

2.3. *Evaluation of the charge density*

The assumption of an electron-repelling sheath, $\chi(\xi) \geqslant \chi(\infty) = 0$, requires that the charge density χ'' in (2.5) become positive in the sheath. The sign of χ'' is established in the following cases sequentially: $-1 < p < 1$, $p = 1$, $1 < p < 3$, $p = 3$ and $p > 3$, thus covering all $p > -1$ in (2.3). It will be convenient to first evaluate the density contributions (2.7)–(2.9). Equations (2.7) and (2.8) for Q_1 and Q_2 become

$$Q_1 = \chi \left[1 - \int_0^\infty \left(g''(w) - \sum_{\nu=1}^{N_{1-}} p_\nu (p_\nu - 1) g_{p_\nu} w^{p_\nu - 2} \right) \ln(1/w)\, \mathrm{d}w \right] - \chi g_1 \ln(1/w_c) + \chi \sum_{\nu=1, p_\nu \neq 1}^{N_{3+}-1} \frac{p_\nu g_{p_\nu}}{p_\nu - 1} w_c^{p_\nu - 1} + O(\chi w_c^{p_{N_{3+}}-1}), \tag{2.10}$$

$$Q_2 = \frac{1}{4}\chi^2 \left[\int_0^\infty \left(g''''(w) - \sum_{\nu=1}^{N_{3-}} g_{p_\nu} p_\nu (p_\nu - 1)(p_\nu - 2)(p_\nu - 3) w^{p_\nu - 4} \right) \ln(1/w)\, \mathrm{d}w - 2 \right] + \frac{3}{4}\chi^2 g_3 (1 + 2\ln(1/w_c)) - \chi^2 \sum_{\nu=1}^{N_{3-}} \frac{p_\nu (p_\nu - 2)}{2(p_\nu - 3)} g_{p_\nu} w_c^{p_\nu - 3} + O(\chi^2 w_c^{p_{N_{3+}}-3}). \tag{2.11}$$

In (2.10), N_{1-} is the index corresponding to the largest exponent in the expansion satisfying $p_{N_{1-}} < 1$, while N_{1+} is the index corresponding to the smallest exponent in the expansion satisfying $p_{N_{1+}} > 1$. In (2.11), N_{3-} is the index corresponding to the largest exponent in the expansion satisfying $p_{N_{3-}} < 3$. The derivation of (2.10) and (2.11) uses the following steps: (i) substitute $g(w) = g(w) - \sum_{\nu=1}^{N_{1-}} g_{p_\nu} w^{p_\nu} + \sum_{\nu=1}^{N_{1-}} g_{p_\nu} w^{p_\nu}$ in (2.7) and $g(w) = g(w) - \sum_{\nu=1}^{N_{3-}} g_{p_\nu} w^{p_\nu} + \sum_{\nu=1}^{N_{3-}} g_{p_\nu} w^{p_\nu}$ in (2.8); (ii) integrate explicitly the integrals involving $\sum g_{p_\nu} w^{p_\nu}$ to obtain the terms proportional to $1/w_c^a$ with $a > 0$; (iii) integrate by parts (more than once for (2.8)) the integrals involving $g(w) - \sum g_{p_\nu} w^p$ and use (2.3)

[3]Hall (1962) expands the expression $\int_0^\infty \mathrm{d}w' w' g(w')(w'^2 + 2\chi)^{-1/2}$ for the ion density, which, from $w' = \sqrt{w^2 - 2\chi}$, is equivalent to the integral in (2.2), by considering separately the regions $w' \leqslant \sqrt{2\chi}$ and $w' \geqslant \sqrt{2\chi}$. This is akin to taking $w_c = 2\sqrt{\chi}$ in (2.5).

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to extract the boundary terms containing $\ln w_c$ (present if $p_\nu = 1$ in (2.10) or $p_\nu = 3$ in (2.11) for some ν); (iv) split the remaining integrals using $\int_{w_c}^{\infty} dw(\ldots) = \int_0^{\infty} dw(\ldots) - \int_0^{w_c} dw(\ldots)$ and substitute (2.3) to evaluate the integrals $\int_0^{w_c} dw(\ldots)$. Equation (2.9) for $\bar{Q}_{(p_\nu+1)/2}$ becomes

$$\bar{Q}_{(p_\nu+1)/2} = A_{(p_\nu+1)/2}\chi^{(p_\nu+1)/2} - \frac{p_\nu g_{p_\nu}\chi w_c^{p_\nu-1}}{p_\nu-1} + \frac{p_\nu(p_\nu-2)}{2(p_\nu-3)} g_{p_\nu}\chi^2 w_c^{p_\nu-3} + O(\chi^3 w_c^{p_\nu-5}) \quad \text{for } p_\nu > -1,\ p_\nu \neq 2n-1 \text{ with } n \in \mathbb{N}^\star, \tag{2.12}$$

$$\bar{Q}_1 = -g_1\chi\left[1 + \frac{1}{2}\sum_{n=2}^{\infty}\frac{\prod_{m=1}^{n-1}\left(m-\frac{1}{2}\right)}{n!(n-1)} + \ln\left(\frac{w_c}{\sqrt{2\chi}}\right)\right] + \frac{1}{4}g_1\chi^2 w_c^{-2} + O(\chi^3 w_c^{-4}), \tag{2.13}$$

$$\bar{Q}_2 = g_3\chi^2\left[2 + 3\sum_{n=3}^{\infty}\frac{\prod_{m=2}^{n-1}\left(m-\frac{3}{2}\right)}{n!(2n-4)} + \frac{3}{2}\ln\left(\frac{w_c}{\sqrt{2\chi}}\right)\right] - \frac{3}{2}g_3\chi w_c^2 + O(\chi^3 w_c^{-2}), \tag{2.14}$$

where $A_{(p+1)/2}$ in (2.12) is defined by

$$\frac{A_{(p_\nu+1)/2}}{2^{(p_\nu+1)/2}} = g_{p_\nu}\left[\sum_{n=1}^{\infty}\frac{\prod_{m=0}^{n-1}\left(m-\frac{p_\nu}{2}\right)}{n!(2n-p_\nu-1)} - \frac{1}{p_\nu+1}\right] \quad \text{for } p_\nu > -1, p_\nu \neq 2n-1 \text{ with } n \in \mathbb{N}^\star. \tag{2.15}$$

Gauss' test can be used to show that the series in (2.13)–(2.15) are convergent for the given values of p_ν.

2.4. *Electron-repelling potential in Poisson's equation: kinetic Bohm criterion*

If $-1 < p < 1$, (2.10) for Q_1, (2.11) for Q_2, (2.12) for $\bar{Q}_{(p+1)/2}$ and (2.12), (2.13) or (2.14) for $\bar{Q}_{(p_\nu+1)/2}$ with $\nu \geqslant 2$ are inserted into (2.5) to obtain

$$\chi''(\xi) = A_{(p+1)/2}\chi^{(p+1)/2} + O(\chi^3 w_c^{p-5}, \chi w_c^{p_{N_{3+}}-1}, \chi^{(p_2+1)/2}). \tag{2.16}$$

This equation is accurate for $\chi^{1/2} \ll w_c \lesssim 1$, where the restriction $w_c \ll 1$ is not necessary because the integral in (2.2) is dominated by the region near $w = \sqrt{2\chi}$ regardless of whether g is approximated by (2.3) or (2.4) for $w \sim 1$. Figure 1 is a plot of the numerical evaluation of $A_{(p+1)/2}$ as a function of $p \in (-1, 1) \cup (1, 3)$, illustrating that $A_{(p+1)/2} < 0$ for $p \in (-1, 1)$. This assumed interval for p has therefore led to a negative charge density, $\chi'' \leqslant 0$, in contradiction with an electron-repelling sheath. Hence, the form (2.3) requires

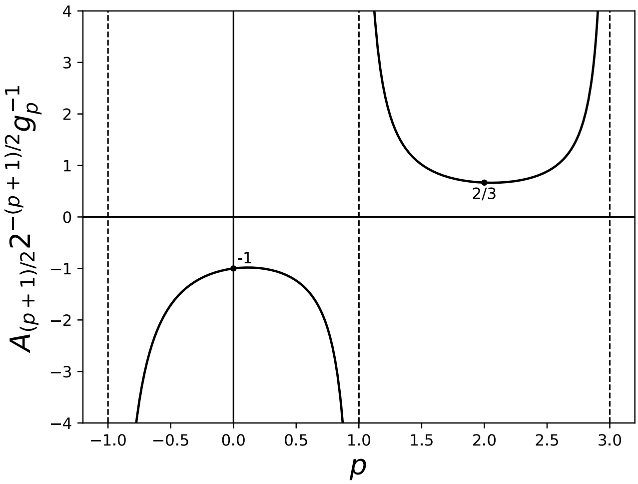

FIGURE 1. The quantity $A_{(p+1)/2}/(2^{(p+1)/2}g_p)$ is plotted as a function of p in the intervals $p \in (-1, 1)$ and $p \in (1, 3)$. The asymptotes at $p = -1$, 1 and 3 are shown as dashed lines. The value of the curve at $p = 0$ and 2 is marked.

$p \geqslant 1$: the distribution function satisfies $g(0^+) = 0$, corresponding to (1.8), and has a non-divergent (zero or finite) first derivative $g'(0^+)$.

If $p = 1$, (2.10) for Q_1, (2.11) for Q_2, (2.13) for $\bar{Q}_1$ and (2.12) or (2.14) for $\bar{Q}_{(p_\nu+1)/2}$, $\nu \geqslant 2$, are inserted into (2.5) to obtain

$$\chi'' = B_1\chi \ln(1/\chi) + A_1\chi + O(\chi^3 w_c^{-4}, \chi w_c^{p_{N_{3+}}-1}, \chi^{(p_2+1)/2}), \tag{2.17}$$

where A_1 and B_1 are defined by

$$A_1 = g_1\left[\frac{1}{2}\ln 2 - \frac{1}{2}\sum_{n=2}^{\infty}\frac{\prod_{m=1}^{n-1}\left(m-\frac{1}{2}\right)}{n!(n-1)} - 1\right] + 1 - \int_0^\infty g''(w)\ln(1/w)\,\mathrm{d}w, \tag{2.18}$$

$$B_1 = -g_1/2 < 0. \tag{2.19}$$

Equation (2.17) is accurate for $\chi^{1/2} \ll w_c \ll 1$. This equation, which follows from the assumption $p = 1$, is in contradiction with an electron-repelling sheath, since $B_1 < 0$ from (2.19) and $\ln(1/\chi) \gg 1$ for $\chi \ll 1$ imply that $\chi'' \leqslant 0$. Hence, $p > 1$ is required in (2.3), corresponding to $g'(0^+) = 0$. This conclusively proves that condition (1.9) must be satisfied at the sheath entrance.

Considering $p > 1$ and inserting (2.10) for Q_1, (2.11) for Q_2, and (2.12) or (2.14) for $\bar{Q}_{(p_\nu+1)/2}$ in (2.5) gives

$$\chi''(x) = \chi\left[1 - \int_0^\infty g''(w)\ln(1/w)\,\mathrm{d}w\right] + O(\chi^{(p+1)/2}, \chi^3 w_c^{p-5}, \chi w_c^{p_{N_{3+}}-1}, \chi^2). \tag{2.20}$$

Only when $p > 1$ does the asymptotic expansion of Poisson's equation in (2.5) up to and including terms of order $\sim \chi$ become accurate for $\sqrt{2\chi} \leqslant w_c \ll 1$, as seen from (2.20)

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upon enforcing the constraint $w_c \geqslant \sqrt{2\chi}$ arising from (2.2). This is because the part of the integral in (2.2) close to $w = \sqrt{2\chi}$ is negligible compared with terms of order $\sim \chi$ irrespective of whether the approximation (2.3) or (2.4) is used there. Hence, the expansion in Hall (1962), which takes $w_c \sim \sqrt{2\chi}$, is correct only when the distribution function and its first derivative vanish at $w = 0$, just like the expansion in Harrison & Thompson (1959) which was criticised by Hall. For (2.20) to be compatible with an electron-repelling sheath, the condition $\int_0^\infty \ln(1/w) g''(w)\,\mathrm{d}w < 1$ is required, which is the dimensionless form of (1.10) without the equality. In order to prove (1.10), and consequently (1.3), it must still be verified that if the equality holds,

$$\int_0^\infty g''(w) \ln(1/w)\,\mathrm{d}w = \int_0^\infty \frac{g(w)}{w^2}\,\mathrm{d}w = 1, \tag{2.21}$$

the higher-order terms in (2.20) are consistent with an electron-repelling sheath solution with $\chi(\xi) \geqslant 0$.

2.5. *Higher-order analysis for marginally satisfied kinetic Bohm criterion*

If $1 < p < 3$, inserting (2.10) for Q_1, (2.11) for Q_2, (2.12) for $\bar{Q}_{(p+1)/2}$, (2.12) or (2.14) for $\bar{Q}_{(p_\nu+1)/2}$ with $\nu \geqslant 2$ and (2.21) in (2.5) leads to

$$\chi'' = A_{(p+1)/2}\chi^{(p+1)/2} + O(\chi^3 w_c^{p-5}, \chi w_c^{p_{N_{3+}}-1}, \chi^{(p_2+1)/2}). \tag{2.22}$$

This equation is accurate for $\chi^{1/2} \ll w_c \ll \chi^{(p-1)/[2(p_{N_{3+}}-1)]}$. From figure 1 it is seen that $A_{(p+1)/2} > 0$ for $1 < p < 3$, which makes (2.22) consistent with a positive space charge, $\chi'' \geqslant 0$. If $p = 3$, inserting (2.10) for Q_1, (2.11) for Q_2, (2.14) for $\bar{Q}_2$, (2.12) for $\bar{Q}_{(p_\nu+1)/2}$ with $\nu \geqslant 2$ and (2.21) into (2.5) gives (note that $p = p_1 = 3$ implies $N_{3+} = 2$)

$$\chi'' = B_2\chi^2 \ln(1/\chi) + A_2\chi^2 + O(\chi^3 w_c^{-2}, \chi w_c^{p_2-1}), \tag{2.23}$$

with

$$A_2 = g_3\left[\frac{11}{4} + 3\sum_{n=3}^{\infty}\frac{\prod_{m=2}^{n-1}\left(m-\frac{3}{2}\right)}{n!(2n-4)} - \frac{3}{4}\ln 2\right] + \frac{1}{4}\int_0^\infty g''''(w)\ln(1/w)\,\mathrm{d}w - \frac{1}{2}, \tag{2.24}$$

$$B_2 = 3g_3/4 > 0. \tag{2.25}$$

Equation (2.23) is accurate for $\chi^{1/2} \ll w_c \ll \chi^{1/(p_2-1)}$, with $p_2 > 3$, and is again consistent with a positive space charge, since $B_2 > 0$ from (2.25). If $p > 3$, inserting (2.10) for Q_1, (2.11) for Q_2, (2.12) for $\bar{Q}_{(p_\nu+1)/2}$ and (2.21) into (2.5) gives

$$\chi'' = A_2\chi^2 + O(\chi w_c^{p-1}, \chi^3), \tag{2.26}$$

$$A_2 = \frac{1}{4}\int_0^\infty g''''(w)\ln(1/w)\,\mathrm{d}w - \frac{1}{2} = \frac{3}{2}\int_0^\infty \frac{g(w)}{w^4}\,\mathrm{d}w - \frac{1}{2} > 0. \tag{2.27}$$

Equation (2.26) requires $\sqrt{2\chi} \leqslant w_c \ll \chi^{1/(p-1)}$ to be accurate. In (2.27), Schwartz's inequality and the equalities (2.6) and (2.21) constrain $\int_0^\infty \mathrm{d}w g(w)/w^4 \geqslant 1$ (Riemann 1991), such that $A_2 > 0$. Equation (2.26) is compatible with an electron-repelling sheath, and exhausts all possible remaining values of p. This completes the proof that an electron-repelling sheath solution (with adiabatic electrons) exists if (1.8)–(1.10), which can be combined into (1.3), are satisfied.

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3. Discussion and conclusion

The physical interpretation of the kinetic Bohm criterion (1.3) in the limit $\epsilon \to 0$ is as follows. When entering the sheath, slower ions undergo a much larger relative velocity increment than faster ions for a given potential drop, and by continuity they have a larger contribution to the ion density drop. Hence, for mono-energetic ions there is a threshold velocity (1.2) above which the ion density drop is less than the electron density drop, consistent with the positive space charge expected in the sheath. Similarly, with an arbitrary velocity distribution instead of a mono-energetic one, conditions (1.8) and (1.9) limit the number of slow ions with $v_x \approx 0$ to ensure that the ion density drop scales linearly with the potential drop, like the electron density drop, and not more strongly. Condition (1.3) then results from requiring that the ion density drop be smaller than the electron one. At this point, it should be unsurprising that slow ions with $|v_x| \ll v_B$ have a larger weight in this condition, forcing the bulk ions to compensate by moving faster towards the wall and causing the over-satisfaction of (1.2).

If it is desirable to formulate (1.3) in terms of the actual distribution function f_ϵ, this can be done carefully as follows. Presuming that f_ϵ is, just like its limit f in (1.6), differentiable at $v_x = 0$, the two conditions (1.8) and (1.9) imply that $f_\epsilon(0) \sim \epsilon^{r_1(q)} n_e/Zv_B$ and $f'_\epsilon(0) \sim \epsilon^{r_2(q)} n_e/Zv_B^2$, where $r_1(q) > 0$ and $r_2(q) > 0$ depend on q in $x_\epsilon = \epsilon^q L$. The small additional number of slow ions with $v_x \approx 0$ at x_ϵ is due to the replenishment of ions with $v_x \geqslant 0$ occurring in the small region $x \in [0, x_\epsilon]$, caused for example by rare (vanishing for $\epsilon \to 0$) collision or ionisation events in this region. Integrating the third condition (1.10) by parts twice gives

$$\lim_{v_x \to 0^-} \lim_{\epsilon \to 0} \left[\int_{-\infty}^{v_x} \frac{f_\epsilon(v'_x)}{v'^2_x} dv'_x + f'_\epsilon(v_x) \ln(-1/v_x) + \frac{f_\epsilon(v_x)}{v_x} \right] \leqslant \lim_{\epsilon \to 0} \frac{n_{e,\epsilon}}{Zv_B^2}. \tag{3.1}$$

If the limits on the left-hand side of (3.1) are interchanged, all terms diverge; without the interchange, the second and third terms vanish by (1.8) and (1.9), recovering (1.3). The left-hand side of (1.7) results from interchanging the limits in (3.1) without retrieving the additional terms that no longer vanish (and diverge) after the interchange. The limits in (3.1) can be combined without interchange by taking $v_x = -v_\epsilon$ with $v_\epsilon = \epsilon^r v_B$ and $r \in (0, r_1(q))$, such that $\lim_{\epsilon \to 0} f_\epsilon(-v_\epsilon)/v_\epsilon = 0$ and $\lim_{\epsilon \to 0} f'_\epsilon(-v_\epsilon) \ln(1/v_\epsilon) = 0$. This allows subsuming conditions (1.8), (1.9) and (3.1) into the single condition[4]

$$\lim_{\epsilon \to 0} \int_{-\infty}^{-v_\epsilon} \frac{f_\epsilon(v_x)}{v_x^2} \, dv_x \leqslant \lim_{\epsilon \to 0} \frac{n_{e,\epsilon}}{Zv_B^2}. \tag{3.2}$$

A small number (vanishing for $\epsilon \to 0$) of slow ions with $|v_x| < v_\epsilon$ are ignored in (3.2) because their contribution to the density drop would be sharply overestimated due to the neglect of their replenishment in the region $x \leqslant x_\epsilon$. Yet, the form of (3.2) confirms the importance of slow ions with $|v_x| \ll v_B$ in the kinetic Bohm criterion.

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Declaration of interest

The authors report no conflict of interest.

[4]If f_ϵ is not differentiable at $v_x = 0$, such that $f_\epsilon(v_x) \sim \epsilon^{r_3} v_x^{p_\epsilon}$ for $v_x \ll v_B$ with $r_3 > 0$ and $p_\epsilon \in (-1, 0) \cup (0, 1)$, the limits $\epsilon \to 0$ and $v_x \to 0_-$ are non-interchangeable in (1.8) (if $p_\epsilon \in (-1, 0)$), (1.9) and (1.10), and $r < r_3/(1 - p_\epsilon)$ is required for (3.2) to hold.

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REFERENCES

BAALRUD, S. & HEGNA, C. 2011 Kinetic theory of the presheath and the Bohm criterion. *Plasma Sources Sci. Technol.* **20** (2), 025013.

BAALRUD, S. & HEGNA, C. 2012 Reply to comment on 'Kinetic theory of the presheath and the Bohm criterion'. *Plasma Sources Sci. Technol.* **21** (6), 068002.

BAALRUD, S., SCHEINER, B., YEE, B., HOPKINS, M. & BARNAT, E. 2015 Extensions and applications of the Bohm criterion. *Plasma Phys. Control. Fusion* **57** (4), 044003.

BENDER, C.M. & ORSZAG, S.A. 1999 *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory*, vol. 1. Springer Science & Business Media.

BERTOTTI, B. 1961 Theory of an electrostatic probe in a strong magnetic field. *Phys. Fluids* **4** (8), 1047–1052.

BERTOTTI, B. & CAVALIERE, A. 1965 Boundary conditions for plasma ion waves. *Il Nuovo Cimento (1955–1965)* **35** (4), 1244–1246.

BOHM, D. 1949 *The Characteristics of Electrical Discharges in Magnetic Fields* (ed. A. Guthrie & R.K. Wakerling), chap. 3. McGraw–Hill.

CARUSO, A. & CAVALIERE, A. 1962 The structure of the collisionless plasma-sheath transition. *Il Nuovo Cimento (1955–1965)* **26**, 1389–1404.

CHODURA, R. 1982 Plasma–wall transition in an oblique magnetic field. *Phys. Fluids (1958–1988)* **25** (9), 1628–1633.

FERNSLER, R.F., SLINKER, S.P. & JOYCE, G. 2005 Quasineutral plasma models. *Phys. Rev.* E **71** (2), 026401.

FRANKLIN, R. & OCKENDON, J. 1970 Asymptotic matching of plasma and sheath in an active low pressure discharge. *J. Plasma Phys.* **4** (2), 371–385.

GERALDINI, A., BRUNNER, S. & PARRA, F. 2024 Sheath constraints in turbulent magnetised plasmas. Preprint, arXiv:2401.07385.

HALL, L. 1962 Harrison-Thompson generalization of Bohm's sheath condition. In *Proceedings of the Physical Society of London*, vol. 80. University of California.

HARRISON, E. & THOMPSON, W. 1959 The low pressure plane symmetric discharge. *Proc. Phys. Soc.* **74** (2), 145.

LANGMUIR, I. 1923 Positive ion currents from the positive column of mercury arcs. *Science* **58** (1502), 290–291.

LANGMUIR, I. 1929 The interaction of electron and positive ion space charges in cathode sheaths. *Phys. Rev.* **33** (6), 954.

RIEMANN, K.-U. 1991 The Bohm criterion and sheath formation. *J. Phys.* D*: Appl. Phys.* **24** (4), 493.

RIEMANN, K.-U. 1997 The influence of collisions on the plasma sheath transition. *Phys. Plasmas* **4** (11), 4158–4166.

RIEMANN, K.U. 2003 Kinetic analysis of the collisional plasma–sheath transition. *J. Phys.* D*: Appl. Phys.* **36** (22), 2811.

RIEMANN, K.-U. 2006 Plasma-sheath transition in the kinetic Tonks-Langmuir model. *Phys. Plasmas* **13** (6), 063508.

RIEMANN, K.U. 2012 Comment on – 'Kinetic theory of the presheath and the Bohm criterion'. *Plasma Sources Sci. Technol.* **21** (6), 068001.

RIEMANN, K.U., SEEBACHER, J., TSKHAKAYA, D. & KUHN, S. 2005 The plasma–sheath matching problem. *Plasma Phys. Control. Fusion* **47** (11), 1949.

SLEMROD, M. 2002 Monotone increasing solutions of the Painlevé 1 equation $y'' = y^2 + x$ and their role in the stability of the plasma-sheath transition. *Eur. J. Appl. Maths* **13** (6), 663–680.

TONKS, L. & LANGMUIR, I. 1929 A general theory of the plasma of an arc. *Phys. Rev.* **34** (6), 876.