Canad. Math. Bull. Vol. 58 (3), 2015 pp. 507–518
http://dx.doi.org/10.4153/CMB-2015-005-2

VMO Space Associated with Parabolic Sections and its Application

Ming-Hsiu Hsu and Ming-Yi Lee

Abstract. In this paper we define a space $VMO_{\mathcal{P}}$ associated with a family $\mathcal{P}$ of parabolic sections and show that the dual of $VMO_{\mathcal{P}}$ is the Hardy space $H^1_{\mathcal{P}}$. As an application, we prove that almost everywhere convergence of a bounded sequence in $H^1_{\mathcal{P}}$ implies weak* convergence.

1 Introduction

Caffarelli and Gutiérrez [4] introduced a family $\mathcal{F} = \{S(x,r) : x \in \mathbb{R}^n \text{ and } r > 0\}$ of open and bounded convex sets, called *sections*, in $\mathbb{R}^n$ satisfying certain axioms. The axioms are established on the properties of the solutions of the real Monge–Ampère equation,

$$\det D^2 u = f,$$

where $\det D^2 u$ denotes the determinant of the Hessian matrix $D^2 u$ of a function u in $\mathbb{R}^n$. Given a Borel measure μ that is finite on compact sets, $\mu(\mathbb{R}^n) = \infty$ and satisfies the *doubling property* with respect to $\mathcal{F}$; *i.e.,* there is a constant C such that

$$\mu\big(S(x,2r)\big) \le C\mu\big(S(x,r)\big), \quad \forall S(x,r) \in \mathcal{F}. \tag{1.1}$$

They showed a variant of the Calderón–Zygmund decomposition in terms of the elements of $\mathcal{F}$ by proving a Besicovitch-type covering lemma for the family $\mathcal{F}$ and using the doubling property of the measure μ. Sections and the decomposition are very important and useful in the study of the Monge–Ampère equation and the linearized Monge-Ampère equation (see [2, 3, 5]). As an application, they defined the Hardy–Littlewood maximal operator M and $BMO_{\mathcal{F}}(\mathbb{R}^n)$ space associated with a family $\mathcal{F}$ of sections and the Borel measure μ, and then obtained the weak type (1,1) boundedness of M and the John–Nirenberg inequality for $BMO_{\mathcal{F}}(\mathbb{R}^n)$ in [4]. Later, Ding and Lin [8] defined the Hardy space $H^1_{\mathcal{F}}(\mathbb{R}^n)$ associated with a family $\mathcal{F}$ of sections and the measure μ, and then showed that the dual space of $H^1_{\mathcal{F}}(\mathbb{R}^n)$ is the space $BMO_{\mathcal{F}}(\mathbb{R}^n)$. They also proved that the Monge–Ampère singular integral operator is bounded from $H^1_{\mathcal{F}}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n, d\mu)$.

Huang [9] showed a Besicovitch-type covering lemma and a variant of Calderón–Zygmund decomposition in terms of *parabolic sections*. A parabolic section $\widetilde{Q}(z,r)$ is defined by

$$\widetilde{Q}(z,r) = S(x,r) \times (t - r/2, t + r/2),$$

Received by the editors October 27, 2014; revised January 19, 2015.
Published electronically March 3, 2015.
AMS subject classification: **42B30**.
Keywords: Monge-Ampère equation, parabolic section, Hardy space, BMO, VMO.

where $z = (x,t) \in \mathbb{R}^n \times \mathbb{R}$, $r > 0$ and $S(x,r)$ is a section mentioned above. In Huang's article, parabolic sections are used to study the Harnack inequality of nonnegative solutions of the equation

$$L_\phi u = u_t - \operatorname{tr}\Big((D^2\phi(x))^{-1} D^2 u \Big) = 0.$$

Here, $u_t = \partial u/\partial t$, $D^2 u$ denotes the Hessian matrix of u in the x variable, $(D^2\phi(x))^{-1}$ is the inverse of the Hessian matrix of a strictly convex smooth function ϕ defined in $\mathbb{R}^n$, and $\operatorname{tr}(A)$ means the trace of the matrix A.

It is natural that we want to study the theory of Hardy spaces associated with parabolic sections. In fact, some results about Hardy spaces associated with *generalized parabolic sections* have been developed in [11,12]. Let $\phi: [0,\infty) \to [0,\infty)$ be a strictly increasing function such that

$$\phi(0) = 0, \quad \lim_{r\to\infty} \phi(r) = \infty, \quad \text{and} \quad \phi(2r) \le C\phi(r), \tag{1.2}$$

where C is a constant. A generalized parabolic section $Q(z,r)$ is defined by

$$Q(z,r) = S(x,r) \times (t - \phi(r)/2, t + \phi(r)/2),$$

where $z = (x,t) \in \mathbb{R}^{n+1}$, $r > 0$ and $S(x,r)$ is a section. A parabolic section is a generalized parabolic section with $\phi(r) = r$. From now on, we call $Q(z,r)$ a parabolic section for simplicity. The space $BMO_{\mathcal{P}}(\mathbb{R}^{n+1})$ and the Hardy space $H^1_{\mathcal{P}}(\mathbb{R}^{n+1})$ associated with a family $\mathcal{P}$ of parabolic sections have been defined in [12], and it is proved that the dual space of $H^1_{\mathcal{P}}(\mathbb{R}^{n+1})$ is $BMO_{\mathcal{P}}(\mathbb{R}^{n+1})$. In [11], the authors showed the John–Nirenberg inequality for $BMO_{\mathcal{P}}(\mathbb{R}^{n+1})$. In this paper, we will show that the Hardy space $H^1_{\mathcal{P}}(\mathbb{R}^{n+1})$ has a predual (Theorem 2.1), and then we prove that the almost everywhere convergence of a bounded sequence in $H^1_{\mathcal{P}}(\mathbb{R}^{n+1})$ implies weak* convergence (Theorem 2.2).

2 Preliminaries

Let us first recall the definition and some properties of sections. For every x in $\mathbb{R}^n$, denote by $\{S(x,r) : r > 0\}$ the one-parameter of open and bounded convex sets in $\mathbb{R}^n$ containing x. A collection $\mathcal{F} = \{S(x,r) : x \in \mathbb{R}^n \text{ and } r > 0\}$ is called a family of *sections* if it is monotonic increasing in r, *i.e.,* $S(x,r) \subset S(x,r')$ for $r \le r'$, and satisfies the following conditions:

(a) There exist positive constants $K_1, K_2, K_3, \epsilon_1$, and ϵ_2 such that given two sections $S(x_0,r_0)$ and $S(x,r)$ with $r \le r_0$ such that

$$S(x_0,r_0) \cap S(x,r) \ne \varnothing,$$

and given T an affine transformation that *normalizes* $S(x_0,r_0)$, *i.e.,*

$$B(0,1/n) \subset T\big(S(x_0,r_0)\big) \subset B(0,1),$$

where $B(x,r)$ denotes the Euclidean ball centered at x with radius r, there exists $x' \in B(0,K_3)$ depending $S(x_0,r_0)$ and $S(x,r)$ such that

$$B\Big(x', K_2\Big(\frac{r}{r_0}\Big)^{\epsilon_2}\Big) \subset T\big(S(x,r)\big) \subset B\Big(x', K_1\Big(\frac{r}{r_0}\Big)^{\epsilon_1}\Big),$$

https://doi.org/10.4153/CMB-2015-005-2 Published online by Cambridge University Press

and

$$Tx \in B\Big(x', \frac{1}{2}K_2\Big(\frac{r}{r_0}\Big)^{\epsilon_2}\Big).$$

(b) There exists $\delta > 0$ such that given a section $S(x_0, r)$ and $x \notin S(x_0, r)$. If T is an affine transformation that normalizes $S(x_0, r)$, then

$$B(T(x), \epsilon^\delta) \cap T\big(S(x_0, (1-\epsilon)r)\big) = \varnothing, \quad \text{for } 0 < \epsilon < 1.$$

(c) $\bigcap_{r>0} S(x, r) = \{x\}$ and $\bigcup_{r>0} S(x, r) = \mathbb{R}^n$.

Aimar, Forzani and Toledano obtained in [1] the following *engulfing property* for sections, *i.e.*, there is a constant $\theta \geq 1$, depending on δ, K_1 and ϵ_1, such that for $y \in S(x, r)$,

$$S(x, r) \subset S(y, \theta r) \quad \text{and} \quad S(y, r) \subset S(x, \theta r). \tag{2.1}$$

Also, they showed that there is a quasi-metric ρ on $\mathbb{R}^n$, defined by

$$\rho(x, y) = \inf\{t : x \in S(y, t) \text{ and } y \in S(x, t)\}, \tag{2.2}$$

such that

$$S(x, r/2\theta) \subset B_\rho(x, r) \subset S(x, r), \quad \forall S(x, r) \in \mathcal{F},$$

where $B_\rho(x, r) = \{y \in \mathbb{R}^n : \rho(x, y) < r\}$.

Let $\phi: [0, \infty) \to [0, \infty)$ be a strictly increasing function satisfying equation (1.2). For $z = (x, t)$ in $\mathbb{R}^{n+1}$ and $r > 0$, recall that a *parabolic section* $Q(z, r)$ is defined by

$$Q(z, r) = S(x, r) \times \big(t - \phi(r)/2, t + \phi(r)/2\big).$$

Given a parabolic section $Q(z_0, r_0)$, let T be an affine transformation that normalizes $S(x_0, r_0)$. Define a map $T_p: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by $T_p(x, t) = (Tx, (t - t_0)/\phi(r_0))$, we have

$$K(0, 1/n) \subset T_p(Q(z_0, r_0)) \subset K(0, 1),$$

where $K(z, r) = B(x, r) \times (t - r^2/2, t + r^2/2)$ is the usual parabolic cylinder. The set $T_p(Q(z_0, r_0))$ will be called the *normalization* of $Q(z_0, r_0)$ and T_p an affine transformation that *normalizes* $Q(z_0, r_0)$. By the definition of sections, it is clear that each parabolic section $Q(z, r)$ is an open and bounded convex set in $\mathbb{R}^{n+1}$ containing z, and the family $\mathcal{P} = \{Q(z, r) : z \in \mathbb{R}^{n+1} \text{ and } r > 0\}$ of parabolic sections is monotonic increasing in r and satisfies the following conditions:

(A) There exist positive constants $K_1, K_2, K_3, \epsilon_1$, and ϵ_2 such that given two parabolic sections $Q(z_0, r_0)$ and $Q(z, r)$ with $r \leq r_0$ such that

$$Q(z_0, r_0) \cap Q(z, r) \neq \varnothing,$$

and given T_p an affine transformation that normalizes $Q(z_0, r_0)$, there exists $z' = (x', t') \in K(0, K_3)$ such that

$$\begin{aligned} B\Big(x', K_2\Big(\frac{r}{r_0}\Big)^{\epsilon_2}\Big) \times \Big(t' - \frac{\phi(r)}{2\phi(r_0)}, t' + \frac{\phi(r)}{2\phi(r_0)}\Big) &\subset T_p\big(Q(z, r)\big) \\ &\subset B\Big(x', K_1\Big(\frac{r}{r_0}\Big)^{\epsilon_1}\Big) \times \Big(t' - \frac{\phi(r)}{2\phi(r_0)}, t' + \frac{\phi(r)}{2\phi(r_0)}\Big), \end{aligned}$$

and

$$T_p z = (Tx, t') \in B\Big(x', \frac{1}{2}K_2\Big(\frac{r}{r_0}\Big)^{\epsilon_2}\Big) \times \{t'\}.$$

https://doi.org/10.4153/CMB-2015-005-2 Published online by Cambridge University Press

(B) There exists $\delta > 0$ such that given a parabolic section $Q(z_0, r)$ and $z \notin Q(z_0, r)$. If T_p is an affine transformation that normalizes $Q(z_0, r)$, then

$$K(T_p(z), \epsilon^{\delta}) \cap T_p(Q(z_0, (1-\epsilon)r)) = \varnothing, \quad \text{for } 0 < \epsilon < 1.$$

(C) $\bigcap_{r>0} Q(z, r) = \{z\}$ and $\bigcup_{r>0} Q(z, r) = \mathbb{R}^{n+1}$.

Similar to equations (2.1) and (2.2), the engulfing property holds for parabolic sections; *i.e.,* there is a constant $\theta \geq 1$, depending on δ, K_1 and ϵ_1, such that for $z \in Q(z_0, r)$,

$$Q(z_0, r) \subset Q(z, \theta r) \quad \text{and} \quad Q(z, r) \subset Q(z_0, \theta r), \tag{2.3}$$

and there is a quasi-metric d on $\mathbb{R}^{n+1}$ such that

$$Q(z, r/2\theta) \subset B^d(z, r) \subset Q(z, r), \quad \forall Q(z, r) \in \mathcal{P}, \tag{2.4}$$

where $B^d(z, r) = \{w \in \mathbb{R}^{n+1} : d(z, w) < r\}$.

Denote by Lip $:=$ Lip$(\mathbb{R}^{n+1})$ the collection of functions on $\mathbb{R}^{n+1}$ satisfying that there is a constant C such that

$$|f(z) - f(w)| \leq C d(z, w), \quad \forall z, w \in \mathbb{R}^{n+1}.$$

We assumed that a Borel measure μ which is finite on compact sets, $\mu(\mathbb{R}^n) = \infty$ and satisfies the *doubling property* (equation (1.1)) is given. Let $\mathcal{M}$ be a measure on $\mathbb{R}^{n+1}$ defined by $d\mathcal{M} = d\mu dt$. It is easy to see that the measure $\mathcal{M}$ is finite on compact sets, $\mathcal{M}(\mathbb{R}^{n+1}) = \infty$ and satisfies the *doubling property with respect to* $\mathcal{P}$; *i.e.,* there is a constant C such that

$$\mathcal{M}\big(Q(z, 2r)\big) \leq C_{\mathcal{M}}\big(Q(z, r)\big), \quad \forall Q(z, r) \in \mathcal{P}. \tag{2.5}$$

A function f defined on $\mathbb{R}^{n+1}$ is said to be in $BMO_{\mathcal{P}} := BMO_{\mathcal{P}}(\mathbb{R}^{n+1})$ if

$$\|f\|_{BMO_{\mathcal{P}}} := \sup_{Q \in \mathcal{P}} \frac{1}{\mathcal{M}(Q)} \int_Q |f(z) - m_Q(f)| \, d\mathcal{M}(z) < \infty,$$

where $m_Q(f)$ denotes the mean of f over the parabolic section Q defined by

$$m_Q(f) = \frac{1}{\mathcal{M}(Q)} \int_Q f(z) \, d\mathcal{M}(z).$$

A function a in $L^\infty(d\mathcal{M}) := L^\infty(\mathbb{R}^{n+1}, d\mathcal{M})$ is called an *atom* if there exists a parabolic section $Q(z_0, r_0) \in \mathcal{P}$ such that

(a) $\operatorname{supp}(a) \subseteq Q(z_0, r_0)$;
(b) $\int_{\mathbb{R}^{n+1}} a(z) \, d\mathcal{M}(z) = 0$;
(c) $\|a\|_{L^\infty(d\mathcal{M})} \leq [\mathcal{M}(Q(z_0, r_0))]^{-1}$.

The Hardy space $H^1_{\mathcal{P}} := H^1_{\mathcal{P}}(\mathbb{R}^{n+1})$ is defined by

$$H^1_{\mathcal{P}} = \Big\{ \sum_j \lambda_j a_j : \text{ each } a_j \text{ is an atom and } \sum_j |\lambda_j| < \infty \Big\}.$$

The norm of f in $H^1_{\mathcal{P}}$ is defined by

$$\|f\|_{H^1_{\mathcal{P}}} = \inf \sum_j |\lambda_j|,$$

where the infimum is taken over all decomposition of $f = \sum_j \lambda_j a_j$ above.

Denote by $C_c := C_c(\mathbb{R}^{n+1})$ the space of continuous functions on $\mathbb{R}^{n+1}$ with compact support. Let $VMO_{\mathcal{P}} := VMO_{\mathcal{P}}(\mathbb{R}^{n+1})$ be the closure of $C_c \cap \mathrm{Lip}$ with respect to the seminorm $\|\cdot\|_{BMO_{\mathcal{P}}}$. Our main result follows.

Theorem 2.1 *$H^1_{\mathcal{P}}(\mathbb{R}^{n+1})$ is the dual space of $VMO_{\mathcal{P}}(\mathbb{R}^{n+1})$.*

As an application, we prove that almost everywhere convergence of a bounded sequence in $H^1_{\mathcal{P}}$ implies weak* convergence. This is an $H^1_{\mathcal{P}}$ version of the Jones–Journé theorem [10]. The Jones–Journé theorem is useful in the application of Hardy spaces to compensated compactness (see [7]).

Theorem 2.2 *Let $\{f_k\}$ be a bounded sequence in $H^1_{\mathcal{P}}(\mathbb{R}^{n+1})$. If f_k converges to f $\mathcal{M}$-almost everywhere, then $f \in H^1_{\mathcal{P}}(\mathbb{R}^{n+1})$ and f_k weak* converges to f;* i.e.,

$$\int_{\mathbb{R}^{n+1}} f_k(x)\phi(x)\,d\mathcal{M}(x) \longrightarrow \int_{\mathbb{R}^{n+1}} f(x)\phi(x)\,d\mathcal{M}(x), \quad \forall\, \phi \in VMO_{\mathcal{P}}(\mathbb{R}^{n+1}).$$

3 Proofs

Lemma 3.1 *For each $m \in \mathbb{Z}$, there is a sequence $\{z_j^m\}_{j\in\mathbb{N}}$ such that $\mathbb{R}^{n+1}$ is the union of parabolic sections $\{Q(z_j^m, \theta^{2m}) : j \in \mathbb{N}\}$ that are finitely overlapping. Moreover, every $f \in H^1_{\mathcal{P}}$ has the representation*

$$f = \sum_{j=1}^{\infty} \sum_{m=-\infty}^{\infty} \lambda_j^m a_j^m,$$

where a_j^m is an atom with support in $Q(z_j^m, \theta^{2m+2})$ and $\sum_{j\in\mathbb{N}, m\in\mathbb{Z}} |\lambda_j^m| \le C\|f\|_{H^1_{\mathcal{P}}}$.

Proof For $m \in \mathbb{Z}$, let z_1^m be an arbitrary point in $\mathbb{R}^{n+1}$. By the engulfing property of the parabolic sections (equation (2.3)), if $Q(z, \theta^{2m-2}) \cap Q(z_1^m, \theta^{2m-2}) \neq \varnothing$, then

$$Q(z, \theta^{2m-2}) \subset Q(z', \theta^{2m-1}) \subset Q(z_1^m, \theta^{2m}), \quad \forall z' \in Q(z, \theta^{2m-2}) \cap Q(z_1^m, \theta^{2m-2}).$$

Let $z_2^m \in \mathbb{R}^{n+1}$ such that $Q(z_2^m, \theta^{2m-2}) \cap Q(z_1^m, \theta^{2m-2}) = \varnothing$. By the engulfing property again, we have, for all $Q(z, \theta^{2m-2})$ with $(z, \theta^{2m-2}) \cap Q(z_2^m, \theta^{2m-2}) \neq \varnothing$,

$$Q(z, \theta^{2m-2}) \subset Q(z_2^m, \theta^{2m}).$$

Let $z_j^m \in \mathbb{R}^{n+1}$ such that $Q(z_j^m, \theta^{2m-2}) \cap [\bigcup_{i=1}^{j-1} Q(z_i^m, \theta^{2m-2})] = \varnothing$. By the engulfing property again, we have, for all $Q(z, \theta^{2m-2})$ with $(z, \theta^{2m-2}) \cap Q(z_j^m, \theta^{2m-2}) \neq \varnothing$,

$$Q(z, \theta^{2m-2}) \subset Q(z_2^m, \theta^{2m}).$$

Note that if no such z_j^m exists then the parabolic sections $\{Q(z_i^m, \theta^{2m})\}_{i=1}^{j-1}$ are finitely overlapping by equation (2.4) and the disjointness of the collection

$$\{Q(z_i^m, \theta^{2m-2})\}_{i=1}^{j-1},$$

whose union is $\mathbb{R}^{n+1}$. Otherwise, continue the same argument to select z_{j+1}^m. Thus, we can find $\{z_j^m\}_{i=1}^{N_m}$, for all m in $\mathbb{Z}$, such that the parabolic sections $\{Q(z_i^m, \theta^{2m})\}_{i=1}^{N_m}$ are finitely overlapping and whose union is $\mathbb{R}^{n+1}$, where N_m can be finite or infinite.

https://doi.org/10.4153/CMB-2015-005-2 Published online by Cambridge University Press

Let $f \in H^1_{\mathcal{P}}$ with representation $f = \sum_k \lambda_k a_k$, where $\sum_k |\lambda_k| < \infty$ and each a_k is an atom with support contained in $Q(z_k, t_k)$. Let $m = m(k)$ be the smallest integer such that $Q(z_k, t_k) \subset Q(z_k, \theta^{2m})$. Let $i = i(k)$ be the integer such that

$$Q(z_k, \theta^{2m}) \cap \Big[\bigcup_{j=1}^{i-1} Q(z_j^m, \theta^{2m})\Big] = \varnothing \quad \text{and} \quad Q(z_k, \theta^{2m}) \cap Q(z_i^m, \theta^{2m}) \neq \varnothing.$$

Let $\psi: \mathbb{N} \to \mathbb{N} \times \mathbb{Z}$ be a function defined by $\psi(k) = (i(k), m(k))$. If $\psi^{-1}(i, m) = \varnothing$, define $\lambda_i^m = 0$. Otherwise, let $\lambda_k a_k = \lambda_i^m a_i^m$, where

$$\lambda_i^m = \mathcal{M}\big(Q(z_i^m, \theta^{2m+2})\big) \frac{\lambda_k}{\mathcal{M}(Q(z_k, t_k))}.$$

Then $\operatorname{supp}(a_i^m) \subset Q(z_i^m, \theta^{2m+2})$,

$$|a_i^m| = \frac{1}{\lambda_i^m} |\lambda_k||a_k| \le \frac{1}{\lambda_i^m} \frac{|\lambda_k|}{\mathcal{M}(Q(z_k, t_k))} = \mathcal{M}(Q(z_i^m, \theta^{2m+2}))^{-1},$$

and hence a_i^m is an atom with $\operatorname{supp}(a_i^m) \subset Q(z_i^m, \theta^{2m+2})$. By the engulfing property, for $Q(z_k, \theta^{2m}) \cap Q(z_i^m, \theta^{2m}) \neq \varnothing$, we have $Q(z_k, \theta^{2m+2}) \cap Q(z_i^m, \theta^{2m+2}) \neq \varnothing$ and hence $Q(z_i^m, \theta^{2m+2}) \subset Q(z_k, \theta^{2m+4})$. By the doubling property (equation (2.5)), there is a constant C' such that

$$\begin{aligned} |\lambda_i^m| &= \mathcal{M}\big(Q(z_i^m, \theta^{2m+2})\big) \frac{|\lambda_k|}{\mathcal{M}(Q(z_k, \theta^{2m}))} \\ &\le \mathcal{M}\big(Q(z_k, \theta^{2m+4})\big) \frac{|\lambda_k|}{\mathcal{M}(Q(z_k, \theta^{2m}))} \le C'|\lambda_k|. \end{aligned}$$

Therefore,

$$\sum_{i\in\mathbb{N}, m\in\mathbb{Z}} |\lambda_i^m| \le C' \sum_{k:\psi(k)=(i,m)} |\lambda_k| \le C' \sum_{k\in\mathbb{N}} |\lambda_k| \le C' \|f\|_{H^1_{\mathcal{P}}}.$$

This completes the proof. ∎

Lemma 3.2 *Let $\{f_k\}$ be a bounded sequence in $H^1_{\mathcal{P}}$. Then there is a subsequence $\{f_{k_l}\}$ and $f \in H^1_{\mathcal{P}}$ such that*

$$\lim_{l\to\infty} \int_{\mathbb{R}^{n+1}} f_{k_l} g \, d\mathcal{M} = \int_{\mathbb{R}^{n+1}} f g \, d\mathcal{M} \quad \text{for all } g \in C_c(\mathbb{R}^{n+1}). \tag{3.1}$$

Proof We can assume that $\|f_k\|_{H^1_{\mathcal{P}}} \le 1$ for all k. By Lemma 3.1, let

$$f_k = \sum_{i=1}^{\infty} \sum_{m=-\infty}^{\infty} \lambda_i^m(k) a_i^m(k),$$

where

$$\sum_{i\in\mathbb{N}, m\in\mathbb{Z}} |\lambda_i^m(k)| \le C\|f_k\|_{H^1_{\mathcal{P}}} \le C,$$

each $a_i^m(k)$ is an atom with support contained in $Q(z_i^m, \theta^{2m+2})$, and

$$\|a_i^m(k)\|_{L^\infty(d\mathcal{M})} \le \mathcal{M}(Q(z_i^m, \theta^{2m+2}))^{-1} \quad \text{for all } k.$$

By [6, Lemma 4.3], there is a subsequence $\lambda_i^m(k_l)$ such that $\lim_{l\to\infty} \lambda_i^m(k_l) = \lambda_i^m$ for each $(i, m) \in \mathbb{N} \times \mathbb{Z}$ and $\sum_{i,m} |\lambda_i^m| \le C$. Since $\{a_i^m(k)\}$ is bounded in $L^\infty(d\mathcal{M})$, which is the dual of $L^1(d\mathcal{M})$, the Banach–Alaoglu theorem shows that there exists a subsequence $\{a_i^m(k_l)\}$ that weak* converges to a function a_i^m with $\|a_i^m\|_{L^\infty(d\mathcal{M})} \leq \mathcal{M}(Q(z_i^m,\theta^{2m+2}))^{-1}$. By [6, Lemma 4.3] again, there exists a subsequence of $\{k_l\}$ (still denoted by $\{k_l\}$ for simplicity) such that $\{a_i^m(k_l)\}_{l\in\mathbb{N}}$ converges to a_i^m, as $l \to \infty$, for all (i,m). It is easy to check that each a_i^m is an atom. Let $f = \sum_{i,m} \lambda_i^m a_i^m$. Since $\sum_{i,m} |\lambda_i^m| \leq C$, we have $f \in H^1_{\mathcal{P}}(\mathbb{R}^{n+1})$.

To show (3.1), we write

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} f_{k_l} g\, d\mathcal{M} &= \int_{\mathbb{R}^{n+1}} \sum_{i,m} \lambda_i^m(k_l) a_i^m(k_l) g\, d\mathcal{M}\\ &= \sum_m \sum_i \lambda_i^m(k_l) \int_{\mathbb{R}^{n+1}} a_i^m(k_l) g\, d\mathcal{M}\\ &= \Big(\sum_{m<-M} + \sum_{-M\leq m\leq M} + \sum_{m>M}\Big) \sum_i \lambda_i^m(k_l) \int_{\mathbb{R}^{n+1}} a_i^m(k_l) g\, d\mathcal{M}.\end{aligned}$$

Given $\epsilon > 0$, let M be a large number such that

$$|g(z) - g(z_i^m)| < \epsilon, \quad \forall x \in Q(z_i^m,\theta^{2m+2}), m < -M.$$

Then

$$\begin{aligned}&\Big|\sum_{m<-M} \sum_i \lambda_i^m(k_l) \int_{Q(z_i^m,\theta^{2m+2})} a_i^m(k_l)(z)\big\{g(z) - g(z_i^m)\big\}\, d\mathcal{M}(z)\Big|\\ &\quad\leq \sum_{m<-M} \sum_i |\lambda_i^m(k_l)| \|a_i^m(k_l)\|_{L^\infty(d\mathcal{M})} \mathcal{M}\big(Q(z_i^m,\theta^{2m+2})\big)\epsilon\\ &\quad\leq C\epsilon.\end{aligned}$$

For each m with $-M \leq m \leq M$, the compact support of g intersects a finite number of $\{Q(z_i^m,\theta^{2m+2})\}_{i\in\mathbb{N}}$, since $\{Q(z_i^m,\theta^{2m})\}_{i\in\mathbb{N}}$ are finitely overlapping by Lemma 3.1. Thus,

$$\begin{aligned}&\sum_{-M\leq m\leq M} \sum_i \lambda_i^m(k_l) \int_{\mathbb{R}^{n+1}} a_i^m(k_l) g\, d\mathcal{M} =\\ &\qquad\qquad \int_{\mathbb{R}^{n+1}} \sum_{-M\leq m\leq M} \sum_i \lambda_i^m(k_l) a_i^m(k_l) g\, d\mathcal{M} \longrightarrow \int_{\mathbb{R}^{n+1}} f g\, d\mathcal{M}\end{aligned}$$

as $l \to \infty$ and $M \to \infty$. Note that

$$\sum_{i,m} |\lambda_i^m(k)| \|a_i^m(k)\|_{L^1(d\mathcal{M})} \|g\|_{L^\infty(d\mathcal{M})} \leq C\|g\|_{L^\infty(d\mathcal{M})}.$$

Given $\epsilon > 0$, we have, for large M,

$$\begin{aligned}&\Big|\sum_{m>M} \sum_i \lambda_i^m(k_l) \int_{\mathbb{R}^{n+1}} a_i^m(k_l) g\, d\mathcal{M}\Big|\\ &\qquad\qquad \leq \sum_{m>M} \sum_i |\lambda_i^m(k)| \|a_i^m(k)\|_{L^1(d\mathcal{M})} \|g\|_{L^\infty(d\mathcal{M})} < \epsilon.\end{aligned}$$

The proof is complete. ∎

Proof of Theorem 2.1 By definition, $VMO_{\mathcal{P}}$ is a subspace of $BMO_{\mathcal{P}}$. Since $BMO_{\mathcal{P}}$ is the dual space of $H^1_{\mathcal{P}}$ by [12, Theorem 1.2], the space $H^1_{\mathcal{P}}$ is a subspace of $VMO^*_{\mathcal{P}}$. Conversely, we note that, if $\langle f, g\rangle = 0$ for all $f \in H^1_{\mathcal{P}}$, then g is the zero element of

https://doi.org/10.4153/CMB-2015-005-2 Published online by Cambridge University Press

$BMO_{\mathcal{P}}$, and hence g is the zero of $VMO_{\mathcal{P}}$. Thus, $H^1_{\mathcal{P}}$ is a total set of functionals on $VMO_{\mathcal{P}}$. This shows that $H^1_{\mathcal{P}}$ is dense in $VMO^*_{\mathcal{P}}$ in the weak*-topology. For each $x^* \in VMO^*_{\mathcal{P}}$, there exists a sequence $\{f_k\}$ in $H^1_{\mathcal{P}}$ such that $\langle f_k, g\rangle \to \langle x^*, g\rangle$ for all $g \in VMO_{\mathcal{P}}$. It follows from the Banach–Steinhaus theorem that $\{\|f_k\|_{H^1_{\mathcal{P}}}\}$ is bounded. By Lemma 3.2, there exists $f \in H^1_{\mathcal{P}}$ and a subsequence $\{f_{k_l}\}_{l\in\mathbb{N}}$ such that

$$\begin{aligned}\langle x^*, g\rangle &= \lim_{l\to\infty}\langle f_{k_l}, g\rangle = \lim_{l\to\infty}\int_{\mathbb{R}^{n+1}} f_{k_l} g\, d\mathcal{M}\\ &= \int_{\mathbb{R}^{n+1}} f g\, d\mathcal{M} = \langle f, g\rangle, \quad \forall g \in C_c(\mathbb{R}^{n+1}).\end{aligned}$$

Thus, the linear functional $x^* \in VMO^*_{\mathcal{P}}$ is represented by $f \in H^1_{\mathcal{P}}$. The proof is complete. ∎

The Hardy–Littlewood maximal function with respect to a family $\mathcal{P}$ and the measure $\mathcal{M}$ is defined as follows:

$$Mf(z) = \sup_{r>0}\frac{1}{\mathcal{M}(Q(z,r))}\int_{Q(z,r)}|f(w)|\, d\mathcal{M}(w).$$

Lemma 3.3 ([12, Lemma 2.2]) *The Hardy–Littlewood maximal operator M is of weak-type (1,1) with respect to the measure $\mathcal{M}$;* i.e., *there exists a constant $C > 0$ such that*

$$\mathcal{M}\big(\{z : Mf(z) > \lambda\}\big) \le \frac{C}{\lambda}\|f\|_{L^1(d\mathcal{M})}.$$

The *noncentered* Hardy-Littlewood maximal operator $\widetilde{M}$ with respect to $\mathcal{P}$ and the measure $\mathcal{M}$ is defined by

$$\widetilde{M}f(z) = \sup_{z\in Q\in\mathcal{P}}\frac{1}{\mathcal{M}(Q)}\int_Q |f(z)|\, d\mathcal{M}(z).$$

By the doubling property (2.5), it is easy to see that there is a constant C such that

$$Mf \le \widetilde{M}f \le CMf, \tag{3.2}$$

and hence $\widetilde{M}$ is of weak type $(1,1)$ with respect to the measure $\mathcal{M}$.

A nonnegative locally integrable function ω is said to belong to $A_{p,\mathcal{P}}$, $1 < p < \infty$, if

$$\sup_{Q\in\mathcal{P}}\Big(\frac{1}{\mathcal{M}(Q)}\int_Q \omega(z)\, d\mathcal{M}(z)\Big)\Big(\frac{1}{\mathcal{M}(Q)}\int_Q \omega(z)^{-\frac{1}{p-1}}\, d\mathcal{M}(z)\Big)^{p-1} < \infty,$$

and ω is said to belong to $A_{1,\mathcal{P}}$ if

$$\sup_{Q\in\mathcal{P}}\Big(\frac{1}{\mathcal{M}(Q)}\int_Q \omega(z)\, d\mathcal{M}(z)\Big)\Big(\operatorname*{ess\,sup}_{z\in Q}\omega^{-1}(z)\Big) < \infty.$$

Lemma 3.4 *Let $f \in L^1_{loc}(\mathbb{R}^{n+1})$ such that $\widetilde{M}f(z) < \infty$ $\mathcal{M}$-almost everywhere. Then $(\widetilde{M}f)^\delta \in A_{1,\mathcal{P}}$ for $0 \le \delta < 1$.*

Proof It suffices to show that there exists a constant C' such that, for any $Q \in \mathcal{P}$ and $\mathcal{M}$-almost every $z \in Q$,

$$\frac{1}{\mathcal{M}(Q)} \int_Q (\widetilde{M} f)^\delta d\mathcal{M} \leq C' \big(\widetilde{M} f(z) \big)^\delta.$$

Let $Q = Q(z_0, r_0) \in \mathcal{P}$. Let $f = f_1 + f_2$, where $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$ with $2Q = Q(z_0, 2r_0)$. Then $\widetilde{M} f \leq \widetilde{M} f_1 + \widetilde{M} f_2$ and

$$(\widetilde{M} f)^\delta \leq (\widetilde{M} f_1)^\delta + (\widetilde{M} f_2)^\delta, \quad \forall\, 0 \leq \delta < 1.$$

Since $\widetilde{M}$ is weak $(1,1)$ with respect to the measure $\mathcal{M}$, Kolmogorov's inequality shows that

$$\begin{aligned} \frac{1}{\mathcal{M}(Q)} \int_Q (\widetilde{M} f_1)^\delta d\mathcal{M} &\leq \frac{C}{\mathcal{M}(Q)} \mathcal{M}(Q)^{1-\delta} \|f_1\|^\delta_{L^1(d\mathcal{M})} \\ &\leq C\Big(\frac{1}{\mathcal{M}(Q)} \int_{2Q} f \, d\mathcal{M} \Big)^\delta \leq C\big(\widetilde{M} f(z) \big)^\delta. \end{aligned}$$

To estimate $\widetilde{M} f_2$, given $w \in Q$ and for any $Q(w_0, R) \in \mathcal{P}$ that contains w, we have $Q \subset Q(w_0, \theta^2 \max\{r_0, R\})$. If $R < r_0$, we have $Q(w_0, r_0) \cap Q(z_0, r_0) \neq \varnothing$, and hence $Q(w_0, r_0) \subset Q(z_0, \theta^2 r_0)$. By equation (2.4), we have $B^d(w_0, r_0) \subset B^d(z_0, 2\theta^3 r_0)$, and hence $B^d(w_0, \frac{r_0}{\theta^3}) \subset B^d(z_0, 2r_0) \subset Q(z_0, 2r_0)$. Then the inequality $\int_{Q(w_0,R)} |f_2| \, d\mathcal{M} > 0$ implies that $R > \frac{r_0}{\theta^3}$, and hence $Q \subset Q(w_0, \theta^5 R)$ when $R < r_0$. It is clear that $Q \subset Q(w_0, \theta^5 R)$ when $R \geq r_0$. Thus,

$$\frac{1}{\mathcal{M}(Q(w_0,R))} \int_{Q(w_0,R)} |f_2| \, d\mathcal{M} \leq \frac{C}{\mathcal{M}(Q(w_0, \theta^5 R))} \int_{Q(w_0,\theta^5 R)} |f_2| \, d\mathcal{M} \leq C \widetilde{M} f(z),$$

so that $\widetilde{M} f_2(w) \leq C \widetilde{M} f(z)$ for any $w \in Q$. Therefore,

$$\frac{1}{\mathcal{M}(Q)} \int_Q \big(\widetilde{M} f_2(w) \big)^\delta d\mathcal{M}(w) \leq C\big(\widetilde{M} f(z) \big)^\delta.$$

The proof is complete. ∎

Lemma 3.5 *If $\omega \in A_{2,\mathcal{P}}$, then $\log \omega \in BMO_{\mathcal{P}}$.*

Proof Let $f = \log \omega$. Then $\exp(f) \in A_{2,\mathcal{P}}$. By Jensen's inequality, for any $Q \in \mathcal{P}$,

$$1 = \exp\Big(\frac{1}{\mathcal{M}(Q)} \int_Q (f - m_Q(f)) d\mathcal{M} \Big) \leq \frac{1}{\mathcal{M}(Q)} \int_Q \exp\big(f - m_Q(f) \big) \, d\mathcal{M},$$

and hence

$$\begin{aligned} &\frac{1}{\mathcal{M}(Q)} \int_Q \exp\big(f - m_Q(f) \big) \, d\mathcal{M} \\ &\quad \leq \Big(\frac{1}{\mathcal{M}(Q)} \int_Q \exp(f - m_Q(f)) d\mathcal{M} \Big) \Big(\frac{1}{\mathcal{M}(Q)} \int_Q \exp(m_Q(f) - f) d\mathcal{M} \Big) \\ &\quad = \Big(\frac{1}{\mathcal{M}(Q)} \int_Q \exp(f) d\mathcal{M} \Big) \Big(\frac{1}{\mathcal{M}(Q)} \int_Q \exp(-f) d\mathcal{M} \Big) \leq C. \end{aligned}$$

Similarly,

$$\frac{1}{\mathcal{M}(Q)} \int_Q \exp\big(m_Q(f) - f \big) \, d\mathcal{M} \leq C.$$

https://doi.org/10.4153/CMB-2015-005-2 Published online by Cambridge University Press

Therefore,

$$\begin{aligned}\frac{1}{\mathcal{M}(Q)}\int_Q |f-m_Q(f)|\,d\mathcal{M}\\ &\le \frac{1}{\mathcal{M}(Q)}\int_Q \exp\big(|f-m_Q(f)|\big)\,d\mathcal{M}\\ &\le \frac{1}{\mathcal{M}(Q)}\int_Q \exp\big(f-m_Q(f)\big)\,d\mathcal{M}+\frac{1}{\mathcal{M}(Q)}\int_Q \exp\big(m_Q(f)-f\big)\,d\mathcal{M}\le 2C.\end{aligned}$$

Hence, $f\in BMO_{\mathcal{P}}$. ∎

Proof of Theorem 2.2 It is suffices to show that

$$\int_{\mathbb{R}^{n+1}} f_k\phi\,d\mathcal{M}\longrightarrow\int_{\mathbb{R}^{n+1}} f\phi\,d\mathcal{M},\quad \forall\phi\in\phi\in C_c\cap\mathrm{Lip}. \tag{3.3}$$

Assume that $\|f_k\|\le 1$. Let $\phi\in C_c\cap\mathrm{Lip}$. Without loss of generality, we can assume that $\|\phi\|_{L^1(d\mathcal{M})}\le 1$, $\|\phi\|_{L^\infty(d\mathcal{M})}\le 1$, and $|\phi(z)-\phi(z')|\le d(z,z')$ for all $z,z'\in\mathbb{R}^{n+1}$. Let $\delta\in(0,1/2\theta)$ and $\eta>0$ such that $\eta\exp(\delta^{-1})\le\delta C_{\mathcal{M}}^{\log_2\delta}$ and $\int_E|f|\,d\mathcal{M}\le\delta$ whenever $\mathcal{M}(E)\le C\eta\exp(\delta^{-1})$. Choose k large enough such that

$$\mathcal{M}(E_k):=\mathcal{M}\big(\{z\in\mathrm{supp}(\phi):|f_k(z)-f(z)|>\eta\}\big)\le\eta.$$

Define

$$\tau(z):=\max\big\{0,1+\delta\log(\widetilde{M}\chi_{E_k})(z)\big\}.$$

It is clear that $0<\tau(z)\le 1$ and $\tau=1$ $\mathcal{M}$-almost everywhere on E_k. By Lemmas 3.4 and 3.5, we have $\|\tau\|_{BMO_{\mathcal{P}}}\le 2\delta\|\log(\widetilde{M}\chi_{E_k})^{1/2}\|_{BMO_{\mathcal{P}}}\le C\delta$. By Lemma 3.3 and equation (3.2), we have

$$\mathcal{M}(\{z:\widetilde{M}\chi_{E_k}(z)>e^{-\delta^{-1}}\})\le\frac{C}{e^{-\delta^{-1}}}\int_{E_k}d\mathcal{M}=Ce^{\delta^{-1}}\mathcal{M}(E_k),$$

and therefore,

$$\int_{\mathrm{supp}(\tau)}|f|\,d\mathcal{M}\le\delta.$$

Observe that

$$\begin{aligned}\Big|\int_{\mathbb{R}^{n+1}}(f-f_k)\phi\,d\mathcal{M}\Big|&\le\Big|\int_{\mathbb{R}^{n+1}}(f-f_k)\phi(1-\tau)d\mathcal{M}\Big|+\Big|\int_{\mathbb{R}^{n+1}}(f-f_k)\phi\tau\,d\mathcal{M}\Big|\\ &=\Big|\int_{\mathbb{R}^{n+1}\setminus\mathbb{E}_k}(f-f_k)\phi(1-\tau)d\mathcal{M}\Big|+\Big|\int_{\mathbb{R}^{n+1}}(f-f_k)\phi\tau\,d\mathcal{M}\Big|\\ &\le\eta\|\phi\|_{L^1(d\mathcal{M})}+\|\phi\|_{L^\infty(d\mathcal{M})}\int_{\mathrm{supp}(\tau)}|f|\,d\mathcal{M}+\Big|\int_{\mathbb{R}^{n+1}}f_k\phi\tau\,d\mathcal{M}\Big|\\ &\le 2\delta+\|f_k\|_{H^1_{\mathcal{P}}}\|\phi\tau\|_{BMO_{\mathcal{P}}}\le 2\delta+\|\phi\tau\|_{BMO_{\mathcal{P}}}.\end{aligned}$$

Equation (3.3) will be established if we have

$$\|\phi\tau\|_{BMO_{\mathcal{P}}}\le C\delta. \tag{3.4}$$

Let $Q=Q(z_0,r_0)$. Note that

$$\begin{aligned}|\phi\tau-m_Q(\phi\tau)|&\le|\phi\tau-m_Q(\phi)m_Q(\tau)|+|m_Q(\phi)m_Q(\tau)-m_Q(\phi\tau)|\\ &\le|\phi\tau-m_Q(\phi)m_Q(\tau)|+\frac{1}{\mathcal{M}(Q)}\int_Q|\phi\tau-m_Q(\phi)m_Q(\tau)|\,d\mathcal{M}.\end{aligned}$$

https://doi.org/10.4153/CMB-2015-005-2 Published online by Cambridge University Press

Suppose that $r_0 < \delta$, then

$$\begin{aligned}\frac{1}{\mathcal{M}(Q)}\int_Q|\phi\tau - m_Q(\phi\tau)|\,d\mathcal{M} &\le \frac{2}{\mathcal{M}(Q)}\int_Q|\phi\tau - m_Q(\phi)m_Q(\tau)|\,d\mathcal{M}\\ &\le \frac{2}{\mathcal{M}(Q)}\int_Q|\phi\tau - m_Q(\phi)\tau|\,d\mathcal{M} + \frac{2|m_Q(\phi)|}{\mathcal{M}(Q)}\int_Q|\tau - m_Q(\tau)|\,d\mathcal{M}\\ &\le C\delta^2 + 2\|\phi\|_{L^\infty(d\mathcal{M})}\|\tau\|_{BMO\mathcal{P}} \le C(\delta^2+2\delta) < C\delta.\end{aligned}$$

For $r_0 > \delta$ with $Q(z_0,\delta)\cap Q(w_0,\delta^{-1}) = \varnothing$, we have

$$\frac{1}{\mathcal{M}(Q)}\int_Q|\phi\tau - m_Q(\phi\tau)|\,d\mathcal{M} \le \frac{2}{\mathcal{M}(Q)}\int_Q|\phi\tau|\,d\mathcal{M} \le C\delta.$$

For $r_0 > \delta$ with $Q(z_0,\delta)\cap Q(w_0,\delta^{-1}) \neq \varnothing$, we have $Q(z_0,\delta^{-1}) \subset Q(w_0,\theta\delta^{-1})$, and hence $\mathcal{M}(Q(w_0,\delta^{-1})) \le \mathcal{M}(Q(z_0,\theta\delta^{-1}))$. The doubling condition shows that

$$\mathcal{M}\big(Q(z_0,\theta\delta^{-1})\big) \le C_{\mathcal{M}}^{\log_2(\theta\delta^{-2})}\mathcal{M}\big(Q(z_0,\delta)\big).$$

Thus,

$$\frac{1}{\mathcal{M}(Q)} \le \frac{C_{\mathcal{M}}^{\log_2(\theta\delta^{-2})}}{\mathcal{M}(Q(z_0,\theta\delta^{-1}))} \le \frac{C_{\mathcal{M}}^{\log_2(\theta\delta^{-2})}}{\mathcal{M}(Q(w_0,\delta^{-1}))} \le \frac{C_{\mathcal{M}}^{\log_2(\theta\delta^{-2})}}{\mathcal{M}(Q(w_0,1))},$$

and hence

$$\begin{aligned}\frac{1}{\mathcal{M}(Q)}\int_Q|\phi\tau - m_Q(\phi\tau)|\,d\mathcal{M} &\le \frac{2}{\mathcal{M}(Q)}\int_Q|\phi\tau|\,d\mathcal{M}\\ &\le \frac{2C_{\mathcal{M}}^{\log_2(\theta\delta^{-2})}}{\mathcal{M}(Q(w_0,1))}\mathcal{M}(\operatorname{supp}(\tau))\\ &\le \frac{2C_{\mathcal{M}}^{\log_2(\theta\delta^{-2})}}{\mathcal{M}(Q(w_0,1))}\eta\exp(\delta^{-1}) \le C\delta.\end{aligned}$$

Therefore,

$$\frac{1}{\mathcal{M}(Q)}\int_Q|\phi\tau - m_Q(\phi\tau)|\,d\mathcal{M} \le C\delta$$

and hence equation (3.4) follows. To show that f is in $H^1_{\mathcal{P}}$, by weak* compactness of the unit ball in $H^1_{\mathcal{P}}$, there exists a subsequence $\{f_{k_l}\}$ and $g \in H^1_{\mathcal{P}}$ with $\|g\|_{H^1_{\mathcal{P}}} \le 1$ such that $\{f_{k_l}\}$ weak* converges to g. By equation (3.4), we have $\int f\phi = \int g\phi$ for all $\phi \in C_c \cap \mathrm{Lip}$, and hence $f = g \in H^1_{\mathcal{P}}$. ■

References

[1] H. Aimar, L. Forzani, and R. Toledano, *Balls and quasi-metrics: a space of homogeneous type modeling the real analysis related to the Monge-Ampère equation.* J. Fourier Anal. Appl. **4**(1998), no. 4–5, 377–381. http://dx.doi.org/10.1007/BF02498215

[2] L. A. Caffarelli, *Some regularity properties of solutions of Monge-Ampère equation.* Comm. Pure Appl. Math. **44**(1991), no. 8–9, 965–969. http://dx.doi.org/10.1002/cpa.3160440809

[3] ———, *Boundary regularity of maps with convex potentials.* Comm. Pure Appl. Math. **45**(1992), no. 9, 1141–1151. http://dx.doi.org/10.1002/cpa.3160450905

[4] L. A. Caffarelli and C. E. Gutiérrez, *Real analysis related to the Monge-Ampère equation.* Trans. Amer. Math. Soc. **348**(1996), no. 3, 1075–1092. http://dx.doi.org/10.1090/S0002-9947-96-01473-0
[5] ———, *Properties of the solutions of the linearized Monge-Ampère equation.* Amer. J. Math. **119**(1997), no. 2, 423–465. http://dx.doi.org/10.1353/ajm.1997.0010
[6] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis.* Bull. Amer. Math. Soc. **83**(1977), no. 4, 569–645. http://dx.doi.org/10.1090/S0002-9904-1977-14325-5
[7] R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes, *Compensated compactness and Hardy sapces.* J. Math. Pures Appl. 72(1993), no. 3, 247–286.
[8] Y. Ding and C.-C. Lin, *Hardy spaces associated to the sections.* Tohoku Math. J. **57**(2005), no. 2, 147–170. http://dx.doi.org/10.2748/tmj/1119888333
[9] Q. Huang, *Harnack inequality for the linearized parabolic Monge-Ampère equation.* Trans. Amer. Math. Soc. **351**(1999), 2025–2054. http://dx.doi.org/10.1090/S0002-9947-99-02142-X
[10] P. W. Jones and J.-L. Journé, *On weak convergence in* $H^1(\mathbb{R}^d)$. Proc. Amer. Math. Soc. **120**(1994), 137–138.
[11] M. Qu and X. Wu, *BMO spaces associated to generalized parabolic sections.* Anal. Theory Appl. **27**(2011), no. 1, 1–9. http://dx.doi.org/10.1007/s10496-011-0001-2
[12] X. Wu, *Hardy spaces associated to generalized parabolic sections.* Panamer. Math. J. **18**(2008), no. 2, 33–51.

Department of Mathematics, National Central University, Chung-Li, 32054, Taiwan
e-mail: hsumh@math.ncu.edu.tw mylee@math.ncu.edu.tw