The number n is multiply perfect if and only if $\sigma_1(n) \equiv 0 \pmod{n}$. By (1) this is equivalent to

$$(2) \qquad T_1(n) \equiv S_1(n) - \varphi_1(n) + 1 \pmod{n}.$$

The right hand side of (2) is congruent to

$$-\sum_{d|n, d>1} \mu(d) d S_1(n/d) + 1 \equiv -\sum_{d|n, d>1} \mu(d) n \tfrac{1}{2}(1+n/d) + 1 \pmod{n}.$$

If n is odd, each $1 + n/d$ is even and $n | n\frac{1}{2}(1+n/d)$. Thus an odd n is multiply perfect if and only if $T_1(n) \equiv 1 \pmod{n}$.

Now let $n = \prod_{p|n} p^{\alpha}$ be even. Correcting the statement of our problem we have to assume $n \neq 2$. We wish to show that n is multiply perfect if and only if $T_1(n) \equiv 1 + n/2 \pmod{n}$. Thus we have to show $\sum_{d|n, d>1} \mu(d) n\frac{1}{2}(1+n/d) \equiv n/2 \pmod{n}$ or $\sum_{d|n, d>1} \mu(d)(1+n/d) + 1 \equiv 0 \pmod{2}$. This is equivalent to

$$(4) \qquad 2 \mid \Sigma = \sum_{d|n} \mu(d)(1+n/d).$$

But $\Sigma = \sum_{d|n} \mu(d)(n/d) + \sum_{d|n} \mu(d) = \sum_{d|n} \mu(d)(n/d)$

$$= \varphi(n) = \prod_{p|n} (p^{\alpha} - p^{\alpha - 1}).$$

Thus Σ is even unless $n = 2$. This proves (4).

P 3. Let F be a finite field of characteristic p. Let V_n be an n-dimensional vector space over F. In V_n a symmetric bilinear form (a, b) is given. Let $n \geq 2$ if $p = 2$ and $n \geq 3$ if p is odd. Show that there is a vector $a \neq 0$ in V_n such that $(a, a) = 0$.

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Solution by the proposer. Let $F = \{\xi, \eta, \ldots\}$ be a finite field of characteristic p. Let G denote the multiplicative group of all the squares $\neq 0$. If $p = 2$, $\xi^2 = \eta^2$ if and only if $\xi = \eta$. Thus the mapping of the elements $\neq 0$ of F onto G is one-one and G is the multiplicative group of F. If $p > 2$, this mapping is two-one and G is a subgroup of index two in the multiplicative group of F. Let $\bar{G}$ denote the complement of G in this group.

If $1 + G = G$, $1 \in G$ would successively imply $2, 3, \ldots, p-1 \in G$ and finally $p = 0 \in G$. Thus

$$(1) \qquad 1 + G \neq G.$$

Let $V_n = \{a, b, \ldots\}$ denote a vector space of dimension n over F with a symmetric bilinear form (x,y). If (a,a) = 0, the vector a is called isotropic.

If p = 2 and $n \geqslant 2$, V_n will contain two linearly independent vectors b and c. We may assume they are non-isotropic. The equation $\xi^2 = (b,b)/(c,c)$ has a solution $\xi \in F$. It follows that $(b + \xi c, b + \xi c) = (b,b) + 2\xi \cdot (b,c) + \xi^2 \cdot (c,c) =$ $(b,b) + \xi^2 \cdot (c,c) = 0$.

From now on let p > 2, $n \geqslant 3$. For every vector a let M_a denote the set of the norms $(\lambda a, \lambda a) = \lambda^2 (a,a)$ with $\lambda \neq 0$. Thus either a is isotropic or $M_a = G$ or $M_a = \overline{G}$.

We choose any three mutually orthogonal vectors $\neq 0$. if none of them is isotropic, two of them, say b and c satisfy $M_b = M_c$. We may assume (b,b) = (c,c). Thus

$$(b + \xi c, b + \xi c) = (b,b) + 2\xi \cdot (b,c) + \xi^2 \cdot (c,c)$$

$$= (b,b) + 2\xi \cdot 0 + \xi^2 \cdot (b,b) = (1 + \xi^2)(b,b).$$

Case (i): $-1 \in G$. Then let ξ be a solution of $1 + \xi^2 = 0$. The vector $b + \xi c$ will be isotropic.

Case (ii): $-1 \in \overline{G}$. By (1) there is a ξ such that $1 + \xi^2 \in \overline{G}$. Thus there is a vector d such that $M_b \neq M_d$.

Since $n \geqslant 3$, there is a vector $e \neq 0$ such that (e,b) = (e,d) = 0. Since M_e must be distinct from either M_b or M_d, we have found two vectors, say e and f such that (e,f) = 0, $M_e \neq M_f$. We may assume $1 \in M_e$, $-1 \in M_f$ and hence (e,e) = 1, (f,f) = -1. This yields (e + f, e + f) = (e,e) + (f,f) = 0.

NOTES

ON THE DISCRIMINANTS OF A BILINEAR FORM

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Let E denote a vector space of dimension n over a field of characteristic $\neq 2$. In E a symmetric bilinear form f(x,y) is given. Define E_f^* as the subspace of those vectors x for which f(x,y) = 0 for all $y \in E$. Thus rank f = n - dim E_f^*. Furthermore, define ind f = maximum dimension of a subspace in which

https://doi.org/10.1017/S0008439500024942 Published online by Cambridge University Press