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SURJECTIVITY OF MULTIFUNCTIONS UNDER GENERALIZED DIFFERENTIABILITY ASSUMPTIONS

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The aim of the present paper is to give some general surjectivity theorems for multifunctions using tangent cones and generalized differentiability assumptions.

1.

The surjectivity theorems for multifunctions are used in existence results for differential inclusions, in the study of stability, in the study of dynamical economical systems [*14*], in optimization problems, and so on.

Many results on these subjects can be found in: [*1*], [*19*], [*21*], [*25*], [*26*], [*27*].

In this note we present a necessary and sufficient condition of surjectivity.

This differential condition is given with respect to the range and from this result we obtain several sufficient conditions of surjectivity under differentiability assumptions.

The fundamental result of this paper, Theorem 3, is a generalization and a substantial improvement of the principal result of the paper [*11*].

Many and diverse differentiability concepts can be found in: [*2*], [*3*], [*9*], [*10*], [*12*], [*15*], [*16*], [*17*].

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2.

We will use the following notations: $F : X \to Y$ denotes a multifunction from a Banach space X into a Banach space Y ; we identify F with its graph writing $(x, y) \in F$ if $y \in F(x)$; $B(b, r)$ denotes the closed r-ball centred at b ; if A and B are closed sets we denote by $e(A, B) = \sup_{x\in A} d(x, B)$ their excess, where $d(x, B) = \inf_{y\in B} d(x, y)$.

DEFINITION 1. Let R be a closed subset of Y and $b \in R$. The *tangent cone* T_bR to R at the point b is the set of vectors v of Y such that there exist a sequence (t_n) of positive real numbers convergent to 0 and a sequence (b_n) of points of R such that $\lim_{n\to\infty} t_n^{-1}(b_n-b) = v$. This means that $\liminf_{t\to 0^+} t^{-1}d(b+tv, R) = 0$.

FUNDAMENTAL RESULT. *If R is a closed subset of Y such that for each $b \in R$ we have $T_bR = Y$, then $R = Y$.*

Since for all $y \in Y$ and all $b \in R$ we have $T_bR = Y$ and $d(y, b+T_bR) = 0$ we observe that the fundamental result is a consequence of the following theorem proved by Browder [4].

THEOREM 1. *Let R be a closed subset of a Banach space Y and $y \in Y$.*

The point y is a point of R if there exist $c \in]0, 1[$ and $r > d(y, R)$ such that for each $b \in R \cap B(y, r)$ the following inequality is true:

$$\text{(i)} \qquad d(y, b+T_bR) \le c\,d(b, y) .$$

Our proof for Theorem 1 is very simple and it is based on the following theorem proved by Daneš [8].

THEOREM 2 [Drop Theorem]. *Let R be a closed subset of Y , let y be a point of $Y\setminus R$ and let r be a real number verifying $0 < r < d(y, R)$.*

Then for each point $b_0 \in R$ there exists a point $\bar{b} \in R \cap G(b_0)$ such *that* $R \cap G(\bar{b}) = \{\bar{b}\}$ *, where* $G(b)$ *denotes the convex hull of* $\{b\} \cup B(y, r)$.

Proof of the Browder Theorem. If $y \notin R$ we denote $d = d(y, R) > 0$ and we choose r verifying *(i)* and such that $cr < d < r$ (this is possible since if r' satisfies *(i)* then all r such that $d < r < r'$ also satisfy *(i)*).

Let s be a real number such that $cr < s < d$ and let b_0 be a point of R such that $d(y, b_0) < r$; then there exists $\bar{b} \in R \cap G(b_0)$ such that $R \cap G(\bar{b}) = \{\bar{b}\}$.

Since $\bar{b} \in G(b_0)$ we have $d(\bar{b}, y) \le \max(s, d(b_0, y)) < r$ and hence $d(y, \bar{b}+T_{\bar{b}}R) \le cd(\bar{b}, y) < cr < s$.

Thus there exists a sequence (t_n) in $]0, 1[$ convergent to 0 and a sequence $(\bar{b}_n)$ of points in R such that

$$c_n = \bar{b} + t_n^{-1}(\bar{b}_n - \bar{b}) \in B(y, s) .$$

We observe now that $\bar{b}_n = t_n c_n + (1-t_n)\bar{b}$ is a convex combination of $\bar{b}$ and a point of $B(y, s)$ and hence $\bar{b}_n \in G(\bar{b}) \cap R = \{\bar{b}\}$.

Then we have $\bar{b} = c_n \in B(y, s)$, but this is impossible because $s < d < r$. □

The following surjectivity theorem for multifunctions is an immediate consequence of the fundamental result.

THEOREM 3. *Let* $F : X \to Y$ *be a multifunction from* X *into* Y *such that* $R = F(X)$ *is a closed subset of* Y .

Then F *is onto if and only if for each* b *in* R , $T_bR = Y$.

The necessary and sufficient condition given by Theorem 3 deals with the range of the multifunction.

We will give now several sufficient conditions that do not use the range directly.

DEFINITION 2 [*20*]. Let F be a multifunction with closed range and

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let $c = (a, b)$ be a point of F.

The set $\overline{D}F(c)(x) = \{y \in Y;\ (x, y) \in T_c F\}$ is called the upper Dini derivative of the multifunction F at the point c in the direction x.

PROPOSITION 1. *Let F be a multifunction. If b is a point of the range R of F then*

$$T_b R \supset \mathrm{cl}\Big[\bigcup_{a \in F^{-1}(b)} \overline{D}F(a, b)(X)\Big] .$$

Proof. Suppose $(a, b) \in F$ and $y \in \overline{D}F(a, b)(X)$. There exists $x \in X$ such that $y \in \overline{D}F(a, b)(x)$ and hence there exist a sequence (t_n) of positive real numbers convergent to 0 and a sequence $((x_n, y_n))$ of points of F such that $t_n^{-1}(x_n - a)$ is convergent to x and $t_n^{-1}(y_n - b)$ is convergent to y.

Thus $y \in T_b R$ and the proposition is proved because $T_b R$ is a closed set. □

COROLLARY 1. *Let F be a multifunction from X into Y with closed graph and closed range.*

If for each $b \in F(X)$ and each $y \in Y$ there exists $a \in X$ such that $y \in \mathrm{cl}[\overline{D}F(a, b)(X)]$, then $F(X) = Y$.

Under certain supplementary assumptions, which will be specified, the condition of Corollary 1 is a condition of open mapping.

DEFINITION 3. If $F : X \to Y$ is a multifunction and $c = (a, b)$ a point of F, we say that F is open at the point c if for each positive real number η there exists a positive real number ε such that $F(B(a, \eta)) \supset B(b, \varepsilon)$.

PROPOSITION 2. *Let F be a multifunction from X into Y and $(a, b) \in F$. Suppose the following assumption: there exists a sequence (η_n) of positive real numbers convergent to 0 such that $F(B(a, \eta_n))$ is a convex set with nonempty interior.*

Then the condition $\mathrm{cl}[\overline{D}F(a, b)(X)] = Y$ is equivalent to the condition that F is open at the point (a, b).

Proof. If F is not open at the point (a, b) then it is possible to find a real number $\eta > 0$ and a sequence (y_n) convergent to b such that $y_n \notin F[B(a, \eta)]$.

There exists p such that $\eta_p < \eta$ and we observe that b is a boundary point of the convex set $F[B(a, \eta_p)]$ with non-empty interior.

Let H be a supporting hyperplane at the point b of this convex set. The set $\overline{D}F(a, b)(X)$ is contained in the closed half space containing $F[B(a, \eta_p)]$. Thus $\text{cl}[\overline{D}F(a, b)(X)] \neq Y$. The converse is immediate. □

EXAMPLE. Let $f : R \to R$ be the function defined by

$$f(0) = 0 ,$$

$$f(x) = \left(|x| - \frac{1}{n+1}\right)2n(n+1) , \text{ if } \frac{1}{n+1} \leq |x| \leq \frac{2n+1}{2n(n+1)} ,$$

and

$$f(x) = \left(\frac{1}{n} - |x|\right)2n(n+1) , \text{ if } \frac{2n+1}{2n(n+1)} \leq |x| \leq \frac{1}{n} .$$

The multifunction $F(x) = [-f(x), f(x)]$ verifies at the point $(0, 0)$ all conditions of Proposition 2.

Corollary 1 implies the verification of the surjectivity condition at every point of the range, but we will propose now a sufficient condition which must be verified at every point of the domain.

DEFINITION 4. If F and G are multifunctions from X into Y and $a \in \text{dom } F \cap \text{dom } G$, then G is said to be inferiorly semi-tangent to F at the point a if

$$G(a) = F(a) \quad \text{and} \quad \lim_{x \to a} \frac{e[G(x), F(x)]}{\|x-a\|} = 0 ,$$

where $e[G(x), F(x)] = 0$ if $G(x) = \emptyset$ and $e[G(x), F(x)] = +\infty$ if $F(x) = \emptyset$ but $G(x) \neq \emptyset$.

PROPOSITION 3. *Suppose F and G are multifunctions with closed graph. If G is inferiorly semi-tangent to F at the point a then for each $b \in F(a)$ and each $x \in X$ we have $\overline{D}F(a, b)(x) \supset \overline{D}G(a, b)(x)$.*

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Proof. Let $y \in \overline{D}G(a, b)(x)$; there exist a sequence of positive real numbers (t_n) convergent to 0 and a sequence $((x_n, y_n))$ of points of G such that $t_n^{-1}(x_n-a)$ is convergent to x and $t_n^{-1}(y_n-b)$ is convergent to y .

By the tangency assumption we have that there exists a sequence (z_n) of Y such that $(x_n, y_n) \in F$ and $\lim_{n\to\infty} (\|y_n-z_n\|)/(\|x_n-a\|) = 0$ (taking $y_n = z_n$ if $x_n = a$).

Thus

$$\lim_{n\to\infty} \left\|\frac{z_n-b}{t_n} - y\right\| \le \lim_{n\to\infty} \frac{\|z_n-y_n\|}{\|x_n-a\|} \cdot \frac{\|x_n-a\|}{t_n} + \lim_{n\to\infty} \left\|\frac{y_n-b}{t_n} - y\right\| = 0 . \qquad \square$$

COROLLARY 2. *Let* F *be a multifunction with closed graph and closed range. If for each* $c = (a, b) \in F$ *there exists an inferiorly semi-tangent multifunction* G_c *to* F *at the point* a *such that* $\mathrm{cl}\left[DG_c(a, b)(X)\right] = Y$ *then* $F(X) = Y$. $\square$

REMARK 1. In particular Corollary 2 can be used for multifunctions F differentiable at every point of their definition domain in the De Blasi, Gautier or Methlouti sense, since these differentiability concepts supply inferiorly semi-tangent multifunctions ([9], [10], [16], [17]).

REMARK 2. We find again here some results proved by Browder and Pohožaev ([4], [5], [6], [7], [20], [22], [23]).

REMARK 3. In [18] Nirenberg stated the following problem.

Suppose $f : H \to H$ (H is a Hilbert space) is a continuous map having the following properties:

(1°) $\|f(x)-f(y)\| \ge \|x-y\|$ for all $x, y \in H$;

(2°) $f(0) = 0$;

(3°) f maps a neighborhood of the origin onto a neighborhood of the origin.

Does f map H onto H ?

A solution of this problem is proposed in [13] by Kung-Ching Chang and Li Shujie in the case where $f : X \to Y$ (X and Y are Banach spaces) is a

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Fréchet-differentiable map verifying (1°), (2°), (3°) and such that for all $x_0 \in X$, $\lim_{x \to x_0} \|f'(x)-f'(x_0)\| < 1$.

We remark that our results are applicable in the case of expanding maps $\left(\|f(x)-f(y)\| \geq \alpha\|x-y\|;\ \forall x,\ y \in X\right)$ under generalized differentiability assumptions since every continuous expanding map between Banach space has a closed range. This will be shown elsewhere. □

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