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RESEARCH ARTICLE

Resolving an open problem on the hazard rate ordering of p-spacings

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Abstract

Let $V^{(p)}_{(r,n,\tilde{m}_n,k)}$ and $W^{(p)}_{(r,n,\tilde{m}_n,k)}$ be the p-spacings of generalized order statistics based on absolutely continuous distribution functions F and G, respectively. Imposing some conditions on F and G and assuming that $m_1 = \cdots = m_{n-1}$, Hu and Zhuang (2006. Stochastic orderings between p-spacings of generalized order statistics from two samples. *Probability in the Engineering and Informational Sciences* 20: 475) established $V^{(p)}_{(r,n,\tilde{m}_n,k)} \leq_{\text{hr}} W^{(p)}_{(r,n,\tilde{m}_n,k)}$ for $p = 1$ and left the case $p \geq 2$ as an open problem. In this article, we not only resolve it but also give the result for unequal m_i's. It is worth mentioning that this problem has not been proved even for ordinary order statistics so far.

1. Introduction

In the last two decades, a great attention has been put on stochastic orderings of order statistics and other ordered random variables. In another direction, such random variables can be embedded in the concept of generalized order statistics (GOS), introduced by Kamps [16], as a general framework for models of ordered random variables. Since then, the researchers attempt to obtain their results for ordered data into the model of GOS.

Let X and Y be two nonnegative variables with absolutely continuous cumulative distribution functions (cdfs) F and G, survival functions (sfs) $\bar{F} = 1-F$ and $\bar{G} = 1-G$, and probability density functions (pdfs) f and g, respectively, in which $F^{-1}(0) = G^{-1}(0)$ (F^{-1} is the right continuous inverse of F). Let $h_X = f/\bar{F}$ and $h_Y = g/\bar{G}$ denote the hazard rate functions of X and Y, respectively.

The random variables $X_{(r,n,\tilde{m}_n,k)}, r = 1, 2, \ldots, n$, arising from independent and identically distributed random variables, are referred to as GOS if their joint density function is given by

$$\mathbf{f}(x_1, \ldots, x_n) = k\left(\prod_{j=1}^{n-1} \gamma_{(j,n,\tilde{m}_n,k)}\right)\left(\prod_{i=1}^{n-1}[\bar{F}(x_i)]^{m_i} f(x_i)\right)[\bar{F}(x_n)]^{k-1} f(x_n),$$

for all $F^{-1}(0) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1-)$, where $n \in \mathbb{N}$, $k > 0$, and $m_1, \ldots, m_{n-1} \in \mathbb{R}$ are such that $\gamma_{(i,n,\tilde{m}_n,k)} = k + n - i + \sum_{j=i}^{n-1} m_j \geq 1$ for all $i \in \{1, \ldots, n-1\}$, and $\tilde{m}_n = (m_1, \ldots, m_{n-1})$ if $n \geq 2$ ($\tilde{m}_n \in \mathbb{R}$ is arbitrary if $n = 1$). Indeed, special choices of parameters k and m_i correspond to some well-known submodels such as order statistics, record values, and sequential order statistics. We refer the readers to Table 1 of Kamps [17] for complete information on various submodels.

Throughout this paper, we shall use the word increasing (decreasing) for nondecreasing (nonincreasing). Furthermore, ratios are supposed to be well defined whenever they are used. We say that X is smaller than Y in the

- likelihood ratio order (denoted by $X \leq_{\mathrm{lr}} Y$) if $g(x)/f(x)$ is increasing in x;
- hazard rate order (denoted by $X \leq_{\mathrm{hr}} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x or, equivalently, $h_Y(x) \leq h_X(x)$.

It is well-known that $X \leq_{\mathrm{lr}} Y \Rightarrow X \leq_{\mathrm{hr}} Y$ (cf. [23]). We say that X is DFR (decreasing failure rate) if $\bar{F}$ is logconvex in $x \in \mathbb{R}_+$ or, equivalently, $\bar{F}(x+\epsilon)/\bar{F}(x)$ is an increasing function of x for each fixed $\epsilon > 0$.

Let $Y_{(r,n,\tilde{m}_n,k)}$, $r = 1, \ldots, n$, be GOS based on G. We denote the p-spacings of GOS from F and G by $V^{(p)}_{(r,n,\tilde{m}_n,k)} = X_{(r+p-1,n,\tilde{m}_n,k)} - X_{(r-1,n,\tilde{m}_n,k)}$ and $W^{(p)}_{(r,n,\tilde{m}_n,k)} = Y_{(r+p-1,n,\tilde{m}_n,k)} - Y_{(r-1,n,\tilde{m}_n,k)}$, for $2 \leq r \leq n-p+1$ and $p \geq 1$, respectively.

For one-sample problem of p-spacings, some stochastic properties of order statistics and GOS have been studied by several authors such as Misra and van der Meulen [21], Hu and Zhuang [13,14], Xie and Hu [25], and Alimohammadi *et al.* [4]. For two-sample problem of p-spacings, a study of stochastic comparisons was initiated by Kochar [20]. He investigated the usual stochastic, hazard rate, and likelihood ratio orderings of order statistics for $p = 1$. Franco *et al.* [12] studied the usual stochastic, hazard rate, and dispersive orderings of GOS under the condition $m_1 = m_2 = \cdots = m_{n-1}$. Then, without this condition, Belzunce *et al.* [7] gave the usual stochastic and likelihood ratio orderings of GOS for $p = 1$. Finally, in an interesting article, Hu and Zhuang [15] established the likelihood ratio ordering of GOS for any $p \geq 1$ and the hazard rate ordering of GOS for $p = 1$ under the condition $m_1 = m_2 = \cdots = m_{n-1}$ and left the case $p \geq 2$ as an open problem. In particular, they proved

$$V^{(1)}_{(r,n,\tilde{m}_n,k)} \leq_{\mathrm{hr}} W^{(1)}_{(r,n,\tilde{m}_n,k)}$$

provided that any one of the following conditions is satisfied:

i. $m_i \geq 0 \; \forall i$, $X \leq_{\mathrm{lr}} Y$, and X or Y is DFR;
ii. $-1 \leq m_i < 0 \; \forall i$, $X \leq_{\mathrm{hr}} Y$, $h_Y(x)/h_X(x)$ is increasing in x, and X or Y is DFR;

and said that whether this ordering holds for $p \geq 2$ under the same assumptions as those of above. In this article, we first answer to this open problem in the affirmative without the condition $m_1 = m_2 = \cdots = m_{n-1}$. Finally, the applications of this result are demonstrated for sequential systems, progressive Type-II censored order statistics with arbitrary censoring schemes and record values.

2. Preliminaries

There exist several representations for the marginal density functions of GOS (see, e.g., [10,16]). Cramer *et al.* [11] obtained the expression

$$f_{X_{(r,n,\tilde{m}_n,k)}}(x) = c_{r-1}[\bar{F}(x)]^{\gamma_r - 1}\xi_r(F(x))f(x), \quad x \in \mathbb{R}, \tag{1}$$

where $c_{r-1} = \prod_{i=1}^{r}\gamma_i$, $r = 1, \ldots, n$, $\gamma_n = k$, and ξ_r is a particular Meijer's G-function. For the joint pdf of $X_{(r,n,\tilde{m}_n,k)}$ and $X_{(s,n,\tilde{m}_n,k)}$, $1 \leq r < s \leq n$, Tavangar and Asadi [24] established the expression

$$\begin{aligned} f_{X_{(r,n,\tilde{m}_n,k)},X_{(s,n,\tilde{m}_n,k)}}(x_1,x_2) &= c_{s-1}[\bar{F}(x_1)]^{\gamma_r-\gamma_s-1}\xi_r(F(x_1)) \\ &\quad \times [\bar{F}(x_2)]^{\gamma_s-1}\psi_{s-r-1}\left(\frac{\bar{F}(x_2)}{\bar{F}(x_1)}\right)f(x_1)f(x_2), \quad x_1 < x_2, \end{aligned} \tag{2}$$

(zero elsewhere), where $\psi_0(t) = 1$, $\psi_1(t) = \delta_{m_{r+1}}(1-t)$,

$$\psi_\alpha(t) = \int_t^1 \int_{u_{\alpha-1}}^1 \cdots \int_{u_2}^1 \delta_{m_{r+1}}(1-u_1)\prod_{i=1}^{\alpha-1} u_i^{m_{r+i+1}}\, du_1 \ldots du_{\alpha-2}\, du_{\alpha-1}, \quad 0 \leq t \leq 1,\ \alpha = 2, 3, \ldots,$$

and

$$\delta_m(t) = \begin{cases} \frac{1}{m+1}(1-(1-t)^{m+1}), & m \neq -1 \\ -\ln(1-t), & m = -1 \end{cases}, \quad t \in (0,1).$$

According to Lemmas 2.1 and 3.1 of Alimohammadi and Alamatsaz [1], we have the following recursive formulas:

$$\xi_1(t) = 1, \quad \xi_r(t) = \int_0^t \xi_{r-1}(u)[1-u]^{m_{r-1}}\, du, \quad 0 \le t \le 1,\ r = 2, \dots, n, \tag{3}$$

and

$$\psi_0(t) = 1, \quad \psi_\alpha(t) = \int_t^1 \psi_{\alpha-1}(u)u^{m_{r+\alpha}}\, du, \quad 0 \le t \le 1,\ \alpha = 1, 2, \dots . \tag{4}$$

Some convexity properties of the function ξ_r and GOS have been studied by Cramer *et al.* [11] and Alimohammadi *et al.* [2,3].

Now, substituting r with $r-1$ and s with $r+p-1$ in (2) and after some calculations, for $2 \le r \le n-p+1$, we obtain

$$\begin{aligned} f_{V^{(p)}_{(r,n,\tilde{m}_n,k)}}(x) = c_{r+p-2} \int_0^{+\infty} & [\bar{F}(x+u)]^{\gamma_{r+p-1}-1} \psi_{p-1}\left(\frac{\bar{F}(x+u)}{\bar{F}(u)}\right) f(x+u) \\ & \times [\bar{F}(u)]^{\gamma_{r-1}-\gamma_{r+p-1}-1} \xi_{r-1}(F(u)) f(u)\, du, \quad x \ge 0, \end{aligned} \tag{5}$$

where, according to (4) for $r-1$,

$$\psi_{p-1}\left(\frac{\bar{F}(x+u)}{\bar{F}(u)}\right) = \int_{\bar{F}(x+u)/\bar{F}(u)}^1 \psi_{p-2}(u)u^{m_{r+p-2}}\, du, \quad 2 \le p \le n-r+1, \tag{6}$$

with $\psi_0(t) = 1$ and, for $r = 1$, we have $f_{V^{(p)}_{(1,n,\tilde{m}_n,k)}}(x) = f_{X_{(r+p-1,n,\tilde{m}_n,k)}}(x)$. Also, for $2 \le r \le n-p+1$, from (5) we arrive at

$$\begin{aligned} \bar{F}_{V^{(p)}_{(r,n,\tilde{m}_n,k)}}(x) = c_{r+p-2} \int_0^{+\infty} & [\bar{F}(x+u)]^{\gamma_{r+p-1}} \left[\int_0^1 z^{\gamma_{r+p-1}-1} \psi_{p-1}\left(z \cdot \left(\frac{\bar{F}(x+u)}{\bar{F}(u)}\right)\right) dz\right] \\ & \times [\bar{F}(u)]^{\gamma_{r-1}-\gamma_{r+p-1}-1} \xi_{r-1}(F(u)) f(u)\, du, \quad x \ge 0. \end{aligned} \tag{7}$$

Now, we recall the following definition about the very useful concept of total positivity (cf. [19]).

Definition 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of the real line $\mathbb{R}$. A function $\lambda : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is said to be totally positive of order 2 (TP_2) (reverse regular of order 2 (RR_2)) if*

$$\lambda(x_1, y_1)\lambda(x_2, y_2) - \lambda(x_1, y_2)\lambda(x_2, y_1) \ge (\le) 0, \tag{8}$$

for all $x_1 \le x_2$ in $\mathcal{X}$ and all $y_1 \le y_2$ in $\mathcal{Y}$.

Note that the TP_2 (RR_2) property is equivalent to $\lambda(x_2, y)/\lambda(x_1, y)$ is increasing (decreasing) in y when $x_1 \le x_2$, whenever this ratio exists. Also, note that the product of two TP_2 (RR_2) functions is TP_2 (RR_2). Moreover, if $\lambda(x, y)$ is TP_2 (RR_2) in (x, y), then $\lambda_1(x)\lambda(x, y)\lambda_2(y)$ is TP_2 (RR_2) in (x, y) when λ_1 and λ_2 are two nonnegative functions (cf. [19]).

The lemma below, due to Misra and van der Meulen [21], is often used in establishing the monotonicity of a fraction in which the numerator and denominator are integrals or summations.

Lemma 2.2. *Assume that Θ is a subset of the real line $\mathbb{R}$, and let U be a nonnegative random variable having a cdf belonging to the family $\mathcal{P} = \{\Xi(\cdot \mid \theta), \theta \in \Theta\}$ which satisfies that, for $\theta_1, \theta_2 \in \Theta$,*

$$\Xi(\cdot \mid \theta_1) \leq_{\mathrm{st}} (\geq_{\mathrm{st}}) \Xi(\cdot \mid \theta_2), \quad \text{whenever } \theta_1 \leq \theta_2.$$

Let $\phi(u, \theta)$ be a real-valued function defined on $\mathbb{R} \times \Theta$, which is measurable in u for each θ such that $E_\theta[\phi(U, \theta)]$ exists. Then, $E_\theta[\phi(U, \theta)]$ is

(i) increasing in θ, if $\phi(u, \theta)$ is increasing in θ and increasing (decreasing) in u;
(ii) decreasing in θ, if $\phi(u, \theta)$ is decreasing in θ and decreasing (increasing) in u.

3. Main result

Now, we are ready to resolve the mentioned problem.

Theorem 3.1. *Let $X_{(r,n,\tilde{m}_n,k)}$ and $Y_{(r,n,\tilde{m}_n,k)}$, $r = 1, \ldots, n$, be GOSs based on absolutely continuous cdfs F and G, respectively. Then, for all $2 \leq r \leq n - p + 1$ and all $p \geq 1$,*

$$V^{(p)}_{(r,n,\tilde{m}_n,k)} \leq_{\mathrm{hr}} W^{(p)}_{(r,n,\tilde{m}_n,k)}$$

provided that any one of the following conditions is satisfied:

(i) $m_i \geq 0\ \forall i$, $X \leq_{\mathrm{lr}} Y$, and X or Y is DFR;
(ii) $-1 \leq m_i < 0\ \forall i$, $X \leq_{\mathrm{hr}} Y$, $h_Y(x)/h_X(x)$ is increasing in x, and X or Y is DFR.

Proof. First note that by changing variable $z = \bar{F}(t)/\bar{F}(x+u)$ in (7), we have

$$\begin{aligned}\bar{F}_{V^{(p)}_{(r,n,\tilde{m}_n,k)}}(x) = c_{r+p-2} \int_0^{+\infty} &\left[\int_{x+u}^{+\infty} [\bar{F}(t)]^{\gamma_{r+p-1}-1} \psi_{p-1}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right) f(t)\, dt \right] \\ &\times [\bar{F}(u)]^{\gamma_{r-1}-\gamma_{r+p-1}-1} \xi_{r-1}(F(u)) f(u)\, du. \end{aligned} \tag{9}$$

We give the proof for the case $p \geq 2$ while the case $p = 1$ can be proved in an analogous and simpler manner (because some terms will be vanished for $p = 1$). Assume that X is DFR. Let's define

$$\phi_2(t, x, u) = \left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{\gamma_{r+p-1}-1} \frac{\psi_{p-1}\left(\frac{\bar{G}(t)}{\bar{G}(u)}\right)}{\psi_{p-1}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)} \frac{g(u)}{f(u)}.$$

From (9), we have

$$\frac{\bar{F}_{W^{(p)}_{(r,n,\tilde{m}_n,k)}}(x)}{\bar{F}_{V^{(p)}_{(r,n,\tilde{m}_n,k)}}(x)} = E[\phi_1(U, x)],$$

where

$$\begin{aligned}\phi_1(u, x) &= E[\phi_2(T, x, u)] \left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{\gamma_{r-1}-\gamma_{r+p-1}-1} \frac{\xi_{r-1}(G(u))}{\xi_{r-1}(F(u))} \frac{g(u)}{f(u)} \\ &= \left(\left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{m_{r-1}} \frac{g(u)}{f(u)}\right) \left(\frac{\xi_{r-1}(G(u))}{\xi_{r-1}(F(u))}\right) \left(E[\phi_2(T, x, u)] \cdot \left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{\sum_{j=r}^{r+p-2}(m_j+1)}\right)\end{aligned} \tag{10}$$

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$$= \left(\left[\frac{\bar{G}(u)}{\bar{F}(u)} \right]^{m_{r-1}+1} \frac{h_Y(u)}{h_X(u)} \right) \left(\frac{\xi_{r-1}(G(u))}{\xi_{r-1}(F(u))} \right) \left(E[\phi_2(T, x, u)] \cdot \left[\frac{\bar{G}(u)}{\bar{F}(u)} \right]^{\sum_{j=r}^{r+p-2}(m_j+1)} \right), \tag{11}$$

U and T are nonnegative random variables having the respective cdfs belonging to the families $\mathcal{P}_1 = \{\mathcal{U}(\cdot \mid x), x \in \mathbb{R}_+\}$ and $\mathcal{P}_2 = \{\mathcal{T}(\cdot \mid x, u), x, u \in \mathbb{R}_+\}$ with the respective pdfs

$$\begin{aligned} l_1(u \mid x) = c_1(x) I_{\{0 \le u\}} &\left[\int_{x+u}^{+\infty} [\bar{F}(t)]^{\gamma_{r+p-1}-1} \psi_{p-1}\left(\frac{\bar{F}(t)}{\bar{F}(u)} \right) f(t)\, dt \right] \\ &\times [\bar{F}(u)]^{\gamma_{r-1}-\gamma_{r+p-1}-1} \xi_{r-1}(F(u)) f(u) \end{aligned}$$

and

$$l_2(t \mid x, u) = c_2(x, u) I_{\{x+u \le t\}} [\bar{F}(t)]^{\gamma_{r+p-1}-1} \psi_{p-1}\left(\frac{\bar{F}(t)}{\bar{F}(u)} \right) f(t),$$

in which

$$c_1(x) = \left[\int_0^{+\infty} \left[\int_{x+u}^{+\infty} [\bar{F}(t)]^{\gamma_{r+p-1}-1} \psi_{p-1}\left(\frac{\bar{F}(t)}{\bar{F}(u)} \right) f(t)\, dt \right] [\bar{F}(u)]^{\gamma_{r-1}-\gamma_{r+p-1}-1} \xi_{r-1}(F(u)) f(u)\, du \right]^{-1}$$

and

$$c_2(x, u) = \left[\int_{x+u}^{+\infty} [\bar{F}(t)]^{\gamma_{r+p-1}-1} \psi_{p-1}\left(\frac{\bar{F}(t)}{\bar{F}(u)} \right) f(t)\, dt \right]^{-1}$$

are the normalizing constants and I_A is the indicator function.

Now, we show that $\phi_1(u, x)$ is increasing in u and x in either (10) and (11).

The first parentheses is increasing in u: It is obvious according to the conditions (i) and (ii) of theorem.

The second parentheses is increasing in u: We prove it by induction on r. It is clearly valid for $r = 2$. For $r \ge 3$, let us assume that $\xi_{j-1}(G(u))/\xi_{j-1}(F(u))$ is increasing in u for $j = 3, \ldots, r-1$. According to (3), we then have

$$\begin{aligned} &\frac{d}{du}\left(\frac{\xi_{r-1}(G(u))}{\xi_{r-1}(F(u))} \right) \ge 0 \\ \iff\ & g(u)\xi_{r-2}(G(u))(\bar{G}(u))^{m_{r-2}} \cdot \xi_{r-1}(F(u)) \ge f(u)\xi_{r-2}(F(u))(\bar{F}(u))^{m_{r-2}} \cdot \xi_{r-1}(G(u)) \\ \iff\ & \frac{g(u)}{f(u)} \frac{\xi_{r-2}(G(u))}{\xi_{r-2}(F(u))} \left(\frac{\bar{G}(u)}{\bar{F}(u)} \right)^{m_{r-2}} \ge \frac{\xi_{r-1}(G(u))}{\xi_{r-1}(F(u))}. \end{aligned} \tag{12}$$

Let us define

$$\nu_1(u) = \int_u^1 \xi_{r-2}(G(z))(\bar{G}(z))^{m_{r-2}} g(z)\, dz, \quad \nu_2(u) = \int_u^1 \xi_{r-2}(F(z))(\bar{F}(z))^{m_{r-2}} f(z)\, dz.$$

Then, we have the right-hand side of (12) to be

$$\begin{aligned} \frac{\xi_{r-1}(G(u))}{\xi_{r-1}(F(u))} &= \frac{\int_{F^{-1}(0)}^1 \xi_{r-2}(G(z))(\bar{G}(z))^{m_{r-2}} g(z)\, dz - \int_u^1 \xi_{r-2}(G(z))(\bar{G}(z))^{m_{r-2}} g(z)\, dz}{\int_{F^{-1}(0)}^1 \xi_{r-2}(F(z))(\bar{F}(z))^{m_{r-2}} f(z)\, dz - \int_u^1 \xi_{r-2}(F(z))(\bar{F}(z))^{m_{r-2}} f(z)\, dz} \\ &= \frac{\nu_1(u) - \nu_1(F^{-1}(0))}{\nu_2(u) - \nu_2(F^{-1}(0))}. \end{aligned}$$

Because of the integral form of $\nu_1(x)$ and $\nu_2(x)$, they are continuous on $x \in [F^{-1}(0), u]$ and differentiable on $x \in (F^{-1}(0), u)$. Also, we have $\nu_2'(x) \ne 0$ for all $x \in (F^{-1}(0), u)$. Because we consider the pdfs on

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their support and if X is DFR, then its support is $(F^{-1}(0), \infty)$ with finite $F^{-1}(0)$ (cf. [8]) and, thus, we have $(F^{-1}(0), u) \subset (F^{-1}(0), \infty)$. So, according to Cauchy's mean value theorem, there exists some $\theta \in (F^{-1}(0), u)$ such that

$$\frac{\nu_1(u) - \nu_1(F^{-1}(0))}{\nu_2(u) - \nu_2(F^{-1}(0))} = \frac{\nu_1'(\theta)}{\nu_2'(\theta)}.$$

Also, we have

$$\frac{\nu_1'(\theta)}{\nu_2'(\theta)} = \frac{g(\theta)}{f(\theta)} \frac{\xi_{r-2}(G(\theta))}{\xi_{r-2}(F(\theta))} \left(\frac{\bar{G}(\theta)}{\bar{F}(\theta)}\right)^{m_{r-2}} = \frac{h_Y(\theta)}{h_X(\theta)} \frac{\xi_{r-2}(G(\theta))}{\xi_{r-2}(F(\theta))} \left(\frac{\bar{G}(\theta)}{\bar{F}(\theta)}\right)^{m_{r-2}+1}.$$

Now, as $\theta \leq u$ and according to the conditions (i) and (ii) of theorem, respectively $(g(u)/f(u))(c\bar{G}(u)/\bar{F}(u))^{m_{r-2}}$ and $(h_Y(u)/h_X(u))(\bar{G}(u)/\bar{F}(u))^{m_{r-2}+1}$ is increasing in u, the right-hand side of (12) becomes less than or equal to the left-hand side by induction.

The third parentheses is increasing in u: We first prove that

$$\frac{\psi_{p-1}\left(\frac{\bar{G}(t)}{\bar{G}(u)}\right)}{\psi_{p-1}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)} \cdot \left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{\sum_{j=r}^{r+p-2}(m_j+1)} \tag{13}$$

is increasing in u by induction on p. For $p = 2$, from (6) we have

$$\frac{\psi_1\left(\frac{\bar{G}(t)}{\bar{G}(u)}\right)}{\psi_1\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)} \cdot \left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{m_r+1} = \frac{\int_u^t (\bar{G}(z))^{m_r} g(z)\, dz}{\int_u^t (\bar{F}(z))^{m_r} f(z)\, dz} = E[\phi_3(Z, t, u)],$$

where

$$\begin{aligned} \phi_3(z, t, u) &= \left[\frac{\bar{G}(z)}{\bar{F}(z)}\right]^{m_r} \frac{g(z)}{f(z)} \\ &= \left[\frac{\bar{G}(z)}{\bar{F}(z)}\right]^{m_r+1} \frac{h_Y(z)}{h_X(z)}, \end{aligned}$$

and Z is a nonnegative random variable having the cdf belonging to the family $\mathcal{P}_3 = \{\mathcal{Z}(\cdot \mid t, u), t, u \in \mathbb{R}_+\}$ with the pdf

$$l_3(z \mid t, u) = c_3(t, u) I_{\{u \leq z \leq t\}} [\bar{F}(z)]^{m_r} f(z),$$

in which $c_3(t, u)$ is the normalizing constant. According to the conditions (i) and (ii) of theorem, $\phi_3(z, t, u)$ is increasing in z. Also, it is constant with respect to u. Since, $I_{\{u \leq z \leq t\}}$ is TP_2 in (z, u), we have $\mathcal{Z}(\cdot \mid t, u_1) \leq_{\text{lr}} \mathcal{Z}(\cdot \mid t, u_2)$ for $u_1 \leq u_2$. Now, part (i) of Lemma 2.2 implies that $E[\phi_3(Z, t, u)]$ is increasing in u. By the same manner and according to

$$\begin{aligned} &\frac{\psi_{p-1}\left(\frac{\bar{G}(t)}{\bar{G}(u)}\right)}{\psi_{p-1}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)} \cdot \left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{\sum_{j=r}^{r+p-2}(m_j+1)} \\ &\quad = \frac{\int_u^t \psi_{p-2}\left(\frac{\bar{G}(z)}{\bar{G}(u)}\right) (\bar{G}(u))^{\sum_{j=r}^{r+p-3}(m_j+1)} (\bar{G}(z))^{m_{r+p-2}} g(z)\, dz}{\int_u^t \psi_{p-2}\left(\frac{\bar{F}(z)}{\bar{F}(u)}\right) (\bar{F}(u))^{\sum_{j=r}^{r+p-3}(m_j+1)} (\bar{F}(z))^{m_{r+p-2}} f(z)\, dz}, \end{aligned}$$

the term in (13) is increasing in u by induction.

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Then, since X is DFR and $I_{\{x+u\leq t\}}$ is TP_2 in (t,u), one can similarly see that $\mathcal{T}(\cdot\,|\,x,u_1) \leq_{\text{lr}} \mathcal{T}(\cdot\,|\,x,u_2)$ for $u_1 \leq u_2$. Thus, part (i) of Lemma 2.2 implies that the third parentheses is increasing in u.

The third parentheses is increasing in x: Similar to the previous step, we can show that $\psi_{p-1}(\bar{G}(t)/\bar{G}(u))/\psi_{p-1}(\bar{F}(t)/\bar{F}(u))$ is increasing in t and $\mathcal{T}(\cdot\,|\,x_1,u) \leq_{\text{lr}} \mathcal{T}(\cdot\,|\,x_2,u)$ for $x_1 \leq x_2$. Again, part (i) of Lemma 2.2 implies that $E[\phi_2(T,x,u)]$ is increasing in x.

Finally, since $\mathcal{U}(\cdot\,|\,x_1) \leq_{\text{lr}} \mathcal{U}(\cdot\,|\,x_2)$ for $x_1 \leq x_2$, part (i) of Lemma 2.2 implies that $E[\phi_1(U,x)]$ is increasing in x.

For the case that Y is DFR, analogously one can see that $\bar{F}_{V^{(p)}_{(r,n,\tilde{m}_n,k)}}(x)/\bar{F}_{W^{(p)}_{(r,n,\tilde{m}_n,k)}}(x)$ is decreasing in x by part (ii) of Lemma 2.2. Therefore, the proof is completed. □ □

It is worthwhile mentioning that there are two crucial points in resolving this problem. The first one is the choosing an appropriate change of variable in the structure of $\bar{F}_{V^{(p)}_{(r,n,\tilde{m}_n,k)}}(x)$. The second one is the noting that $\psi_{p-1}(\bar{G}(t)/\bar{G}(u))/\psi_{p-1}(\bar{F}(t)/\bar{F}(u))$ is not increasing in u on its own (while it is increasing in t), and thus, it is needed to borrow some increasing terms from the other parts to make it increasing.

Remark 3.2. By a similar approach, one can see that all results and all corollaries of Hu and Zhuang [15] are now valid for any $p \geq 1$ and for unequal m_i's.

4. Applications in submodels

The hazard rate of spacings is an important measure for studying lifetime random variables in reliability theory and survival analysis. According to the previous findings, one could compare the simple spacings of ordered random variables in terms of the hazard rate ordering. But, now, one can do that for p-spacings. Here, we present the applications for three useful submodels and the other ones can be considered similarly.

4.1. Sequential (n − r + 1)-out-of-n systems

In this system (in which contains the ordinary $(n-r+1)$-out-of-n systems), successive failure times of components are observed which are called sequential order statistics (SOS). The system collapses after the rth failure so that the rth SOS describes the system lifetime. After the failure of the ith component, the distribution of the lifetimes of the remaining components in the system is adjusted by a parameter α_i (cf. [9]). This reflects both a damage caused by the previous failures and a higher load imposed on the remaining components leading possibly to shorter residual life. SOS under proportional hazard rates are included in GOS (cf. [16,17]). Indeed, the specific choice of distribution functions

$$F_i(x) = 1 - (1 - F(x))^{\alpha_i}, \quad i = 1, \ldots, n,$$

with a cdf F and positive real numbers $\alpha_1, \ldots, \alpha_n$ leads to the model of GOS with parameters $k = \alpha_n$, $m_i = (n-i+1)\alpha_i - (n-i)\alpha_{i+1} - 1$, $i = 1, \ldots, n-1$, and hence, $\gamma_i = (n-i+1)\alpha_i$, $i = 1, \ldots, n$, ($\alpha_1 = \cdots = \alpha_n = 1$ leads to the ordinary $(n-r+1)$-out-of-n systems). The main result of the paper enables us to compare the p-spacings of failures of components in two different sequential systems in hazard ratio orders. Let $V^{\text{SOS}}_{(r,n,\tilde{\alpha})}$ and $W^{\text{SOS}}_{(r,n,\tilde{\alpha})}$ represent the p-spacings of two sequential systems when the components have the lifetime distributions F and G, respectively. Then, we have

$$V^{\text{SOS}}_{(r,n,\tilde{\alpha})} \leq_{\text{hr}} W^{\text{SOS}}_{(r,n,\tilde{\alpha})}$$

provided that the corresponding conditions in Theorem 3.1 are satisfied.

4.2. *Progressive Type-II censored order statistics*

A progressively censored life test involves N items with i.i.d. lifetimes placed simultaneously on test. At the time of the ith failure ($i = 1, \ldots, n$), R_i surviving units are randomly withdrawn from the test. Progressively Type-II censored order statistics (PCOS) arising from such a reliability experiment correspond to GOS with parameters $m_i = R_i \in \mathbb{N}_0$, $i = 1, \ldots, n-1$, and $k = R_n + 1$. The vector $\tilde{R} = (R_1, \ldots, R_n)$ is called censoring plan (cf. [5], Section 3.2). Our result can be applied to compare the hazard rate of p-spacings of failures in two life tests when the components have different lifetime distributions. Let $V^{\mathrm{PCOS}}_{(r,n,\tilde{R})}$ and $W^{\mathrm{PCOS}}_{(r,n,\tilde{R})}$ represent the p-spacings of PCOS with item lifetime distributions F and G, respectively. If the conditions in part (i) of Theorem 3.1 are satisfied, we then have

$$V^{\mathrm{PCOS}}_{(r,n,\tilde{R})} \leq_{\mathrm{hr}} W^{\mathrm{PCOS}}_{(r,n,\tilde{R})}.$$

4.3. *Record values*

Record values are defined as a model of successive extremes in a sequence of i.i.d. random variables. Pellerey *et al.* [22] and Belzunce *et al.* [6] investigated the inter-epoch intervals of nonhomogeneous Poisson processes, which can be regarded as spacings of record values. Choosing $m_1 = \cdots = m_{n-1} = -1$, GOS can be viewed as record values (cf. [16,17]). Now, one can compare the p-spacings of record values arising from different distributions in hazard ratio orders. Let $V^*_{(1)}, V^*_{(2)}, \ldots$ and $W^*_{(1)}, W^*_{(2)}, \ldots$ represent the p-spacings of record values based on F and G, respectively. If the conditions in part (ii) of Theorem 3.1 are satisfied, we then have

$$V^*_{(r)} \leq_{\mathrm{hr}} W^*_{(r)}.$$

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