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Control theorems for Hilbert modular varieties

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Abstract. We prove an exact control theorem, in the sense of Hida theory, for the ordinary part of the middle degree étale cohomology of certain Hilbert modular varieties, after localizing at a suitable maximal ideal of the Hecke algebra. Our method of proof builds upon the techniques introduced by Loeffler–Rockwood–Zerbes (2023, *Spherical varieties and p-adic families of cohomology classes*); another important ingredient in our proof is the recent work of Caraiani–Tamiozzo (2023, *Compositio Mathematica* 159, 2279–2325) on the vanishing of the étale cohomology of Hilbert modular varieties with torsion coefficients outside the middle degree. This work will be used in forthcoming work of the author to show that the Asai–Flach Euler system corresponding to a quadratic Hilbert modular form varies in Hida families.

1 Introduction

Let p be an odd prime and let N be a positive integer coprime to p. A fundamental theme in Hida theory is to consider the tower of modular curves

$$\cdots \to Y_1(Np^r) \to \cdots \to Y_1(Np),$$

corresponding to the chain of congruence subgroups

$$\cdots \subseteq \Gamma_1(Np^r) \subseteq \cdots \subseteq \Gamma_1(Np).$$

The étale cohomology groups of this tower are packaged into the following inverse limit:

$$H^1(Np^\infty) := \varprojlim_r H^1_{\text{ét}}(Y_1(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)),$$

where the transition maps are taken to be the corresponding pushfoward maps in étale cohomology. The module $H^1(Np^\infty)$ is equipped with an action of the adjoint Hecke operators T'_ℓ for $\ell \nmid N$ as well as the adjoint Atkin operator U'_p (the usual Hecke operators T_ℓ and U_p do not commute with the pushforward maps and hence do not act on the inverse limit). Analogous to the usual Hida projector, we may define the adjoint or "anti-ordinary" Hida projector by $e'_{\text{ord}} = \lim_{n\to\infty}(U'_p)^{n!}$. Building on the theory of Hida on p-adic families of modular forms, Ohta [18] proved a control

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theorem for the anti-ordinary part of $H^1(Np^\infty)$. To state the theorem, we note that $H^1(Np^\infty)$ is a module over the Iwasawa algebra $\Lambda := \mathbb{Z}_p[[1+p\mathbb{Z}_p]] \cong \mathbb{Z}_p[[X]]$ via the diamond operators. Let Σ denote the set of primes dividing Np and let $G_{\mathbb{Q},\Sigma}$ denote the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside the primes in Σ.

Theorem 1.1 (Ohta) *The following hold.*

(a) *We have that $e'_{\mathrm{ord}}H^1(Np^\infty)$ is finite and free as a Λ-module.*

(b) *For $r \geq 1$ and $k \geq 0$, let $\mathfrak{p}_{r,k}$ denote the ideal of Λ generated by $(1+X)^{p^{r-1}} - (1+p)^{kp^{r-1}}$. Then there is a canonical isomorphism*

$$e'_{\mathrm{ord}}H^1(Np^\infty)/\mathfrak{p}_{r,k} \cong e'_{\mathrm{ord}}H^1_{\acute{e}t}(Y_1(Np^r)_{\overline{\mathbb{Q}}}, \mathrm{Sym}^k(\mathbb{Z}_p)(1))$$

of $\mathbb{Z}_p$-modules that is compatible with the action of $G_{\mathbb{Q},\Sigma}$ and the Hecke operators.

A remarkable aspect of the above theorem is that the module $H^1(Np^\infty)$, which is built only from étale cohomology groups with constant coefficients, also embodies information about étale cohomology with nonconstant coefficients. Indeed, Theorem 1.1 can be thought of as a cohomological version of the landmark work of Hida [9], where he constructed a space of Lambda-adic modular forms which p-adically interpolate, as the weights vary, the ordinary parts of spaces of classical modular forms. Ohta's control theorem has since been used in a wide variety of different contexts. For instance, building on the ideas introduced in [18], Ohta gave a new and streamlined proof of the theorem of Mazur–Wiles [16] (the Iwasawa main conjecture over $\mathbb{Q}$) in a subsequent article [19]. Ohta's control theorem has also been used as a crucial input by Lei–Loeffler–Zerbes [12] and Kings–Loeffler–Zerbes [11] to show that the Beilinson–Flach Euler system associated with the tensor product of two modular forms varies in Hida families.

The main goal of this article is to prove an analogous control theorem for the (anti)-ordinary part of certain Hilbert modular varieties after localizing at a suitable maximal ideal of the Hecke algebra.

1.1 Main results.

Let F be a totally real number field of degree $g = [F : \mathbb{Q}]$ with ring of integers $\mathcal{O}_F$ and discriminant Δ_F. We fix an odd prime p which is unramified in F. Let $G = \mathrm{Res}_{F/\mathbb{Q}}\ \mathrm{GL}_2$ and let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$ such that the maximal torus of diagonal matrices splits over E. For each $n \geq 1$, we let

$$U_{n,p} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} x & * \\ 0 & x \end{pmatrix} \bmod p^n \text{ for some } x \in \mathcal{O}_F \otimes \mathbb{Z}_p \right\}.$$

When $n = 0$, we let $U_{n,p}$ denote the Iwahori subgroup *i.e.* the subgroup of $\mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p)$ consisting of those matrices which are upper triangular mod p. Let $\mathfrak{N}$ be an ideal of $\mathcal{O}_F$ which does not divide 2, 3, or Δ_F. We fix a prime-to-p open compact subgroup $U^{(p)} = \{g \in G(\mathbb{A}_f^{(p)}) : g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{N}\}$ and for all $n \geq 0$, we set $U_n = U^{(p)}U_{n,p}$

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to be an open compact subgroup of $G(\mathbb{A}_f)$. We denote by $Y_G(U_n)$ the corresponding Hilbert modular variety. The reason for working with this particular level group is explained in Remark 2.3; briefly, since our reductive group G does not satisfy the SV5 axiom for Shimura varieties, we need to work with level groups having fixed and sufficiently large intersection with the center Z_G.

As above, we wish to study the étale cohomology of the Hilbert modular varieties in the tower

$$\cdots \to Y_G(U_r) \to \cdots \to Y_G(U_1),$$

which we similarly package in the Iwasawa cohomology module

$$H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O}) := \varprojlim_n H^g_{\text{ét}}(Y_G(U_n)_{\overline{\mathbb{Q}}}, \mathcal{O}),$$

where the transition maps are taken to be the corresponding pushforward maps in étale cohomology.

Let Σ be the set of places of F containing all primes dividing $\mathfrak{N}$ and all primes above p. Let $\mathbb{T}$ denote the spherical Hecke algebra generated by the standard Hecke operators $\mathcal{T}_v$ and $\mathcal{S}_v^{\pm 1}$ for $v \notin \Sigma$ (henceforth, we use calligraphic font for our Hecke operators to avoid confusion with various level groups appearing in the article). We let e'_{ord} denote Hida's anti-ordinary projection operator; as above, $e'_{\mathrm{ord}} = \lim_{n\to\infty} (\mathcal{U}'_p)^{n!}$, where $\mathcal{U}'_p$ is the Hecke operator corresponding to the double coset $U_{n,p}\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} U_{n,p}$(see Section 3.1). Both $\mathbb{T}$ and e'_{ord} act on the étale cohomology of $Y_G(U_n)$. Let $\mathfrak{m}$ be a maximal ideal in the support of $H^g_{\text{ét}}(Y_G(U_n)_{\overline{\mathbb{Q}}}, \mathbb{F}_p)$ such that the image of the associated Galois representation $\overline{\rho_{\mathfrak{m}}}$ is not solvable (see Section 4).

Let $\Lambda = \mathcal{O}[\![(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times]\!]$ *i.e.* Λ is the Iwasawa algebra over $\mathcal{O}$ corresponding to $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$ and for each $n \geq 1$, let Λ_n denote the Iwasawa algebra over $\mathcal{O}$ corresponding to elements of $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$ which are congruent to 1 mod p^n. Let λ denote a weight of G that is self-dual in the sense of Section 3.3; in other words λ is a character of the maximal torus of diagonal matrices of G that is trivial on the subgroup of scalar matrices. Let V_λ denote the irreducible representation of G of highest weight λ. Let $\mathcal{O}[-\lambda]$ denote $\mathcal{O}$ with Λ acting via the inverse of the character λ.

Theorem 1.2 (Theorem 4.7) *With the notation as above, the following hold:*

(a) *For all $n \geq 1$, we have that $e'_{\mathrm{ord}} H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}}$ is free as a Λ_n-module.*

(b) *For all $n \geq 1$, we have an isomorphism of $\mathcal{O}$-modules*

$$e'_{\mathrm{ord}} H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}} \otimes_{\Lambda_n} \mathcal{O}[-\lambda] \cong e'_{\mathrm{ord}} H^g_{\acute{e}t}(Y_G(U_n)_{\overline{\mathbb{Q}}}, V_\lambda)_{\mathfrak{m}},$$

that is compatible with the action of $G_{\mathbb{Q},\Sigma}$ and $\mathbb{T}$.

(c) *When $n = 0$, we have a similar isomorphism*

$$e'_{\mathrm{ord}} H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}} \otimes_{\Lambda} \mathcal{O}[-\lambda] \cong e'_{\mathrm{ord}} H^g_{\acute{e}t}(Y_G(U_0)_{\overline{\mathbb{Q}}}, V_\lambda)_{\mathfrak{m}}.$$

This theorem can be regarded as a generalization to étale cohomology of Hilbert varieties of Ohta's work on étale cohomology of modular curves. In [15], Loeffler–Rockwood–Zerbes further generalize Ohta's result to Shimura varieties associated

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with arbitrary reductive groups. However, their results do not apply in this setting since they assume the SV5 axiom for Shimura varieties, which does not hold for our reductive group $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$. The techniques used in this article build on the methods introduced in [15], but we face additional technical difficulties due to the lack of the SV5 axiom in our setting. Throughout this article, we have tried to emphasize the places where the SV5 axiom was needed in *op.cit*, and the alternative arguments we make in the absence of this axiom. A key ingredient to obtaining the perfect control (after localizing at $\mathfrak{m}$) in our main theorem is the recent work of Caraiani–Tamiozzo [4] where they show that the étale cohomology of Hilbert modular varieties with torsion coefficients is concentrated in the middle degree after localizing at suitable maximal ideal of the Hecke algebra. As a corollary of our main theorem, we can extend the vanishing results of Caraiani–Tamiozzo to étale cohomology of the Hilbert varieties $Y_G(U_n)$ with nontrivial coefficients.

Corollary 1.3 (Corollary 4.8) *For all* $n \geq 0$, *we have* $H^i_{ét}(Y_G(U_n)_{\overline{\mathbb{Q}}}, V_\lambda)_{\mathfrak{m}} = 0$ *when* $i \neq g$.

1.2 Arithmetic applications.

The results of the article will be used in forthcoming work of the author [21] to show that the Asai–Flach Euler system associated with a quadratic Hilbert modular form, constructed by Lei–Loeffler–Zerbes in [13], varies in Hida families. This in turn is an important ingredient in recent work of Grossi–Loeffler–Zerbes [8] on the proof of the Bloch–Kato conjecture in analytic rank zero for the Asai representation of a quadratic Hilbert modular form. We also expect that this work can find applications in the study of p-adic families of various other global cohomology classes in the Hilbert setting such as, for instance, the Hirzebruch–Zagier cycles considered in [2] and [7].

1.3 Comparison with other work.

We note that there is related work of Dimitrov [6] which also establishes control theorems for certain Hilbert modular varieties (see Section 3 of *op.cit.*), but the results in *op.cit.* make stronger hypotheses on the relevant Galois representations in consideration. In particular, the results in [6] are conditional on two hypotheses (a certain global big image assumption and a Fontaine–Laffaille type assumption on local weights) stated in Section 0.3 of *op.cit.*; while we assume that the image $\overline{\rho_{\mathfrak{m}}}$ is not solvable to prove our main theorem, we do not make any assumption similar to the second hypothesis referenced above.

2 Background on Hilbert modular varieties

In this section, we establish some basic properties of the Hilbert modular varieties $Y_G(U_n)$ which we work with. We also follow the method in [22] to establish a relation between the Betti cohomology of these varieties and the group cohomology of their corresponding arithmetic subgroups.

2.1 Notation

We begin by setting some notation that will remain fixed in the article. Let F be a totally real number field of degree g with ring of integers $\mathcal{O}_F$ and discriminant Δ_F. We fix a numbering $\{\sigma_1, \ldots, \sigma_g\}$ of real embeddings of F into $\mathbb{C}$. We let $F^{\times+}$ (resp. $\mathcal{O}_F^{\times+}$) denote the totally positive elements in $F^\times$ (resp. $\mathcal{O}_F^\times$). Let $\mathcal{H}$ denote the upper half plane and let $\mathcal{H}_F$ denote the set of elements of $F \otimes \mathbb{C}$ of totally positive imaginary part; note that $\mathcal{H}_F$ can be identified with the product of g copies of $\mathcal{H}$. We let p be an odd prime that is unramified in F. We let $\mathbb{A}_f$ denote the finite adeles of $\mathbb{Q}$, $\mathbb{A}_f^{(p)}$ the finite adeles away from p and $\mathbb{A}_{F,f}$ the finite adeles of F. We let G to be the algebraic group $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$ over $\mathbb{Q}$.

2.2 Shimura varieties for G

If $K \subseteq G(\mathbb{A}_f)$ is an open compact subgroup, its corresponding Shimura variety $Y_G(K)$ is a quasi-projective variety with a canonical model over the reflex field $\mathbb{Q}$ whose complex points are given by

$$Y_G(K)(\mathbb{C}) = G(\mathbb{Q})^+ \backslash [\mathcal{H}_F \times G(\mathbb{A}_f)]/K.$$

The Shimura varieties $Y_G(K)$ are called Hilbert modular varieties.

Definition 2.1 We say than an open compact subgroup $K \subseteq G(\mathbb{A}_f)$ is sufficiently small if for every $h \in G(\mathbb{A}_f)$ the quotient

$$\frac{G(\mathbb{Q})^+ \cap hKh^{-1}}{K \cap \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} : u \in \mathcal{O}_F^\times \right\}}$$

acts without fixed points on $\mathcal{H}_F$.

Remark 2.1 The above definition is slightly different from [13, Definition 2.2.1]; we have used $\mathcal{O}_F^\times$ in the denominator rather than $\mathcal{O}_F^{\times+}$ used in *op.cit.*

If $K \subseteq G(\mathbb{A}_f)$ is sufficiently small, then $Y_G(K)$ is smooth. We also note that if $K_1 \subseteq K_2$ is an inclusion of open compact subgroups, K_2 is sufficiently small and K_1 is normal in K_2, then the map $Y_G(K_1) \to Y_G(K_2)$ is a finite étale Galois cover.

2.3 The Hilbert modular variety $Y_G(U_n)$

Definition 2.2 For each $n \geq 1$, we let

$$U_{n,p} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} x & * \\ 0 & x \end{pmatrix} \bmod p^n \text{ for some } x \in \mathcal{O}_F \otimes \mathbb{Z}_p \right\}.$$

When $n = 0$, we let $U_{n,p}$ denote the Iwahori subgroup *i.e.* the subgroup of $\mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p)$ consisting of those matrices which are upper triangular mod p. Let $\mathfrak{N}$

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be an ideal of $\mathcal{O}_F$ which does not divide 2, 3, or Δ_F. We fix a prime-to-p open compact subgroup

$$U^{(p)} = \left\{ g \in G(\mathbb{A}_f^{(p)}) : g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{N} \right\},$$

and let $U_n = U^{(p)} U_{n,p}$. By [5, Lemma 2.1], we have that U_n is sufficiently small for all $n \geq 1$ and that the determinant map $\det : U_n \to (\mathcal{O}_F \otimes \widehat{\mathbb{Z}})^\times$ is surjective away from p.

The inclusion $U_{n+1} \hookrightarrow U_n$ induces a map of Shimura varieties $\phi_n : Y_G(U_{n+1}) \to Y_G(U_n)$.

Proposition 2.2 *The degree of the map ϕ_n is $[U_n : U_{n+1}]$.*

Proof Pick any element $[(x, g)] \in Y_G(U_n)(\mathbb{C}) = G(\mathbb{Q})^+ \backslash [\mathcal{H}_F \times G(\mathbb{A}_f)]/U_n$ and note that

$$\phi_n^{-1}([(x, g)]) = \{[(x, gu_i)] : i \in I\},$$

where $\{u_i\}_{i \in I}$ is a set of representatives of U_n/U_{n+1}. It suffices to prove that $[(x, gu_i)] \neq [(x, gu_j)]$ when $i \neq j$. Suppose for contradiction that $[(x, gu_i)] = [(x, gu_j)]$ when $i \neq j$. Then there exist $h \in G(\mathbb{Q})^+$ and $k \in U_{n+1}$ such that $(hx, hgu_i k) = (x, gu_j)$. In particular, we conclude that $h \in G^+(\mathbb{Q}) \cap gU_n g^{-1}$. Since $hx = x$ and since U_n is sufficiently small, we have by definition that

$$h \in U_n \cap \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} : u \in \mathcal{O}_F^\times \right\} = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} : u \in \mathcal{O}_F^\times \text{ and } u \equiv 1 \mod \mathfrak{N} \right\}.$$

Thus, $h \in U_{n+1}$ as well. Using the equality $hgu_i k = gu_j$ and the fact that h lies in the center of $G(\mathbb{A}_f)$, we can now conclude that $u_i hk = u_j$. Since hk lies in U_{n+1}, this contradicts the fact that $\{u_i\}_{i \in I}$ is a set of representatives for U_n/U_{n+1}. ■

Remark 2.3 The SV5 axiom for Shimura varieties (see [17, p. 75]) states that if (G, X) is a Shimura datum, then the center Z is isogenous to the product of a $\mathbb{Q}$-split torus and an $\mathbb{R}$-anisotropic torus. An equivalent formulation is that $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_f)$. In [14], Loeffler showed that if the SV5 axiom is satisfied, and if K_1 and K_2 are open compact subgroups of $G(\mathbb{A}_f)$ with $K_1 \subseteq K_2$, then the degree of the corresponding map of Shimura varieties $Y_G(K_1) \to Y_G(K_2)$ equals the index $[K_2 : K_1]$. The group $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$ that we are working with does not satisfy the SV5 axiom (this is essentially because the unit group $\mathcal{O}_F^\times$ is infinite). Nevertheless, with our choice of level groups U_n, Proposition 2.2 shows that the desired claim still holds.

2.4 The number of components of $Y_G(U_n)$

Let $I(F)$ denote the group of fractional ideals of F, and let $\mathrm{Cl}^+(F) := I(F)/\{(\beta) : \beta$ totally positive$\}$ denote the narrow class group of F. Let h^+ denote the narrow class number of F.

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Proposition 2.4 *The Hilbert modular variety* $Y_G(U_n)$ *has* $h^+ \cdot |\mathcal{O}_F^{\times +} \backslash (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times / \det(U_{n,p})|$ *connected components.*

Proof Note that the map $\mathbb{A}_{F,f}^\times \to \mathrm{Cl}^+(F)$ defined via $(\alpha)_\mathfrak{p} \mapsto \prod_\mathfrak{p} \mathfrak{p}^{v_\mathfrak{p}(\alpha_\mathfrak{p})}$ has kernel $\prod_\mathfrak{p} \mathcal{O}_{F,\mathfrak{p}}^\times \cdot F^{\times +}$. For each $x \in \mathrm{Cl}^+(F)$, choose a preimage $\alpha_x \in \mathbb{A}_{F,f}^\times$. Hence, we have a decomposition $\mathbb{A}_{F,f}^\times = \bigsqcup_{x \in \mathrm{Cl}^+(F)} (\alpha_x \cdot \prod_\mathfrak{p} \mathcal{O}_{F,\mathfrak{p}}^\times \cdot F^{\times +})$ as sets. By strong approximation, the connected components of $Y_G(U_n)$ are indexed by $F_+^\times \backslash \mathbb{A}_{F,f}^\times / \det(U_n)$. This set is in bijection with

$$\begin{aligned} \bigsqcup_{x \in \mathrm{Cl}^+(F)} F^{\times +} \backslash \prod_\mathfrak{p} (\alpha_x \cdot \mathcal{O}_{F,\mathfrak{p}}^\times \cdot F^{\times +}) / \det(U_n) &\cong \bigsqcup_{x \in \mathrm{Cl}^+(F)} \mathcal{O}_F^{\times^+} \backslash (\alpha_x \cdot \prod_\mathfrak{p} \mathcal{O}_{F,\mathfrak{p}}^\times) / \det(U_n) \\ &\cong \bigsqcup_{x \in \mathrm{Cl}^+(F)} \mathcal{O}_F^{\times +} \backslash (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times / \det(U_{n,p}), \end{aligned}$$

where the last equality follows since the determinant map $\det : U_n \to (\mathcal{O}_F \otimes \widehat{\mathbb{Z}})^\times$ is surjective away from p. ∎

2.5 Hecke action

For $K \subseteq G(\mathbb{A}_f)$ an open compact subgroup, we let $\mathbb{T}_K(G) = \mathbb{Z}[K \backslash G(\mathbb{A}_f)/K]$ be the Hecke-algebra of compactly supported bi-invariant functions on $G(\mathbb{A}_f)$ with multiplication given by convolution. Let $g \in G(\mathbb{A}_f)$ and let $K_g = K \cap gKg^{-1}$; we have a correspondence $[KgK]$

$$\begin{array}{ccc} Y_G(K_g) & \xrightarrow{\ f\ } & Y_G(K_{g^{-1}}) \\ \downarrow & & \downarrow \\ Y_G(K) & \overset{[KgK]}{\dashrightarrow} & Y_G(K), \end{array}$$

where the vertical maps are canonical projections and the upper-half horizontal map f on complex points is induced by multiplication by g. We obtain an action of $\mathbb{T}_K(G)$ on $H^i_{\text{ét}}(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{F}_p)$ and $H^i_{\text{ét}}(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)$. Let Σ be the set of places of F containing all primes dividing $\mathfrak{N}$ and all primes above p. Let

$$\mathbb{T} := {\bigotimes_{v \notin \Sigma}}' \mathbb{Z}[\mathrm{GL}_2(\mathcal{O}_v) \backslash \mathrm{GL}_2(F_v) / \mathrm{GL}_2(\mathcal{O}_v)]$$

denote the abstract spherical algebra away from Σ. We note that $\mathbb{T}$ is a subalgebra of $\mathbb{T}_K(G)$; it is also commutative, generated by the following Hecke operators $\mathcal{T}_v$ and $\mathcal{S}_v^{\pm 1}$ for every finite place $v \notin \Sigma$; for every such v we choose a uniformizer ϖ_v of $\mathcal{O}_v$ and define

- $\mathcal{T}_v$ to be the double coset $\mathrm{GL}_2(\mathcal{O}_v) \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_v)$.
- $\mathcal{S}_v$ to the double coset $\mathrm{GL}_2(\mathcal{O}_v) \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_v)$.

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2.6 Betti cohomology

Let Γ be a subgroup of $G^+(\mathbb{Q})$ and let $\overline{\Gamma}$ be its image in $G(\mathbb{Q})/Z_G(\mathbb{Q})$. We assume that $\overline{\Gamma}$ has no nontrivial elements of finite order; hence, it acts freely and continuously on $\mathcal{H}_F$. In this subsection, we closely follow [22] to establish a relation between the group cohomology of $\overline{\Gamma}$ and the Betti cohomology of the corresponding Hilbert modular variety.

By the work of Borel–Serre, there exists a canonical compactification $\overline{\Gamma}\backslash\overline{\mathcal{H}_F}$, where $\overline{\mathcal{H}_F}$ is a contractible real manifold with corners. Since $\overline{\Gamma}\backslash\overline{\mathcal{H}_F}$ is compact, we may choose a finite triangulation of $\overline{\Gamma}\backslash\overline{\mathcal{H}_F}$. We may pull it back to $\overline{\mathcal{H}_F}$ via the canonical projection $\overline{\mathcal{H}_F} \to \overline{\Gamma}\backslash\overline{\mathcal{H}_F}$. Let $C_q(\overline{\Gamma})$ be the free $\mathbb{Z}$-module over the set of q-dimensional simplices of the triangulation obtained by pull-back to $\overline{\mathcal{H}_F}$. Since the action of $\overline{\Gamma}$ on $\overline{\mathcal{H}_F}$ is free, and since the triangulation of $\overline{\Gamma}\backslash\overline{\mathcal{H}_F}$ is finite, the $C_q(\overline{\Gamma})$'s are free $\mathbb{Z}[\overline{\Gamma}]$-modules of finite type. We also note that

$$0 \to C_d(\overline{\Gamma}) \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_1} C_0(\overline{\Gamma}) \to 0$$

is a complex computing the homology of $\overline{\mathcal{H}_F}$. Since $\overline{\mathcal{H}_F}$ is contractible, this complex is exact except in degree zero and $H_0(\overline{\mathcal{H}_F}, \mathbb{Z}) = C_0(\overline{\Gamma})/\partial_0(C_1(\overline{\Gamma})) = \mathbb{Z}$. Thus, in summary, we have that

$$0 \to C_d(\overline{\Gamma}) \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_1} C_0(\overline{\Gamma}) \to \mathbb{Z} \to 0$$

is an exact sequence of finite free $\mathbb{Z}[\overline{\Gamma}]$-modules. If M is a Γ-module, we let $\mathcal{C}^\bullet(\overline{\Gamma}, M)$ denote the complex

$$0 \to \mathrm{Hom}(C_0(\overline{\Gamma}), M) \to \cdots \to \mathrm{Hom}(C_d(\overline{\Gamma}), M).$$

Thus, $H^i(\overline{\Gamma}, M)$ is the ith cohomology group of the complex $\mathcal{C}^\bullet(\overline{\Gamma}, M)$.

Let K be an open compact subgroup of $G(\mathbb{A}_f)$ which is sufficently small and let M be a left K-module acting via its projection to K_p, the image of K in $G(\mathbb{Q}_p)$. The corresponding Hilbert modular variety $Y_G(K)$ satisfies

$$Y_G(K) \cong \bigsqcup_{j \in J} \Gamma_j \backslash \mathcal{H}_F,$$

where J is a finite set and for each $j \in J$, $\Gamma_j = g_j K g_j^{-1} \cap G^+(\mathbb{Q})$ for some $g_j \in G(\mathbb{A}_f)$. As before, we let $\overline{\Gamma_j}$ denote the image of Γ_j in $G(\mathbb{Q})/Z_G(\mathbb{Q})$. We set

$$C^\bullet(K, M) := \bigoplus_{j \in J} C^\bullet(\overline{\Gamma_j}, M). \tag{2.1}$$

Let $\overline{Y_G} := G(\mathbb{Q})^+\backslash G(\mathbb{A}_f) \times \overline{\mathcal{H}_F}$. Then $\overline{Y_G}(K) := \overline{Y_G}/K$ is the Borel–Serre compactification of $Y_G(K)$. Let $\pi : \overline{Y_G} \to \overline{Y_G}(K)$ denote the canonical projection. We choose a finite triangulation of $\overline{Y_G}(K)$ and pull it back via π. Let $C^q(K)$ denote the corresponding chain complex equipped with a right action of K. Then

$$C^\bullet(K, M) = \mathrm{Hom}_K(C^q(K), M).$$

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Thus, $C^\bullet(K, M)$ also computes the cohomology of the local system M on $Y_G(K)$ and so we have an isomorphism

$$H^i(Y_G(K), M) \cong \bigoplus_{j\in J} H^i(\overline{\Gamma}_j, M). \tag{2.2}$$

3 Construction of a Tor spectral sequence

In this section, we construct a Tor descent spectral sequence which will be an important tool to relate the Iwasawa cohomology module $H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})$ to the cohomology of $Y_G(U_n)$ at finite layers.

3.1 General notation

We consider the group scheme GL_2 over $\mathbb{Z}_p$ and we let B_2, N_2, and T_2 to be the subgroups of upper-triangular, unipotent, and diagonal matrices, respectively. Following [15], we set the following notation:

- $Q = \mathrm{Res}_{\mathcal{O}_F\otimes\mathbb{Z}_p/\mathbb{Z}_p} B_2$.
- $N = \mathrm{Res}_{\mathcal{O}_F\otimes\mathbb{Z}_p/\mathbb{Z}_p} N_2$.
- $S = \mathrm{Res}_{\mathcal{O}_F\otimes\mathbb{Z}_p/\mathbb{Z}_p} T_2$.
- $E/\mathbb{Q}_p$ is a finite extension with ring of integers $\mathcal{O}$ such that S splits over E.
- $S^0 = \mathrm{Res}_{\mathcal{O}_F\otimes\mathbb{Z}_p/\mathbb{Z}_p} \mathbb{G}_m$, viewed as a subgroup of S via the diagonal embedding.
- Q^0 denotes the preimage of S^0 under the projection $Q \to S$
  i.e. $Q^0(\mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p) | a, b \in \mathcal{O}_F \otimes \mathbb{Z}_p \right\}$.
- $\mathfrak{S} = S(\mathbb{Z}_p)/S^0(\mathbb{Z}_p)$ identified with $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$ via the short exact sequence

  $$1 \to S^0(\mathbb{Z}_p) \to S(\mathbb{Z}_p) \to (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \to 1,$$

  where the map

  $$S(\mathbb{Z}_p) \to (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \quad \text{is given by} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto ab^{-1}.$$

- $\Lambda = \mathcal{O}[\![\mathfrak{S}]\!]$.
- $\tau = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p \otimes \mathcal{O}_F)$.
- $N_r = \tau^r N(\mathbb{Z}_p)\tau^{-r} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p) |\ x \equiv 0 \bmod p^r \right\}$.
- $\overline{N_r} = \tau^{-r}\overline{N}(\mathbb{Z}_p)\tau^r = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p) |\ x \equiv 0 \bmod p^r \right\}$.
- $L_r = \{\ell \in S(\mathbb{Z}_p) : \ell \in S^0(\mathbb{Z}_p) \bmod p^r\} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p) | a \equiv d \bmod p^r \right\}$.
- $V_r = \overline{N_r} L_r N_0$.

Proposition 3.1 *For every $r \geq 1$, we have that $V_r = U_{r,p}$.*

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Proof This follows from a direct matrix computation. ∎

We also set $V_0 = \mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p)$ and $V_0^{(s)} := \tau^{-s} V_0 \tau^s \cap V_0$ for $s \geq 1$; in other words, $V_0^{(s)}$ is the group of upper-triangular matrices mod p^s :

$$V_0^{(s)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p) | a, b, c, d \in \mathcal{O}_F \otimes \mathbb{Z}_p \text{ and } c \equiv 0 \bmod p^s \right\}.$$

Similarly, for $s \geq 0$ and $n \geq 0$, we set

$$U_{n,p}^{(s)} := \tau^{-s} U_{n,p} \tau^s \cap U_{n,p}.$$

For our level groups $U_{n,p}$, we define the Hecke operator $\mathcal{U}'_p$ to be the double coset $U_{n,p} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} U_{n,p}$ and we define the analogue of Hida's anti-ordinary projection operator to be

$$e'_{\mathrm{ord}} = \lim_{n \to \infty} (\mathcal{U}'_p)^{n!}.$$

The operator e'_{ord} acts on the cohomology of $Y_G(U_n)$. Let M be a $\mathbb{Z}[U_n]$-module (acting via projection to $U_{n,p}$) with a compatible action of $\mathcal{U}'_p$ (in the sense that for any $u \in U_{n,p}^{(s)}$, the action of $\mathcal{U}'_p$ intertwines the action of u and $\tau^{-s} u \tau^s$). For $K \subseteq G(\mathbb{A}_f)$ an open compact subgroup, the operator e'_{ord} also acts on the complexes $C^\bullet(K, M)$ introduced in Section 2.6 by lifting the action on cohomology (see [15, p. 6]).

Lemma 3.2 *Let M be a $\mathbb{Z}[U_n]$-module with a compatible action of $\mathcal{U}'_p$. The following diagram commutes on cohomology*

$$\begin{array}{ccc} \mathcal{C}^\bullet(U^{(p)}U_{n,p}^{(s)}, M) & \xrightarrow{\text{cores}} & \mathcal{C}^\bullet(U_n, M) \\ \downarrow{\scriptstyle (\mathcal{U}'_p)^s} & \swarrow{\scriptstyle [\tau^s]_*} & \downarrow{\scriptstyle (\mathcal{U}'_p)^s} \\ \mathcal{C}^\bullet(U^{(p)}U_{n,p}^{(s)}, M) & \xrightarrow[\text{cores}]{} & \mathcal{C}^\bullet(U_n, M) \end{array}$$

Proof This follows from [15, Lemma 2.7.4] (the SV5 axiom is not needed in the proof of this lemma). ∎

Proposition 3.3 *The corestriction maps induce isomorphisms*

$$e'_{\mathrm{ord}} H^i(Y_G(U^{(p)}U_{n,p}^{(s)}), M) \cong e'_{\mathrm{ord}} H^i(Y_G(U_n), M).$$

Proof As explained in [15, Corollary 2.7.5], this follows from the previous lemma. ∎

3.2 Algebraic representations

Let $X^\bullet(S)$ denote the character lattice of S and $X^\bullet_+(S)$ be the set of dominant weights. For each $\lambda \in X^\bullet_+(S)$, there is a unique isomorphism class of irreducible representations of $(\rho_\lambda, V_\lambda)$ of G (over E) of highest weight λ. A representative of this isomorphism

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class can be constructed using the Borel–Weil–Bott theorem, as the space of all polynomials

$$\{f \in E[G] : f(\overline{n}\ell g) = \lambda(\ell)f(g) \quad \forall \overline{n} \in \overline{N}, \ell \in S, g \in G\},$$

with G acting by right translation. More concretely, each such $\lambda \in X_+^\bullet(S)$ can be identified with an integer tuple $(k_1, \ldots, k_g, t_1, \ldots, t_g)$ such that the associated V_λ is given by the representation $\mathrm{Sym}^{k_i} V \otimes \det^{t_i}$ at the ith embedding, where V is the standard representation of G. All our Hecke operators defined above also act on cohomology with the V_λ's as coefficient systems (see [15, Definition 2.5.1]).

3.3 Modules of measures

In this subsection, for brevity, we let U to be $U_{r,p}$ for some $r \geq 0$. We also fix a character $\lambda = (k_1, \ldots, k_g, t_1, \ldots t_g)$ such that $k_1 + 2t_1 = \cdots = k_g + 2t_g = 0$.

Lemma 3.4 *We have that λ is trivial on the subtorus S^0.*

Proof This follows immediately from our normalization of the weights described in Section 3.2, namely,

$$\lambda\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \sigma_1(a)^{k_1+2t_1} \cdots \sigma_g^{k_g+2t_g}(a) = 1. \qquad \blacksquare$$

Remark 3.5 The weights we have considered above are exactly those which are self-dual *i.e* those for which the dual of V_λ is isomorphic to itself.

Following [15], we set the following notation. We define U- modules of continuous functions

$$C_{\lambda,U} := \{f : U \to \mathcal{O} : f \text{ continuous}, f(\ell n g) = \lambda^{-1}(\ell)f(g) \quad \forall ln \in (S \cap U)N_0, g \in U\}$$

and

$$C_{\mathrm{univ}} := \{f : U \to \mathcal{O} : f \text{ continuous}, f(\ell n g) = f(g) \quad \forall ln \in Q^0(\mathbb{Z}_p), g \in U\},$$

with U acting by right translation. We endow these spaces with an action of τ given by

$$\tau \cdot f(n\ell\bar{n}) = f(n\ell\tau^{-1}\bar{n}\tau).$$

We define modules of bounded distributions

$$D_{\lambda,U} := \mathrm{Hom}_{\mathrm{cts}}(C_{\lambda,U}, \mathcal{O}),$$

and

$$D_{\mathrm{univ}} := \mathrm{Hom}_{\mathrm{cts}}(C_{\mathrm{univ}}, \mathcal{O}),$$

which inherit actions of V_0' and τ^{-1} by duality. We let $\mathfrak{S}_U$ denote the image of U in $\mathfrak{S}$ (with respect to the Iwahori decomposition stated in Section 3.1); note that $\mathfrak{S}_U = \mathfrak{S} \cong (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$ when $U = U_{0,p}$ and $\mathfrak{S}_U$ is the set of elements of $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$ which are congruent to 1 mod p^n when $U = U_{r,p}$ and $r \geq 1$. Similarly, we write $\bar{N}_U$ for

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$\bar{N}_r$ when $U = U_{r,p}$ for $r \geq 0$. We let $\mathcal{O}[\![\mathfrak{S}_U]\!]$ denote the Iwasawa algebra corresponding to $\mathfrak{S}_U$. We have that $D_{\lambda,U}$ and D_{univ} are modules over $\Lambda = \mathcal{O}[\![\mathfrak{S}]\!]$ (with the action given by inverse translation) and this structure is given explicitly by the isomorphisms (see [15, p. 6])

$$D_{\lambda,U} \cong (\mathcal{O}[-\lambda] \otimes_{\mathcal{O}} \Lambda)\hat{\otimes}_{\mathcal{O}}\mathcal{O}[\![\bar{N}_U]\!],$$

and

$$D_{\text{univ}} \cong \Lambda\hat{\otimes}_{\mathcal{O}}\mathcal{O}[\![\bar{N}_U]\!],$$

where $\mathcal{O}[-\lambda]$ denotes $\mathcal{O}$ regarded as an $\mathfrak{S}$ (and hence Λ) module via the inverse of λ. In particular, we have that

$$D_{\text{univ}}\hat{\otimes}_{\mathcal{O}[\![\mathfrak{S}_U]\!]}\mathcal{O}[-\lambda] \cong D_{\lambda,U}$$

as Λ-modules. Since a power series ring in finitely many variables over a Noetherian ring is flat (see [3, p. 146]), we have that D_{univ} is flat as a Λ-module.

Proposition 3.6 *The anti-ordinary projector e'_{ord} acts on $\mathcal{C}^\bullet(U, D_{\text{univ}})$ such that we have a decomposition*

$$C^\bullet(U, D_{\text{univ}}) = e'_{\text{ord}}C^\bullet(U, D_{\text{univ}}) \oplus (1 - e'_{\text{ord}})C^\bullet(U, D_{\text{univ}})$$

with $\mathcal{U}'_p$ acting invertibly on the first component and topologically nilpotently on the second. Moreover, the complex $e'_{\text{ord}}C^\bullet(U, D_{\text{univ}})$ consists of flat Λ-modules.

Proof See [15, Proposition 2.7.2]. ■

3.4 Proof of the Tor spectral sequence

In this subsection, we let λ denote a weight that is self-dual in the sense of Section 3.3. We let s and n be integers with $s \geq n$. Let $\Gamma_1 = G(\mathbb{Q})^+ \cap U^{(p)}V_0^{(s)}$ and $\Gamma_2 = G(\mathbb{Q})^+ \cap (U^{(p)}(U_{n,p} \cap V_0^{(s)}))$. Let $\overline{\Gamma_1}$ and $\overline{\Gamma_2}$ denote the images of Γ_1 and Γ_2 in $G/Z_G(\mathbb{Q})$. Write

$$Y_G(U^{(p)}V_0^{(s)})(\mathbb{C}) = \bigsqcup_{j\in J_1}\Gamma_1\backslash\mathcal{H}_F,$$

and

$$Y_G(U^{(p)}(V_0^{(s)} \cap U_{n,p}))(\mathbb{C}) \cong \bigsqcup_{j\in J_2}\Gamma_2\backslash\mathcal{H}_F.$$

Each $i \in J_1$ corresponds to a matrix $g_i \in G(\mathbb{A}_f)$ whose determinants form a set of representatives for $F^{\times+}\backslash\mathbb{A}_{F,f}^\times/\det(U^{(p)}V_0^{(s)})$. Similarly, each $i \in J_2$ corresponds to a matrix $g_i \in G(\mathbb{A}_f)$ whose determinants form a set of representatives for $F^{\times+}\backslash\mathbb{A}_{F,f}^\times/\det(U^{(p)}(U_{n,p} \cap V_0^{(s)}))$.

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Since $\det(U^{(p)}V_0^{(s)})) = (\mathcal{O}_F \otimes \hat{\mathbb{Z}})^\times$, we have that $|J_1| = h^+$, and by Lemmma 2.4, $|J_2| = h^+ \cdot |\mathcal{O}_F^{\times^+}\backslash(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times/\det(U_{n,p})|$ (as written, Lemma 2.4 can only be applied when $s = n$ but the exact same proof goes through when $s \geq n$ since $\det(U_{n,p}) = \det(U_{n,p} \cap V_0^{(s)})$ in $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$).

Proposition 3.7 *Let φ denote the natural map*

$$\varphi : \Gamma_1/\Gamma_2 \to V_0^{(s)}/(U_{n,p} \cap V_0^{(s)}).$$

Then $\operatorname{im}(\varphi)$ *has index* $|\mathcal{O}_F^{\times+}\backslash(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times/\det(U_{n,p})|$ *in* $V_0^{(s)}/(U_{n,p} \cap V_0^{(s)})$.

Proof By strong approximation for the semisimple group $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{SL}_2$, the diagram

$$\begin{array}{ccc} \Gamma_1/\Gamma_2 & \longrightarrow & V_0^{(s)}/(U_{n,p} \cap V_0^{(s)}) \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} \\ \mathcal{O}_F^{\times+}/(\mathcal{O}_F^\times)^2 & \longrightarrow & (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times/\det(U_{n,p}) \end{array}$$

is cartesian (see [10, Corollary 3.3] for a similar argument). On the other hand, note that $\det(U_{n,p}) = (\mathcal{O}_F^\times)^2$ in $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$ by Hensel's lemma; thus, we conclude that

$$|\operatorname{coker}(\varphi)| = \frac{[(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times : \det(U_{n,p})]}{[\mathcal{O}_F^{\times+} : (\mathcal{O}_F^\times)^2]}$$

and that the natural map $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times/\det(U_{n,p}) \to \mathcal{O}_F^{\times+}\backslash(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times/\det(U_{n,p})$ has kernel $\mathcal{O}_F^{\times+}/(\mathcal{O}_F^\times)^2$. Thus, $\dfrac{[(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times : \det(U_{n,p})]}{[\mathcal{O}_F^{\times+} : (\mathcal{O}_F^\times)^2]} = |\mathcal{O}_F^{\times^+}\backslash(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times/\det(U_{n,p})|$ proving the claim. ■

Proposition 3.8 *We have a Hecke-equivariant isomorphism*

$$\bigoplus_{j\in J_1} H^i(\overline{\Gamma_1}, \mathcal{O}/(p^s)[-\lambda][V_0^{(s)}/V_0^{(s)} \cap U_{n,p}]) \cong \bigoplus_{j\in J_2} H^i(\overline{\Gamma_1}, \mathcal{O}/(p^s)[-\lambda][\overline{\Gamma_1}/\overline{\Gamma_2}]).$$

Proof This follows from the previous proposition and the fact that $|J_1| = h^+$ and $|J_2| = h^+ \cdot |\mathcal{O}_F^{\times+}\backslash(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times/\det(U_{n,p})|$. ■

Theorem 3.9 *We have an isomorphism of $\mathcal{O}$-modules*

$$e'_{\mathrm{ord}}H^i(Y_G(U_0), D_{\lambda,U_{n,p}}) \cong e'_{\mathrm{ord}}H^i(Y_G(U_n), V_\lambda).$$

Proof As explained in [15, Proposition 2.7.7], we have an isomorphism

$$e'_{\mathrm{ord}}H^i(Y_G(U_0), D_{\lambda,U_{n,p}}/p^s) \cong e'_{\mathrm{ord}}H^i(Y_G(U^{(p)}V_0^{(s)}), \mathcal{O}/(p^s)[-\lambda] \otimes_{\mathcal{O}[\mathfrak{S}_{U_{n,p}}]} \mathcal{O}/(p^s)[\mathfrak{S}]).$$

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Hence,

$$\begin{aligned}
&H^i(Y_G(U^{(p)}V_0^{(s)}), \mathcal{O}/(p^s)[-\lambda] \otimes_{\mathcal{O}[\mathfrak{S}_{U_{n,p}}]} \mathcal{O}(p^n)[\mathfrak{S}])\\
&\cong \bigoplus_{j\in J_1} H^i(\overline{\Gamma_1}, \mathcal{O}/(p^s)[-\lambda] \otimes_{\mathcal{O}[\mathfrak{S}_{U_{n,p}}]} \mathcal{O}/(p^s)[\mathfrak{S}])\\
&\cong \bigoplus_{j\in J_1} H^i(\overline{\Gamma_1}, \mathcal{O}/(p^s)[-\lambda][\mathfrak{S}/\mathfrak{S}_{U_{n,p}}])\\
&\cong \bigoplus_{j\in J_1} H^i(\overline{\Gamma_1}, \mathcal{O}/(p^s)[-\lambda][V_0^{(s)}/V_0^{(s)} \cap U_{n,p}])\\
&\cong \bigoplus_{j\in J_2} H^i(\overline{\Gamma_1}, \mathcal{O}/(p^s)[-\lambda][\overline{\Gamma_1}/\overline{\Gamma_2}])\\
&\cong \bigoplus_{j\in J_2} H^i(\overline{\Gamma_2}, \mathcal{O}/(p^s)[-\lambda])\\
&\cong H^i(Y_G(U^{(p)}(V_0^{(s)} \cap U_{n,p})), \mathcal{O}/(p^s)[-\lambda]).
\end{aligned}$$

Here, the first isomorphism follows from Equation (2.2), the second is a general property of tensor products, the third follows the fact that $\mathfrak{S}/\mathfrak{S}_{U_{n,p}} \cong V_0^{(s)}/V_0^{(s)} \cap U_{n,p}$, the fourth follows from Proposition 3.8, the fifth follows from Shapiro's Lemma, and the sixth follows from Equation (2.2) again. As explained in [15, Proposition 2.7.7], we can use Proposition 3.3 to conclude that $e'_{\mathrm{ord}}H^i(Y_G(U^{(p)}(V_0^{(s)} \cap U_{n,p})), \mathcal{O}/(p^s)[-\lambda])$ is in turn isomorphic to $e'_{\mathrm{ord}}H^i(Y_G(U_n), V_\lambda/p^s)$. Thus, combining our isomorphims, we conclude that

$$\begin{aligned}
e'_{\mathrm{ord}}H^i(Y_G(U_0), D_{\lambda,U_{n,p}}) &= \varprojlim_s e'_{\mathrm{ord}}H^i(Y_G(U_0), D_{\lambda,U_{n,p}}/p^s)\\
&\cong \varprojlim_s e'_{\mathrm{ord}}H^i(Y_G(U_n), V_\lambda/p^s)\\
&= e'_{\mathrm{ord}}H^i(Y_G(U_n), V_\lambda).
\end{aligned}$$ ∎

Remark 3.10 The analogue of this theorem in [15] ([15, Proposition 2.7.7]) crucially used the SV5 axiom for Shimura varieties in the application of Shapiro's lemma. Nevertheless, as the proof given above demonstrates, with our choice of level groups, we do not need to invoke this axiom.

We note that Theorem 3.9 is compatible with the comparision isomorphism between Betti and étale cohomology; for the rest of the article, we work with étale cohomology of Hilbert modular varieties.

Corollary 3.11 *We have an isomorphism*

$$e'_{\mathrm{ord}}H^i_{\acute{e}t}(Y_G(U_0), D_{\mathrm{univ}}) \cong e'_{\mathrm{ord}} \varprojlim_n H^i_{\acute{e}t}(Y_G(U_n), \mathcal{O}).$$

Proof Setting λ to be the trivial character, we deduce from the previous theorem that

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$$
\begin{aligned}
e'_{\mathrm{ord}} H^i_{\text{ét}}(Y_G(U_0), D_{\mathrm{univ}}) &\cong \varprojlim_{Q^0 \subseteq U, n} e'_{\mathrm{ord}} H^i_{\text{ét}}(Y_G(U_0), D_{\lambda,U}/(p^n)) \\
&\cong \varprojlim_n e'_{\mathrm{ord}} H^i_{\text{ét}}(Y_G(U_n), \mathcal{O}/(p^n)) \\
&\cong e'_{\mathrm{ord}} \varprojlim_n H^i_{\text{ét}}(Y_G(U_n), \mathcal{O}).
\end{aligned}
$$

Here the last isomorphism follows from the facts that our inverse system satisfies the Mittag–Leffler property because the cohomology groups in the system are finitely generated, and that e'_{ord} commutes with the maps in the inverse limit. ∎

Define $M^\bullet$ to be the image of $e'_{\mathrm{ord}} C^\bullet(U_0, D_{\mathrm{univ}})$ in the subcategory $D^{\mathrm{flat}}(R)$ of the derived category of R-modules generated by flat objects. We also set $\Lambda_n = \mathcal{O}[\![\mathfrak{S}_{U_{n,p}}]\!]$ for all $n \geq 1$.

Theorem 3.12 *For all $n \geq 1$, we have a quasi-isomorphism*

$$
M^\bullet \otimes^{\mathbb{L}}_{\Lambda_n} \mathcal{O}[-\lambda] \cong e'_{\mathrm{ord}} C^\bullet(U_n, V_\lambda).
$$

Proof Since $M^\bullet$ is represented by the flat complex $e'_{\mathrm{ord}} \mathcal{C}^\bullet(U_0, D_{\mathrm{univ}})$, we can compute the derived tensor product as

$$
e'_{\mathrm{ord}} \mathcal{C}^\bullet(U_0, D_{\mathrm{univ}}) \otimes_{\Lambda_n} \mathcal{O}[-\lambda] = e'_{\mathrm{ord}} \mathcal{C}^\bullet(U_0, D_{\mathrm{univ}} \otimes_{\Lambda_n} \mathcal{O}[-\lambda]).
$$

By Theorem 3.9, this complex is isomorphic to $e'_{\mathrm{ord}} C^\bullet(U_n, V_\lambda)$. ∎

Corollary 3.13 *For all $n \geq 1$, there is a spectral sequence*

$$
E_2^{i,j} : \mathrm{Tor}^{\Lambda_n}_{-i}(e'_{\mathrm{ord}} \varprojlim_s H^j_{\acute{e}t}(Y_G(U_s), \mathcal{O})_{\overline{\mathbb{Q}}}, \mathcal{O}[-\lambda]) \implies e'_{\mathrm{ord}} H^{i+j}_{\acute{e}t}(Y_G(U_n)_{\overline{\mathbb{Q}}}, V_\lambda).
$$

Proof This follows from the previous theorem using the spectral sequence for the Tor functor. ∎

An analogous argument to the one given in this section allows us to obtain a similar result at the Iwahori level U_0.

Corollary 3.14 *There is a spectral sequence*

$$
E_2^{i,j} : \mathrm{Tor}^{\Lambda}_{-i}(e'_{\mathrm{ord}} \varprojlim_s H^j_{\acute{e}t}(Y_G(U_s), \mathcal{O})_{\overline{\mathbb{Q}}}, \mathcal{O}[-\lambda]) \implies e'_{\mathrm{ord}} H^{i+j}_{\acute{e}t}(Y_G(U_0)_{\overline{\mathbb{Q}}}, V_\lambda).
$$

4 Proof of the control theorem

In this section, we use the spectral sequences in Corollaries 3.13 and 3.14 to give a proof of our control theorem. We begin by recalling the results of Caraiani–Tamiozzo [4], which play a crucial role in proving our control theorem.

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4.1 The results of Caraiani–Tamiozzo

Let $K \subseteq G(\mathbb{A}_f)$ be a neat compact open subgroup and take a maximal ideal $\mathfrak{m} \subseteq \mathbb{T}$ in the support of $H^i(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{F}_p)$. By the work of Scholze [20], we have a unique continuous semisimple Galois representation

$$\overline{\rho_{\mathfrak{m}}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}_p}),$$

such that $\overline{\rho_{\mathfrak{m}}}$ is unramified for all v not in a finite set of suitable places of F and such that the characteristic polynomial of $\overline{\rho_{\mathfrak{m}}}(\mathrm{Frob}_v)$ equals $X^2 - \mathcal{T}_v X + \mathcal{S}_v N(v)$ mod $\mathfrak{m}$.

Theorem 4.1 (Caraiani–Tamiozzo) *Assume that the image of $\overline{\rho_{\mathfrak{m}}}$ is not solvable. Then $H^i_{ét}(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{F}_p)_{\mathfrak{m}}$ is nonzero only for $i = g$.*

Proof See [4, Theorem 7.1.1]. ∎

Corollary 4.2 *In the above setting, we have $H^i_{ét}(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}} \neq 0$ only for $i = g$. Moreover, $H^g_{ét}(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}}$ is free as a $\mathbb{Z}_p$-module.*

Proof This is explained in [4, Corollary 7.1.2]; we give the proof below for the convenience of the reader. We consider the short exact sequence

$$0 \to \mathbb{Z}_p \xrightarrow{\cdot p} \mathbb{Z}_p \to \mathbb{F}_p \to 0,$$

localize the corresponding long exact sequence at $\mathfrak{m}$ and employ Theorem 4.1 to obtain a surjective map

$$H^i_{\text{ét}}(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}} \xrightarrow{\cdot p} H^i_{\text{ét}}(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}},$$

for all $i \neq g$. Since $H^i_{\text{ét}}(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}}$ is a finitely generated $\mathbb{Z}_p$-module, we can apply Nakayama's lemma to conclude that $H^i_{\text{ét}}(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}} = 0$ for all $i \neq g$. When $i = g$, we get that

$$\frac{H^g_{\text{ét}}(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{F}_p)_{\mathfrak{m}}}{\mathrm{im}(H^g_{\text{ét}}(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}} \to H^g_{\text{ét}}(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{F}_p)_{\mathfrak{m}}} = 0.$$

Since $H^g_{\text{ét}}(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{F}_p)_{\mathfrak{m}} \neq 0$, it follows that $H^g_{\text{ét}}(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}} \neq 0$ as well. Finally, note that the long-exact sequence also yields an injection $H^g_{\text{ét}}(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}} \hookrightarrow \cdot p H^g_{\text{ét}}(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}}$. Hence, $H^g_{\text{ét}}(Y_G(K)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}}$ has no p-torsion, hence is torsion-free and hence free (using the fact that torsion-free modules over a PID are free). ∎

4.2 Proof of the control theorem

We begin by recalling some notation and assumptions. We let $H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O}) = \varprojlim_n H^g_{\text{ét}}(Y_G(U_n)_{\overline{\mathbb{Q}}}, \mathcal{O})$, $\Lambda = [\![(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times]\!]$ and $\Lambda_n = \mathcal{O}[\![\mathfrak{S}_{U_{n,p}}]\!]$. We recall that Λ_n is just the Iwasawa algebra corresponding to elements of $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$ which are congruent to 1 mod p^n. Let $\mathfrak{m}$ be a maximal ideal in the support of $H^g_{\text{ét}}(Y_G(U_n)_{\overline{\mathbb{Q}}}, \mathbb{F}_p)$ such that the image of the associated Galois representation $\overline{\rho}_{\mathfrak{m}}$ is not solvable. Let λ

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denote a weight of G that is self-dual in the sense of Section 3.3. Let $\mathcal{O}[-\lambda]$ denote $\mathcal{O}$ with Λ acting via the inverse of the character λ. The first step in the proof of the control theorem is to analyze the $E_2^{i,j}$ term in Corollary 3.13. To do so, we first recall the following result from commutative algebra.

Proposition 4.3 *Let R be a commutative ring, let $x_1, \dots, x_N$ be elements of R, and let R^N be the free R-module of rank N with basis $\{e_i : 1 \le i \le N\}$. Consider the Koszul complex $K_\bullet(x_1, \dots, x_N)$ associated with $x_1, \dots, x_N$ given by*

$$0 \to \bigwedge^N R^N \xrightarrow{d} \bigwedge^{N-1} R^N \xrightarrow{d} \cdots \xrightarrow{d} \bigwedge^1 R^N \xrightarrow{d} R \to 0,$$

where

$$d(e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{s=1}^{i} (-1)^{s-1} x_{j_s} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_s}} \wedge \cdots \wedge e_{j_i}.$$

If $x_1, \dots, x_N$ form a regular sequence in R, then $K_\bullet(x_1, \dots, x_N)$ is a free-resolution of $R/(x_1, \dots, x_N)$ as an R-module.

Proof See for instance [23, Corollary 4.5.5]. ∎

Proposition 4.4 *We have that the $E_2^{i,j}$ term in Corollary 3.13 is zero unless $i \in \{0, -1, \dots, -g\}$.*

Proof Since $E_2^{i,j} = \mathrm{Tor}_{-i}^{\Lambda_n}(e'_{\mathrm{ord}} \varprojlim_s H^j_{\text{ét}}(Y_G(U_s)_{\overline{\mathbb{Q}}}, \mathcal{O}), \mathcal{O}[-\lambda])$, it suffices to construct a free resolution of $\mathcal{O}[-\lambda]$ as a Λ_n-module of length g. On the other hand, the Iwasawa algebra Λ_n can be identified with products of copies of $\mathcal{O}[[T_1, \dots, T_g]]$; it thus suffices to construct a free resolution of $\mathcal{O}[-\lambda]$ as an $\mathcal{O}[[T_1, \dots, T_g]]$ module of length g. To do this, we note that the sequence $T_1 - \lambda^{-1}(T_1), \dots, T_g - \lambda^{-1}(T_g)$ is a regular sequence in $O[[T_1, \dots, T_g]]$ and that

$$O[[T_1, \dots, T_g]]/(T_1 - \lambda^{-1}(T_1), \dots, T_g - \lambda^{-1}(T_g)) \cong \mathcal{O}[-\lambda]$$

as $\mathcal{O}[[T_1, \dots, T_g]]$-modules. We can thus apply Proposition 4.3 to deduce that the Koszul complex associated with $T_1 - \lambda^{-1}(T_1), \dots, T_g - \lambda^{-1}(T_g)$ provides a free resolution of $\mathcal{O}[-\lambda]$ as an $\mathcal{O}[[T_1, \dots, T_g]]$ module of length g. ∎

Remark 4.5 By using a similar argument as above, we can also deduce that $E_2^{i,j}$ term in Corollary 3.14 is zero unless $i \in \{0, -1, \dots, -g\}$.

Proposition 4.6 *The following hold:*

(a) *For all $n \ge 1$, we have that*

$$\mathrm{Tor}_{-i}^{\Lambda_n}(e'_{\mathrm{ord}} H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}}, \mathcal{O}[-\lambda]) = e'_{\mathrm{ord}} H^{i+g}_{\acute{e}t}(Y_G(U_n)_{\overline{\mathbb{Q}}}, V_\lambda)_{\mathfrak{m}}.$$

(b) *When $n = 0$, we have that*

$$\mathrm{Tor}_{-i}^{\Lambda}(e'_{\mathrm{ord}} H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}}, \mathcal{O}[-\lambda]) = e'_{\mathrm{ord}} H^{i+g}_{\acute{e}t}(Y_G(U_0)_{\overline{\mathbb{Q}}}, V_\lambda)_{\mathfrak{m}}.$$

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Proof (a) Since localization is exact and commutes with the Tor functor, we deduce from Corollary 3.13 the following spectral sequence:

$$F_2^{i,j} := (E_2^{i,j})_{\mathfrak{m}} = \mathrm{Tor}_{-i}^{\Lambda_n}(e'_{\mathrm{ord}}H^j_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}}, \mathcal{O}[-\lambda]) \implies e'_{\mathrm{ord}}H^{i+j}_{\text{ét}}(Y_G(U_n), V_\lambda)_{\mathfrak{m}}.$$

By Proposition 4.4 and Corollary 4.2, we deduce that $F_2^{i,j} = 0$ unless $(i,j) \in \{(0,g), \dots (-g,g)\}$. It follows that the spectral sequence degenerates on the second page, and thus, the desired isomorphism follows directly from applying the filtration theorem.

(b) Using the spectral sequence in Corollary 3.14 and Remark 4.5, this follows by using a similar argument as in part (a). ■

Theorem 4.7 *With the notation as above, the following hold:*

(a) *For all $n \geq 1$, we have that $e'_{ord}H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}}$ is free as a Λ_n-module.*

(b) *For all $n \geq 1$, we have an isomorphism of $\mathcal{O}$-modules*

$$e'_{ord}H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}} \otimes_{\Lambda_n} \mathcal{O}[-\lambda] \cong e'_{ord}H^g_{ét}(Y_G(U_n)_{\overline{\mathbb{Q}}}, V_\lambda)_{\mathfrak{m}}.$$

that is compatible with the action of $G_{\mathbb{Q},\Sigma}$ and $\mathbb{T}$.

(c) *When $n = 0$, we have a similar isomorphism*

$$e'_{\mathrm{ord}}H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}} \otimes_{\Lambda} \mathcal{O}[-\lambda] \cong e'_{\mathrm{ord}}H^g_{ét}(Y_G(U_0)_{\overline{\mathbb{Q}}}, V_\lambda)_{\mathfrak{m}}.$$

Proof We first deduce parts (b) and (c) from the results above.

(b) This follows by setting $i = 0$ in Proposition 4.6(a).

(c) This follows by setting $i = 0$ in Proposition 4.6(b).

We now prove part (a).

(a) Let π and k denote the uniformizer and residue field of $\mathcal{O}$, respectively. Consider the short-exact sequence

$$0 \to \mathcal{O} \xrightarrow{\cdot\pi} \mathcal{O} \twoheadrightarrow k \to 0.$$

This gives rise to a long-exact sequence of Tor groups:

$$\cdots \to \mathrm{Tor}_1^{\Lambda_n}(e'_{\mathrm{ord}}H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}}, \mathcal{O}) \xrightarrow{\cdot\pi} \mathrm{Tor}_1^{\Lambda_n}(e'_{\mathrm{ord}}H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}}, \mathcal{O}) \to$$
$$\mathrm{Tor}_1^{\Lambda}(e'_{\mathrm{ord}}H^g_{\text{ét}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}}, k) \to e'_{\mathrm{ord}}H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}} \otimes_{\Lambda_n} \mathcal{O}$$
$$\xrightarrow{\cdot\pi} H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}} \otimes_{\Lambda_n} \mathcal{O} \to \cdots$$

By setting $i = -1$ and λ to be the trivial character in Proposition 4.6, and then applying Corollary 4.2, we get that

$$\mathrm{Tor}_1^{\Lambda_n}(e'_{\mathrm{ord}}H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}}, \mathcal{O}) = 0.$$

On the other hand, Corollary 4.2 implies that $e'_{\mathrm{ord}}H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}} \otimes_{\Lambda_n} \mathcal{O}$ has no π-torsion, so the long exact sequence gives us that $\mathrm{Tor}_1^{\Lambda_n}(e'_{\mathrm{ord}}H^g_{\text{ét}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}}, k) = 0$. Finally, we note that part (b) and the topological Nakayama's lemma (see [1, p. 226]) imply that $e'_{\mathrm{ord}}H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}}$ is finitely generated as a Λ_n-module; we can now use the local criterion of flatness to conclude that $e'_{\mathrm{ord}}H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}}$ is flat as a Λ_n-module. Since a finitely generated flat module

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over a noetherian local ring is free, we have that $e'_{\mathrm{ord}}H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}}$ is free as a Λ_n-module. ∎

Corollary 4.8 *For all $n \geq 0$, we have $H^i_{\acute{e}t}(Y_G(U_n)_{\overline{\mathbb{Q}}}, V_\lambda)_{\mathfrak{m}} = 0$ when $i \neq g$.*

Proof When $n \geq 1$, we recall from Proposition 4.6(a) that $\mathrm{Tor}^{\Lambda_n}_{-i}(e'_{\mathrm{ord}}H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}}, \mathcal{O}[-\lambda]) = H^{i+g}_{\acute{e}t}(Y_G(U_n)_{\overline{\mathbb{Q}}}, V_\lambda)_{\mathfrak{m}}$. On the other hand, we know from Theorem 4.7 that $e'_{\mathrm{ord}}H^g_{\mathrm{Iw}}(Y_G(U_\infty)_{\overline{\mathbb{Q}}}, \mathcal{O})_{\mathfrak{m}}$ is free as a Λ_n-module. Thus, the above Tor group vanishes when $i \neq 0$ and so we conclude as desired that $H^i_{\acute{e}t}(Y_G(U_n)_{\overline{\mathbb{Q}}}, V_\lambda)_{\mathfrak{m}} = 0$ when $i \neq g$. The case $n = 0$ follows similarly using Proposition 4.6(b). ∎

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