



Eigenvalue Approach to Even Order System Periodic Boundary Value Problems

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Abstract. We study an even order system boundary value problem with periodic boundary conditions. By establishing the existence of a positive eigenvalue of an associated linear system Sturm–Liouville problem, we obtain new conditions for the boundary value problem to have a positive solution. Our major tools are the Krein–Rutman theorem for linear spectra and the fixed point index theory for compact operators.

1 Introduction

In this paper, we consider the higher order system periodic boundary value problem (BVP)

$$(1.1) \quad (AD^2 + B)^n u = W(t)f(t, u), \quad t \in (0, \omega),$$

$$(1.2) \quad u^{(k)}(0) = u^{(k)}(\omega), \quad k = 0, 1, \dots, 2n - 1,$$

with $n \in \mathbb{N}$, $D = \frac{d}{dt}$, $u = (u_1, \dots, u_m)^T \in \mathbb{R}_+^m$ (where $\mathbb{R}_+ = [0, \infty)$); $A = \text{diag}(a_1, \dots, a_m)$ and $B = \text{diag}(b_1, \dots, b_m)$ with $a_i = \pm 1$ and $b_i > 0$, $i = 1, \dots, m$; $W = (w_{ij})_{m \times m} \in C([0, \omega], \mathbb{R}_+^{m \times m})$ such that $\min_{1 \leq i \leq m} w_{ij}(t) \neq 0$ on $[0, \omega]$ for all $j = 1, \dots, m$; and $f = (f_1, \dots, f_m)^T \in C([0, \omega] \times \mathbb{R}_+^m, \mathbb{R}_+^m)$.

The existence of positive solutions for BVPs has been a focus of research for several decades. The main approaches are based on the fixed point theory on cones, the upper and lower solution method, and the variational method. See, for example, [3–5, 8, 10, 12–14, 18] for recent development in this area. Periodic BVPs have special importance in theory and applications, and many results have been obtained on the existence, multiplicity, and nonexistence of positive solutions. For the work on scalar periodic BVPs, see [1, 6, 7, 11, 16, 19–22, 25, 26, 28, 30] and the references therein. Systems of second order BVPs have also been investigated by researchers. For the existence of solutions, see Mawhin and Willem [23], O'Regan and Wang [24], Wang [27], and Zhao and Wu [31].

Recently, the authors investigated the BVP (1.1), (1.2) in [17]. A series of criteria on the existence, multiplicity, and nonexistence of positive solutions of BVP (1.1), (1.2) were obtained. In particular, Theorem 2.2 in [17] showed that BVP (1.1), (1.2) has at least one positive solution if for certain positive numbers α and β , either of the following holds:

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- $\hat{F}^0 < \beta^{-1}$ and $\hat{F}_\infty > (\alpha\beta)^{-1}$,
- $\hat{F}_0 > (\alpha\beta)^{-1}$ and $\hat{F}^\infty < \beta^{-1}$,

where

$$\begin{aligned}
 (1.3) \quad \hat{F}_0 &= \liminf_{\|x\| \rightarrow 0} \min_{\substack{i \in \{1, \dots, m\} \\ t \in [0, \omega]}} f_i(t, x) / \|x\|, \\
 \hat{F}^0 &= \limsup_{\|x\| \rightarrow 0} \max_{\substack{i \in \{1, \dots, m\} \\ t \in [0, \omega]}} f_i(t, x) / \|x\|, \\
 \hat{F}_\infty &= \liminf_{\|x\| \rightarrow \infty} \min_{\substack{i \in \{1, \dots, m\} \\ t \in [0, \omega]}} f_i(t, x) / \|x\|, \\
 \hat{F}^\infty &= \limsup_{\|x\| \rightarrow \infty} \max_{\substack{i \in \{1, \dots, m\} \\ t \in [0, \omega]}} f_i(t, x) / \|x\|.
 \end{aligned}$$

However, when one of $\hat{F}_0, \hat{F}^0, \hat{F}_\infty, \hat{F}^\infty$ is between β^{-1} and $(\alpha\beta)^{-1}$, Theorem 2.2 fails to apply. For second order scalar BVPs, an important approach to the existence of solutions or positive solutions is to utilize certain eigenvalues of associated linear Sturm–Liouville problems (SLP); see, for example, [4, 15]. This method has been applied by the authors to the scalar BVP (1.1), (1.2) with $n = 2$; see [16, Theorem 2.11].

In this paper, we first define a linear system SLP as the SLP associated with BVP (1.1), (1.2) and show that it has a positive eigenvalue, and then use this eigenvalue to establish new conditions for BVP (1.1), (1.2) to have a positive solution. Our major tools are the Krein–Rutman theorem for spectra of linear operators and the fixed point index theory for compact operators. Examples are given to show that our results in certain sense improve the results in the literature including those in [17] by the authors.

This paper is organized as follows: after this introduction, our main results, together with two examples, are stated in Section 2. All proofs are given in Section 3. In Section 4, we extend our results to a class of generalized systems of periodic BVP.

2 Main Results

BVP (1.1), (1.2) can be written componentwise as

$$\begin{aligned}
 (a_i D^2 + b_i) u_i &= \sum_{j=1}^m w_{ij}(t) f_j(t, u_1, \dots, u_m), \quad t \in (0, \omega), \\
 u_i^{(k)}(0) &= u_i^{(k)}(\omega), \quad i = 1, \dots, m, \quad k = 0, 1, \dots, 2n - 1.
 \end{aligned}$$

Throughout this paper, we assume that

(H1) For fixed $1 \leq i \leq m$, the scalar BVP

$$\begin{aligned}
 a_i u_i'' + b_i u_i &= 0, \quad t \in (0, \omega), \\
 u_i(0) &= u_i(\omega), \quad u_i'(0) = u_i'(\omega),
 \end{aligned}$$

has a Green’s function $G_i^{[1]}(t, s)$;

(H2) $G_i^{[1]}(t, s) > 0$ for $0 \leq t, s \leq \omega, i = 1, \dots, m$.

We denote

$$(2.1) \quad L_i = \min_{t,s \in [0,\omega]} G_i^{[1]}(t, s) \quad \text{and} \quad U_i = \max_{t,s \in [0,\omega]} G_i^{[1]}(t, s).$$

Remark 2.1 When $a_i = -1$, let

$$g_i(t) = \frac{e^{\sqrt{b_i}t} + e^{\sqrt{b_i}(\omega-t)}}{2\sqrt{b_i}(e^{\sqrt{b_i}\omega} - 1)}, \quad t \in [0, \omega].$$

Then

$$(2.2) \quad G_i^{[1]}(t, s) = \begin{cases} g_i(t-s) & 0 \leq s \leq t \leq \omega, \\ g_i(s-t) & 0 \leq t \leq s \leq \omega, \end{cases}$$

and $L_i = g_i(\omega/2), U_i = g_i(0)$.

When $a_i = 1$ and $0 < b_i < \pi^2\omega^{-2}, k = 1, 2, \dots$, let

$$g_i(t) = \frac{\sin(\sqrt{b_i}t) + \sin(\sqrt{b_i}(\omega-t))}{2\sqrt{b_i}(1 - \cos(\sqrt{b_i}\omega))}, \quad t \in [0, \omega].$$

Then

$$G_i^{[1]}(t, s) = \begin{cases} g_i(t-s) & 0 \leq s \leq t \leq \omega, \\ g_i(s-t), & 0 \leq t \leq s \leq \omega, \end{cases}$$

and $L_i = g_i(0), U_i = g_i(\omega/2)$.

Definition 2.2 A function $u = (u_1, \dots, u_m)^T \in C([0, \omega], \mathbb{R}^m)$ is said to be a positive solution of BVP (1.1), (1.2) if $u_i \in C^{2n-1}[0, \omega] \cap C^{2n}(0, \omega), u$ satisfies BVP (1.1), (1.2), $u_i(t) \geq 0$ on $[0, \omega], i = 1, \dots, m$, and $\sum_{i=1}^m u_i(t) > 0$ on $[0, \omega]$.

For any $u \in C([0, \omega], \mathbb{R}^m)$, define $\|u\| = \max_{t \in [0,\omega]} \sum_{i=1}^m |u_i(t)|$. To present our main results we need the following SLP associated with BVP (1.1), (1.2), which consists of the equation

$$(2.3) \quad (AD^2 + B)^n u = \lambda W^T(t)u, \quad t \in (0, \omega),$$

and the boundary condition (BC) (1.2), where $W^T = (w_{ij}^T)_{m \times m}$ is the transpose of $W, i.e., w_{ij}^T = w_{ji}, i, j = 1, \dots, m$.

The first result is about the existence of a desired eigenvalue of SLP (2.3), (1.2).

Theorem 2.3 SLP (2.3), (1.2) has a positive eigenvalue $\bar{\lambda}$ with a positive eigenfunction $v(t)$.

In the sequel we will use the following notation for limits where x is restricted to \mathbb{R}_+^m :

$$F_0 = \liminf_{\|x\| \rightarrow 0} \min_{\substack{i \in \{1, \dots, m\} \\ t \in [0, \omega]}} f_i(t, x)/x_i,$$

$$F^0 = \limsup_{\|x\| \rightarrow 0} \max_{\substack{i \in \{1, \dots, m\} \\ t \in [0, \omega]}} f_i(t, x)/x_i,$$

$$F_\infty = \liminf_{\|x\| \rightarrow \infty} \min_{\substack{i \in \{1, \dots, m\} \\ t \in [0, \omega]}} f_i(t, x)/x_i,$$

$$F^\infty = \limsup_{\|x\| \rightarrow \infty} \max_{\substack{i \in \{1, \dots, m\} \\ t \in [0, \omega]}} f_i(t, x)/x_i.$$

Our major result is given below.

Theorem 2.4 *BVP (1.1), (1.2) has at least one positive solution if either $F^0 < \bar{\lambda} < F_\infty$ or $F^\infty < \bar{\lambda} < F_0$.*

To derive some explicit conditions for BVP (1.1), (1.2) to have a positive solution, we introduce the following notation: with $G_i^{[l]}(t, s)$ as the Green's function defined in (H1) and L_i, U_i given by (2.1), we define

$$(2.4) \quad G_i^{[l]}(t, s) = \int_0^\omega G_i^{[1]}(t, \tau) G_i^{[l-1]}(\tau, s) d\tau, \quad l = 2, \dots, n,$$

and $G = \text{diag}(G_1, \dots, G_m)$ with $G_i = G_i^{[n]}$, $i = 1, \dots, m$. Let

$$(2.5) \quad L_* = \min_{i \in \{1, \dots, m\}} \{\omega^{n-1} L_i^n\}, \quad U^* = \max_{i \in \{1, \dots, m\}} \{\omega^{n-1} U_i^n\},$$

and

$$(2.6) \quad \alpha = L_*/U^*,$$

$$(2.7) \quad \beta^* = \max_{\substack{j \in \{1, \dots, m\} \\ t \in [0, \omega]}} \int_0^\omega \sum_{i=1}^m G_i(t, s) w_{ij}(s) ds,$$

$$(2.8) \quad \beta_* = \min_{\substack{j \in \{1, \dots, m\} \\ t \in [0, \omega]}} \int_0^\omega \sum_{i=1}^m G_i(t, s) w_{ij}(s) ds.$$

Clearly, $0 < \alpha < 1$ and $\beta_* < \beta^*$. Now we have an estimate for $\bar{\lambda}$.

Lemma 2.5 *For any positive eigenvalue $\bar{\lambda}$ of SLP (2.3), (1.2) with a positive eigenfunction, we have that $(\beta^*)^{-1} \leq \bar{\lambda} \leq (\alpha\beta_*)^{-1}$.*

Combining Theorem 2.4 and Lemma 2.5, we obtain the following result immediately.

Corollary 2.6 BVP (1.1), (1.2) has at least one positive solution if either of the following holds:

- (i) $F^0 < (\beta^*)^{-1}$ and $F_\infty > (\alpha\beta_*)^{-1}$;
- (ii) $F^\infty < (\beta^*)^{-1}$ and $F_0 > (\alpha\beta_*)^{-1}$.

Example 1 Consider the BVP

$$(2.9) \quad \begin{aligned} Au'' + Bu &= Wf(u), \\ u(0) = u(1), \quad u'(0) &= u'(1), \end{aligned}$$

where $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$, $W = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}$, and $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ with $f_i(x_1, x_2) = x_i[1 + c(\tan^{-1}(x_1 + x_2) - \pi/4)]$, $i = 1, 2$, for $0 < |c| < 4/\pi$. Then BVP (2.9) has at least one positive solution.

In fact, it is easy to see that $\bar{\lambda} = 1$ is a positive eigenvalue of the following SLP associated with BVP (2.9)

$$\begin{aligned} Au'' + Bu &= \lambda W^T u, \\ u(0) = u(1), \quad u'(0) &= u'(1), \end{aligned}$$

with an eigenfunction $v(t) \equiv (1, 2)^T$, $F_0 = F^0 = 1 - c\pi/4$ and $F_\infty = F^\infty = 1 + c\pi/4$. Thus, $0 < F^\infty < 1 < F_0$ for $-4/\pi < c < 0$ and $0 < F^0 < 1 < F_\infty$ for $0 < c < 4/\pi$. Then the conclusion follows from Theorem 2.4.

Note that in this example, when $c \rightarrow 0$, all $F_0, F^0, F_\infty, F^\infty \rightarrow \bar{\lambda}$. Then Theorem 2.2 in [17] and other existing results in the literature fail to apply.

Example 2 Consider the BVP

$$(2.10) \quad \begin{aligned} -u'' + Bu &= Wf(u), \\ u(0) = u(1), \quad u'(0) &= u'(1), \end{aligned}$$

where $B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$, $W = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$, and $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ with

$$f_i(x_1, x_2) = x_i \left(\left(\frac{(1+e)^2}{4e} - c_1 \right) e^{-x_1-x_2} + c_2 \right), \quad i = 1, 2.$$

Then BVP (2.10) has at least one positive solution when $0 < c_1 < c_2 < 1/4$.

In fact, from (2.2) and by a simple computation, we have $L_* = e/[2(e^2 - 1)]$, $U^* = (1 + e)/[2(e - 1)]$ and $\int_0^1 G_i(t, s) = b_i^{-1}$, $i = 1, 2$. By (2.6), (2.7), (2.8), $\alpha = e/(1+e)^2$ and $\beta^* = \beta_* = 4$. Therefore, $(\beta^*)^{-1} = 1/4$, $(\alpha\beta_*)^{-1} = (1+e)^2/(4e)$. It is clear that $F_0 = (1+e)^2/(4e) - c_1 + c_2$ and $F^\infty = c_2$. Then the conclusion follows from Corollary 2.6 (ii).

Note that in this example, for $\hat{F}_0, \hat{F}^0, \hat{F}_\infty, \hat{F}^\infty$ defined in (1.3), we have $\hat{F}_0 = 0$, $\hat{F}^0 = (1+e)^2/(4e) - c_1 + c_2$, $\hat{F}_\infty = 0$, and $\hat{F}^\infty = c_2$. It is easy to see that [17, Theorem 2.2] fails to apply.

Remark 2.7 Results on the existence of more than one positive solution can also be obtained by combining our Theorem 2.3 and the Theorem 2.1 in [17]. We omit the details.

3 Proofs

The first lemma concerns the Green’s function used to deal with BVP (1.1), (1.2). See [17, Lemma 4.2] for the proof.

Lemma 3.1 *Let $G_i^{[n]}(t, s)$ be defined by (2.4). Then $G_i^{[n]}(t, s)$ is the Green’s function of the BVP*

$$(3.1) \quad (a_i D^2 + b_i)^n u_i = 0, \quad t \in (0, \omega),$$

$$(3.2) \quad u_i^{(k)}(0) = u_i^{(k)}(\omega), \quad k = 0, 1, \dots, 2n - 1.$$

Hence $G = \text{diag}(G_1, \dots, G_m)$ with $G_i = G_i^{[n]}$, $i = 1, \dots, m$, is the Green’s function of the BVP consisting of the equation

$$(3.3) \quad (AD^2 + B)^n u = 0$$

and BC (1.2).

Moreover, $0 < \omega^{n-1} L_i^n \leq G_i(t, s) \leq \omega^{n-1} U_i^n$, where L_i and U_i are defined by (2.1).

Let X be a Banach space and $T: X \rightarrow X$ a linear operator. We recall that μ is an eigenvalue of T with a corresponding eigenfunction u if u is a nontrivial solution of the equation $Tu = \mu u$. The radius of the spectrum of T , denoted by $r(T)$, is given by the well-known spectral formula $r(T) = \lim_{k \rightarrow \infty} \|T^k\|^{1/k}$. Recall also that a cone $P \subset X$ is called total cone if $X = \overline{P - P}$.

We refer the reader to [2, Theorem 19.2] or [29, Proposition 7.26] for the following well-known Krein–Rutman Theorem.

Lemma 3.2 *Assume that P is a total cone in X . Let $T: X \rightarrow X$ be a completely continuous linear operator with $T(P) \subset P$ and $r(T) \in (0, \infty)$. Then $r(T)$ is an eigenvalue of T with an eigenfunction in P .*

In the following, let $X = C([0, \omega], \mathbb{R}^m)$, and for each $u = (u_1, \dots, u_m)^T \in X$, define

$$\|u\| = \max_{t \in [0, \omega]} \sum_{i=1}^m |u_i(t)|.$$

It is known that $(X, \|\cdot\|)$ is a Banach space. Define a cone P in X and an operator $T: X \rightarrow X$ by

$$P = \{u = (u_1, \dots, u_m)^T \in X \mid u_i(t) \geq 0 \text{ on } [0, \omega], i = 1, \dots, m\}$$

$$(Tu)(t) = \int_0^\omega G(t, s) W^T(s) u(s) ds, \quad t \in [0, \omega],$$

where $G(t, s)$ is the Green’s function of BVP (3.3), (1.2).

Clearly, P is a total cone in X . By a standard argument we can show that $T: X \rightarrow X$ is a completely continuous linear operator. We omit the details. Now we are ready to prove Theorem 2.3.

Proof of Theorem 2.3 We observe that λ is an eigenvalue of SLP (2.3), (1.2) if and only if $\mu = 1/\lambda$ is an eigenvalue of the operator T . Moreover, up to a constant multiple, the eigenfunction of SLP (2.3), (1.2) is the same as that of T associated with μ .

Let K be a subcone of P given by

$$K = \left\{ u = (u_1, \dots, u_m)^T \in P \mid \sum_{i=1}^m u_i(t) \geq \alpha \|u\| \text{ on } [0, \omega] \right\}.$$

Then $T(P) \subset K$. In fact, for any $u \in P$ and for $i = 1, \dots, m$, we have $(Tu)_i(t) \geq 0$ on $[0, \omega]$. By (2.5) and (2.6)

$$\begin{aligned} \min_{t \in [0, \omega]} (Tu)_i(t) &= \min_{t \in [0, \omega]} \int_0^\omega G_i(t, s) \sum_{j=1}^m w_{ij}^T(s) u_j(s) ds \\ &\geq L_* \int_0^\omega \sum_{j=1}^m w_{ij}^T(s) u_j(s) ds = \alpha U^* \int_0^\omega \sum_{j=1}^m w_{ij}^T(s) u_j(s) ds \\ &\geq \alpha \max_{t \in [0, \omega]} \int_0^\omega G_i(t, s) \sum_{j=1}^m w_{ij}^T(s) u_j(s) ds = \alpha \max_{t \in [0, \omega]} (Tu)_i(t). \end{aligned}$$

Hence

$$\begin{aligned} \min_{t \in [0, \omega]} \sum_{i=1}^m (Tu)_i(t) &\geq \sum_{i=1}^m \min_{t \in [0, \omega]} (Tu)_i(t) \geq \alpha \sum_{i=1}^m \max_{t \in [0, \omega]} (Tu)_i(t) \\ &\geq \alpha \max_{t \in [0, \omega]} \sum_{i=1}^m (Tu)_i(t) = \alpha \|Tu\|. \end{aligned}$$

Therefore, $T(P) \subset K \subset P$.

To prove Theorem 2.3, by Lemma 3.2 we only need to show that $r(T) \in (0, \infty)$, and hence $r(T)$ is an eigenvalue of T with an eigenfunction in P and then in K . As a result, $\lambda = 1/r(T)$ is such an eigenvalue of SLP (2.3), (1.2). By the spectral theory in Banach spaces (see, for example, [29]), we have $r(T) < \infty$. Now we show that $r(T) > 0$. Let $\underline{w}_i^T(t) = \min_{1 \leq j \leq m} \{w_{ij}^T(t)\} = \min_{1 \leq j \leq m} \{w_{ji}(t)\}$, $t \in [0, \omega]$. From the assumption, $\underline{w}_i^T(t) \not\equiv 0$ on $[0, \omega]$ for $i = 1, \dots, m$. Let $u \in K$, $t \in [0, \omega]$, and $i = 1, \dots, m$. By (2.5)

$$\begin{aligned} (Tu)_i(t) &= \int_0^\omega G_i(t, s) \sum_{j=1}^m w_{ij}^T(s) u_j(s) ds \\ &\geq L_* \int_0^\omega \underline{w}_i^T(s) \sum_{j=1}^m u_j(s) ds \geq \alpha L_* \left(\int_0^\omega \underline{w}_i^T(s) ds \right) \|u\|. \end{aligned}$$

Hence

$$\begin{aligned} (T^2u)_i(t) &= [T(Tu)]_i(t) = \int_0^\omega G_i(t,s) \sum_{j=1}^m w_{ij}^T(s)(Tu)_j(s)ds \\ &\geq L_* \int_0^\omega \underline{w}_i^T(s) \sum_{j=1}^m (Tu)_j(s) ds \\ &\geq \alpha L_*^2 \left(\int_0^\omega \underline{w}_i^T(s) ds \right) \left(\sum_{j=1}^m \int_0^\omega \underline{w}_j^T(s) ds \right) \|u\|. \end{aligned}$$

By induction, for $l = 1, 2, \dots$,

$$(T^l u)_i(t) \geq \alpha L_*^l \left(\int_0^\omega \underline{w}_i^T(s) ds \right) \left(\sum_{j=1}^m \int_0^\omega \underline{w}_j^T(s) ds \right)^{l-1} \|u\|.$$

It follows that

$$\|T^l u\| \geq \sum_{i=1}^m (T^l u)_i(t) \geq \alpha \left(L_* \sum_{i=1}^m \int_0^\omega \underline{w}_i^T(s) ds \right)^l \|u\|.$$

Therefore,

$$\|T^l\| = \max_{u \neq 0} \frac{\|T^l u\|}{\|u\|} \geq \alpha \left(L_* \sum_{i=1}^m \int_0^\omega \underline{w}_i^T(s) ds \right)^l.$$

Consequently,

$$r(T) = \lim_{l \rightarrow \infty} \|T^l\|^{1/l} \geq L_* \sum_{i=1}^m \int_0^\omega \underline{w}_i^T(s) ds > 0. \quad \blacksquare$$

To prove Theorem 2.4, define a map $\Gamma : K \rightarrow X$ as

$$(\Gamma u)(t) = \int_0^\omega G(t,s)W(s)f(s,u(s)) ds, \quad t \in [0, \omega].$$

Clearly, $u(t)$ is a solution of BVP (1.1), (1.2) if and only if u is a fixed point of Γ . It is easy to show Γ is completely continuous and $\Gamma(K) \subset K$. For $r > 0$, define

$$K_r = \{u \in K \mid \|u\| < r\} \quad \text{and} \quad \partial K_r = \{u \in K \mid \|u\| = r\}.$$

Let $i(\Gamma, K_r, K)$ be the fixed point index of Γ on K_r with respect to K . The following is a well-known result on fixed-point index; see [2, 9, 27].

Lemma 3.3 *Assume that $\Gamma : K_r \rightarrow K$ is a completely continuous map such that $\Gamma u \neq u$ for $u \in \partial K_r$.*

- (i) If $\|\Gamma u\| \geq \|u\|$ for $u \in \partial K_r$, then $i(\Gamma, K_r, K) = 0$.
- (ii) If $\|\Gamma u\| \leq \|u\|$ for $u \in \partial K_r$, then $i(\Gamma, K_r, K) = 1$.

Our proof of Theorem 2.4 is based on the next lemma, which is established using Lemma 3.3.

Lemma 3.4 *Let $\bar{\lambda}$ be a positive eigenvalue of SLP (2.3), (1.2) with a positive eigenfunction.*

- (i) If $F_0 > \bar{\lambda}$, then $i(\Gamma, K_r, K) = 0$ for sufficiently small $r > 0$.
- (ii) If $F^0 < \bar{\lambda}$, then $i(\Gamma, K_r, K) = 1$ for sufficiently small $r > 0$.
- (iii) If $F_\infty > \bar{\lambda}$, then $i(\Gamma, K_r, K) = 0$ for sufficiently large $r > 0$.
- (iv) If $F^\infty < \bar{\lambda}$, then $i(\Gamma, K_r, K) = 1$ for sufficiently large $r > 0$.

Proof For the purpose of brevity, we only give proofs for parts (i) and (ii). The proofs for the rest are similar and are hence omitted.

(i) Let $l_1: \mathbb{R}_+^m \rightarrow \mathbb{R}_+^m$ be defined by $l_1(u(t)) = (\|u\|^p, \dots, \|u\|^p)^T$ with $0 < p < 1$. Define $\Gamma_1: X \rightarrow X$ by $(\Gamma_1 u)(t) = \int_0^\omega G(t, s)W(s)l_1(u(s)) ds$. Similar to Γ , Γ_1 is completely continuous and $\Gamma_1(K) \subset K$.

It is clear that $\int_0^\omega \sum_{i,j=1}^m w_{ij}(s) ds > 0$. Let

$$r_1 = \left(L_* \int_0^\omega \sum_{i,j=1}^m w_{ij}(s) ds \right)^{1/(1-p)},$$

where L_* is given by (2.5). Then for any $u \in \partial K_r$ with $r < r_1$ and $t \in [0, \omega]$

$$\begin{aligned} \|\Gamma_1 u\| &\geq \sum_{i=1}^m (\Gamma_1 u)_i(t) = \sum_{i=1}^m \int_0^\omega G_i(t, s) \left(\sum_{j=1}^m w_{ij}(s) \right) r^p ds \\ &\geq r^p L_* \int_0^\omega \sum_{i,j=1}^m w_{ij}(s) ds = r^p r_1^{p-1} > r = \|u\|. \end{aligned}$$

Thus $i(\Gamma_1, K_r, K) = 0$.

Since $F_0 > \bar{\lambda}$, there exists $0 < r_2 \leq r_1$ such that for $x \in \mathbb{R}_+^m$ with $\|x\| < r_2$ and $t \in [0, \omega]$,

$$(3.4) \quad f_i(t, x) > \bar{\lambda}x_i \quad \text{and} \quad \|x\|^p > \bar{\lambda}x_i, \quad i = 1, \dots, m.$$

Define a homotopy operator $H_1: [0, 1] \times K \rightarrow K$ by

$$H_1(\theta, u) = (1 - \theta)\Gamma u + \theta\Gamma_1 u.$$

Then $H_1(\theta, \cdot)$ is completely continuous for $0 \leq \theta \leq 1$. We claim that $H_1(\theta, u) \neq u$ for all $0 \leq \theta \leq 1$ and $u \in \partial K_r$ with $r < r_2$. In fact, assume that there exist $\theta_1 \in [0, 1]$ and $u \in \partial K_r$ with $H_1(\theta_1, u) = u$. Then $u(t)$ satisfies the equation

$$(AD^2 + B)^n u = (1 - \theta_1)W(t)f(t, u) + \theta_1 W(t)l_1(u), \quad t \in (0, \omega),$$

and BC (1.2). Let $v(t)$ be the eigenfunction of SLP (2.3), (1.2) associated with $\bar{\lambda}$. Then

$$(3.5) \quad \int_0^\omega v^T(t)(AD^2 + B)^n u(t) dt = \int_0^\omega v^T(t)[(1 - \theta_1)W(t)f(t, u(t)) + \theta_1 W(t)l_1(u(t))] dt.$$

Integrating by parts $2n$ times and using BC (1.2), we obtain that

$$(3.6) \quad \begin{aligned} \int_0^\omega v^T(t)(AD^2 + B)^n u(t) dt &= \int_0^\omega u^T(t)(AD^2 + B)^n v(t) dt \\ &= \bar{\lambda} \int_0^\omega u^T(t)W^T(t)v(t) dt \\ &= \bar{\lambda} \int_0^\omega v^T(t)W(t)u(t) dt. \end{aligned}$$

Therefore by (3.4), (3.5), and (3.6)

$$\begin{aligned} \bar{\lambda} \int_0^\omega v^T(t)W(t)u(t) dt &= \int_0^\omega v^T(t)[(1 - \theta_1)W(t)f(t, u(t)) + \theta_1 W(s)l_1(u(t))] dt \\ &= \int_0^\omega [(1 - \theta_1)(v^T(t)W(t)f(t, u(t)) + \theta_1 v^T(t)W(t)l_1(u(t)))] dt \\ &> (1 - \theta_1) \int_0^\omega \bar{\lambda} v^T(t)W(t)u(t) dt + \theta_1 \int_0^\omega \bar{\lambda} v^T(t)W(t)u(t) dt \\ &= \bar{\lambda} \int_0^\omega v^T(t)W(t)u(t) dt. \end{aligned}$$

Note that $\int_0^\omega v^T(t)W(t)u(t) dt > 0$ from the assumption of W and due to the fact that $u, v \in K$, we have reached a contradiction. This shows that

$$i(\Gamma, K_r, K) = i(H_1(0, \cdot), K_r, K) = i(H_1(1, \cdot), K_r, K) = i(\Gamma_1, K_r, K) = 0.$$

(ii) Let $l_2: \mathbb{R}_+^m \rightarrow \mathbb{R}_+^m$ be defined by $l_2(u) = (u_1^q, \dots, u_m^q)^T$ with $q > 1$. Define $\Gamma_2: X \rightarrow X$ by

$$(\Gamma_2 u)(t) = \int_0^\omega G(t, s)W(s)l_2(u(s)) ds.$$

Similar to Γ , Γ_2 is completely continuous and $\Gamma_2(K) \subset K$.

Let $r_3 = (U^* \int_0^\omega \max_{1 \leq j \leq m} \sum_{i=1}^m w_{ij}(s) ds)^{1/(1-q)}$, where U^* is given by (2.5). Then for any $u \in \partial K_r$ with $r < r_3$

$$\begin{aligned} \|\Gamma_2 u\| &= \max_{t \in [0, \omega]} \sum_{i=1}^m (\Gamma_2 u)_i(t) = \max_{t \in [0, \omega]} \sum_{i=1}^m \int_0^\omega G_i(t, s) \left(\sum_{j=1}^m w_{ij}(s) u_j^q(s) \right) ds \\ &\leq U^* \int_0^\omega \sum_{j=1}^m \left(\sum_{i=1}^m w_{ij}(s) \right) u_j^q(s) ds \\ &\leq U^* \int_0^\omega \max_{1 \leq j \leq m} \sum_{i=1}^m w_{ij}(s) \left(\sum_{j=1}^m u_j(s) \right)^q ds \\ &\leq r^q U^* \int_0^\omega \max_{1 \leq j \leq m} \sum_{i=1}^m w_{ij}(s) ds = r^q r_3^{q-1} < r = \|u\|. \end{aligned}$$

Therefore, $i(\Gamma_2, K_r, K) = 1$.

Since $F^0 < \bar{\lambda}$, there exists $0 < r_4 \leq r_3$ such that for $x \in \mathbb{R}_+^m$ with $\|x\| < r_4$ and $t \in [0, \omega]$,

$$f_i(t, x) < \bar{\lambda} x_i \text{ and } x_i^q < \bar{\lambda} x_i, \quad i = 1, \dots, m.$$

Define a homotopy operator $H_2: [0, 1] \times K \rightarrow K$ by

$$H_2(\theta, u) = (1 - \theta)\Gamma u + \theta\Gamma_2 u.$$

Then $H_2(\theta, \cdot)$ is completely continuous for $0 \leq \theta \leq 1$. With a similar argument to $H_1(\theta, \cdot)$ we can show that $H_2(\theta, u) \neq u$ for all $0 \leq \theta \leq 1$ and $u \in \partial K_r$ with $r < r_4$. We omit the details. This shows that

$$i(\Gamma, K_r, K) = i(H_2(0, \cdot), K_r, K) = i(H_2(1, \cdot), K_r, K) = i(\Gamma_2, K_r, K) = 1. \quad \blacksquare$$

Proof of Theorem 2.4 Assume $F^0 < \bar{\lambda} < F_\infty$. Then by Lemma 3.4, (ii) and (iii), there exist $0 < \bar{r}_1 < \bar{r}_2 < \infty$ such that $(\Gamma, K_{\bar{r}_1}, K) = 1$ and $(\Gamma, K_{\bar{r}_2}, K) = 0$. Hence Γ has a fixed point $u \in K_{\bar{r}_2} \setminus \bar{K}_{\bar{r}_1}$. Therefore, $u(t)$ is a positive solution of the BVP (1.1), (1.2).

Assume $F^\infty < \bar{\lambda} < F_0$. Then by Lemma 3.4, (i) and (iv), there exist $0 < \bar{r}_3 < \bar{r}_4 < \infty$ such that $(\Gamma, K_{\bar{r}_3}, K) = 0$ and $(\Gamma, K_{\bar{r}_4}, K) = 1$. Hence Γ has a fixed point $u \in K_{\bar{r}_4} \setminus \bar{K}_{\bar{r}_3}$. Therefore, $u(t)$ is a positive solution of the BVP (1.1), (1.2). \blacksquare

Proof of Lemma 2.5 For the eigenfunction $v(t) = (v_1, \dots, v_m)^T(t)$ of SLP (2.3), (1.2) associated with $\bar{\lambda}$, we have

$$v(t) = \bar{\lambda} \int_0^\omega G(t, s) W^T(s) v(s) ds, \quad t \in [0, \omega].$$

Thus for any $t \in [0, \omega]$

$$\begin{aligned}
 (3.7) \quad \sum_{i=1}^m v_i(t) &= \bar{\lambda} \sum_{i=1}^m \int_0^\omega G_i(t, s) \sum_{j=1}^m w_{ij}^T(s) v_j(s) ds \\
 &= \bar{\lambda} \sum_{j=1}^m \int_0^\omega \left(\sum_{i=1}^m G_i(t, s) w_{ij}^T(s) \right) v_j(s) ds \\
 &\leq \bar{\lambda} \int_0^\omega \left(\max_{\substack{j \in \{1, \dots, m\} \\ t \in [0, \omega]}} \sum_{i=1}^m G_i(t, s) w_{ij}^T(s) \right) \left(\sum_{j=1}^m v_j(s) \right) ds \\
 &\leq \bar{\lambda} \beta^* \|v\|,
 \end{aligned}$$

which implies that $\bar{\lambda} \geq (\beta^*)^{-1}$.

Note that $v \in K$; we have that $\sum_{i=1}^m v_i(t) \geq \alpha \|v\|$ on $[0, \omega]$. Then from (3.7) we see that for $t \in [0, \omega]$

$$\begin{aligned}
 \sum_{i=1}^m v_i(t) &\geq \bar{\lambda} \int_0^\omega \left(\min_{\substack{j \in \{1, \dots, m\} \\ t \in [0, \omega]}} \sum_{i=1}^m G_i(t, s) w_{ij}^T(s) \right) \left(\sum_{j=1}^m v_j(s) \right) ds \\
 &\geq \bar{\lambda} \beta_* \alpha \|v\|,
 \end{aligned}$$

which implies that $\bar{\lambda} \leq (\alpha \beta_*)^{-1}$. ■

4 Generalized System of Periodic BVPs

In the last section we consider the BVP consisting of the equation

$$(4.1) \quad (AD^2 + \tilde{B})^n y = \tilde{W}(t) \tilde{f}(t, y), \quad t \in (0, \omega),$$

and the BC

$$(4.2) \quad y^{(k)}(0) = y^{(k)}(\omega), \quad k = 0, 1, \dots, 2n - 1.$$

Here we assume the following

- (i) $A = \pm I$;
- (ii) \tilde{B} is a positive definite $m \times m$ matrix, i.e., there exists an $m \times m$ invertible matrix S such that $S^{-1} \tilde{B} S = \text{diag}(b_1, \dots, b_m)$ with $b_i > 0, i = 1, \dots, m$;
- (iii) $S^{-1} \tilde{W} S \in C([0, \omega], \mathbb{R}_+^{m \times m})$;
- (iv) $S^{-1} \tilde{f} = S^{-1}(\tilde{f}_1, \dots, \tilde{f}_m)^T \in C([0, \omega] \times S(\mathbb{R}_+^m), \mathbb{R}_+^m)$, where

$$S(\mathbb{R}_+^m) = \{x \in \mathbb{R}_+^m \mid Sx \in \mathbb{R}_+^m\}.$$

Let $B = S^{-1} \tilde{B} S$, $W = S^{-1} \tilde{W} S$, and $f(t, u) = S^{-1} \tilde{f}(t, Su)$. Then the corresponding BVP (1.1), (1.2) is called the transformed problem of BVP (4.1), (4.2). By a simple computation, we can obtain the result below.

Lemma 4.1 y is a solution of BVP (4.1), (4.2) if and only if $u = S^{-1}y$ is a solution of the transformed problem (1.1), (1.2).

In addition to the conditions (i)–(iv), with $W = S^{-1}\tilde{W}S = (w_{ij})_{m \times m}$ we further assume that $\min_{1 \leq i \leq m} w_{ij}(t) \not\equiv 0$ on $[0, \omega]$ for all $j = 1, \dots, m$. Then all assumptions for BVP (1.1), (1.2) are satisfied by the transformed problem of BVP (4.1), (1.2). Let (H1) and (H2) hold for the transformed problem of BVP (4.1), (4.2). We use the following notation for limits where $S^{-1}x$ is restricted to \mathbb{R}_+^m :

$$\begin{aligned}\tilde{F}_0 &= \liminf_{\|x\| \rightarrow 0} \min_{\substack{i \in \{1, \dots, m\} \\ t \in [0, \omega]}} (S^{-1}\tilde{f})_i(t, x)/(S^{-1}x)_i, \\ \tilde{F}^0 &= \limsup_{\|x\| \rightarrow 0} \max_{\substack{i \in \{1, \dots, m\} \\ t \in [0, \omega]}} (S^{-1}\tilde{f})_i(t, x)/(S^{-1}x)_i, \\ \tilde{F}_\infty &= \liminf_{\|x\| \rightarrow \infty} \min_{\substack{i \in \{1, \dots, m\} \\ t \in [0, \omega]}} (S^{-1}\tilde{f})_i(t, x)/(S^{-1}x)_i, \\ \tilde{F}^\infty &= \limsup_{\|x\| \rightarrow \infty} \max_{\substack{i \in \{1, \dots, m\} \\ t \in [0, \omega]}} (S^{-1}\tilde{f})_i(t, x)/(S^{-1}x)_i.\end{aligned}$$

Applying the results in Section 2 to the transformed problem, we obtain results on existence of a nontrivial solution of BVP (4.1), (4.2) using a positive eigenvalue of the corresponding transformed SLP (2.3), (1.2).

Theorem 4.2 Let $\bar{\lambda}$ be a positive eigenvalue of the transformed SLP (2.3), (1.2) with a positive eigenfunction. Then BVP (4.1), (4.2) has at least one nontrivial solution if either $\tilde{F}^0 < \bar{\lambda} < \tilde{F}_\infty$ or $\tilde{F}^\infty < \bar{\lambda} < \tilde{F}_0$.

Corollary 2.6 can also be extended to BVP (3.1), (3.2) in the same way. We omit the details.

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