CAMBRIDGE
UNIVERSITY PRESS

RESEARCH ARTICLE

Gromov–Witten theory with maximal contacts

Navid Nabijou[1] and Dhruv Ranganathan[2]

[1]Department of Pure Mathematics & Mathematical Statistics, University of Cambridge; E-mail: nn333@cam.ac.uk.
[2]Department of Pure Mathematics & Mathematical Statistics, University of Cambridge; E-mail: dr508@cam.ac.uk.

Abstract

We propose an intersection-theoretic method to reduce questions in genus 0 logarithmic Gromov–Witten theory to questions in the Gromov–Witten theory of smooth pairs, in the presence of positivity. The method is applied to the enumerative geometry of rational curves with maximal contact orders along a simple normal crossings divisor and to recent questions about its relationship to local curve counting. Three results are established. We produce counterexamples to the local-logarithmic conjectures of van Garrel–Graber–Ruddat and Tseng–You. We prove that a weak form of the conjecture holds for product geometries. Finally, we explicitly determine the difference between local and logarithmic theories, in terms of relative invariants for which efficient algorithms are known. The polyhedral geometry of the tropical moduli of maps plays an essential and intricate role in the analysis.

Contents

Introduction

Let X be a smooth projective variety and D a simple normal crossings divisor whose irreducible components $D_1, \ldots, D_k$ are hyperplane sections, hereafter *section pairs*. We examine three genus 0 Gromov–Witten theories: (1) the logarithmic theory of $(X|D)$, (2) the naive logarithmic theory of $(X|D)$ constructed out of the relative theories of $(X|D_i)$ and (3) the local theory of the direct sum of the $\mathcal{O}_X(-D_i)$. The first two encode rational curves in X with tangency conditions along D. The local theory models rational curves in a rigid embedding of X in an ambient variety with split normal bundle $\oplus_{i=1}^{k} \mathcal{O}_X(-D_i)$.

The *naive theory* is defined as follows. First let $X = \mathbb{P}$ be a product of k projective spaces, with $H_i \subseteq \mathbb{P}$ the pullback of a hyperplane from the ith factor and $H = \Sigma_{i=1}^{k} H_i$. The *naive space* is defined as the fibre product of stacks:

$$\begin{array}{ccc}
\mathsf{N}(\mathbb{P}|H) & \longrightarrow & \prod_{i=1}^{k} \mathsf{K}(\mathbb{P}|H_i) \\
\downarrow & \square & \downarrow \\
\mathsf{K}(\mathbb{P}) & \overset{\Delta}{\longhookrightarrow} & \prod_{i=1}^{k} \mathsf{K}(\mathbb{P}).
\end{array}$$

The moduli space $\mathsf{K}(\mathbb{P})$ is smooth and so Δ is a regular embedding. We obtain a virtual class on $\mathsf{N}(\mathbb{P}|H)$ by pullback:

$$[\mathsf{N}(\mathbb{P}|H)]^{\mathrm{vir}} := \Delta^{!} \left(\prod_{i=1}^{k} [\mathsf{K}(\mathbb{P}|H_i)] \right).$$

The pushforward to $\mathsf{K}(\mathbb{P})$ is simply the product of the classes $[\mathsf{K}(\mathbb{P}|H_i)]$. Virtual pullback defines the theory for arbitrary section pairs $(X|D)$; see Section 4.[1]

0.1. Correspondences

If D is smooth, logarithmic Gromov–Witten theory coincides with the relative theory for all tangency orders. If the tangency with D is maximal, it coincides with the local theory by a result of van Garrel–Graber–Ruddat [31], following Takahashi and Gathmann [28, 16].

The local-logarithmic correspondences. Let X be smooth and projective with simple normal crossings divisor D with smooth nef components $D_1, \ldots, D_k$. Let β be a curve class and suppose $d_i := D_i \cdot \beta > 0$ for all i. Consider the moduli of logarithmic stable maps $\mathsf{K}_{0,k}^{\max}(X|D, \beta)$ with maximal contact with each component at distinct points.

Strong form: There is an equality of homology classes on the Kontsevich space $\mathsf{K}_{0,k}(X, \beta)$ of k-pointed stable maps to X, suppressing the relevant pushforwards, given by

$$[\mathsf{K}_{0,k}^{\max}(X|D, \beta)]^{\mathrm{vir}} = \prod_{i=1}^{k} (-1)^{d_i+1} \mathsf{ev}_i^{\star} D_i \cdot [\mathsf{K}_{0,k}(\oplus_{i=1}^{k} \mathcal{O}_X(-D_i), \beta)]^{\mathrm{vir}}.$$

Original form: The equality above holds after pushing forward to the Kontsevich space $\mathsf{K}_{0,0}(X, \beta)$ of unpointed stable maps to X; that is,

$$[\mathsf{K}_{0,k}^{\max}(X|D, \beta)]^{\mathrm{vir}} = \prod_{i=1}^{k} (-1)^{d_i+1} d_i \cdot [\mathsf{K}_{0,0}(\oplus_{i=1}^{k} \mathcal{O}_X(-D_i), \beta)]^{\mathrm{vir}}.$$

The pushforward on the left-hand side is suppressed, while the right-hand side is a naturally a class on the Kontsevich space.

The original form was conjectured by van Garrel–Graber–Ruddat [31]. Fan–Wu and Tseng–You observed that if D is smooth, the original proof yields the strong form. The general strong form was conjectured by Tseng–You [29]. The conjecture holds in numerical form in many cases [11, 12].

We refer to the local theory class cut down by the divisorial evaluations as they appear in the strong form above as the *evaluation local theory* of $(X|D)$. The following observation is elementary.

Observation. The evaluation local theory of $(X|D)$ coincides up to sign with the naive theory of $(X|D)$. After pushforward to $\mathsf{K}_{0,0}(X, \beta)$, the naive theory coincides up to explicit multiplicity with the local theory of $(X|D)$ as a homology class on $\mathsf{K}_{0,0}(X, \beta)$.

With this observation at hand, we dispense with local Gromov–Witten theory and focus on the more general question of when the logarithmic and naive theories coincide.

---

[1]A fourth possibility is the multi-root theory of $(X|D)$ constructed by Tseng–You [29]. This coincides with the naive theory for section pairs; see [7, Corollary 2.2].

0.2. Results

Our first result proves that the naive theory does not coincide with the logarithmic theory, giving counterexamples to both forms of the local-logarithmic correspondence.

Theorem X. *The naive and logarithmic maximal contact theories do not coincide in degree* 2 *for* $\mathbb{P}^2$ *equipped with the divisor consisting of two lines. The strong form of the correspondence fails for this geometry in degree* 2. *The original form of the correspondence fails in degree* 4.

The result is proved by direct geometric analysis. The proof gives a basic flavour of the naive theory and implies that the naive theory is not even enumerative in genus 0 for logarithmically convex pairs. The difference between the theories is controlled by the following result; see Sections 2–3 and in particular Theorem 3.4.

Theorem Y. *The difference between the logarithmic and local/naive maximal contact Gromov–Witten invariants of a section pair* $(X|D)$ *is determined algorithmically in terms of tautological integrals on the moduli space of stable maps to X.*

Numerical consequences can be extracted. For example, the logarithmic theory of $\mathbb{P}^2$ relative to two lines can be computed with primary insertions in degree up to 4, as the corrections will not contribute; see Remark 1.6. A systematic study will appear in future work.

The result is a consequence of the following much stronger result, which implies that the difference between the two theories is captured by Chern classes of tautological bundles, Segre classes of boundary strata in the moduli space of relative maps and descendent integrals thereon.

Let $\mathbb{P}$ denote a product of k projective spaces and H a divisor that is a union of hyperplanes $H_1, \ldots, H_k$ pulled back from each factor.

Theorem Z. *Let* $\mathsf{K}_{0,k}^{\max}(\mathbb{P}|H_i, \beta)$ *be the space of logarithmic stable maps to* $\mathbb{P}$ *that are maximally tangent to* H_i *at the ith marked point. There is an explicit sequence of weighted blowups of the Kontsevich space*

$$\mathsf{K}_{0,k}(\mathbb{P}, \beta)^{\dagger} \to \mathsf{K}_{0,k}(\mathbb{P}, \beta)$$

along smooth centres such that, denoting strict transforms as $\mathsf{K}_{0,k}^{\max}(\mathbb{P}|H_i, \beta)^{\dagger}$ *and suppressing pushforwards, there is an equality of cycles in the Chow group of* $\mathsf{K}_{0,k}(\mathbb{P}, \beta)$:

$$[\mathsf{K}_{0,k}^{\max}(\mathbb{P}|H_1, \beta)^{\dagger}] \cdots [\mathsf{K}_{0,k}^{\max}(\mathbb{P}|H_k, \beta)^{\dagger}] = [\mathsf{K}_{0,k}^{\max}(\mathbb{P}|H, \beta)].$$

Blowups of moduli spaces have appeared in recent work on logarithmic Gromov–Witten theory [26, 25]. The theorem above is considerably stronger. The birational modifications in those papers are not made explicit, while the result above is completely algorithmic, without arbitrary choices. The combinatorics of the maximal contacts situation is leveraged heavily. A reader will find that the combinatorial arguments, manipulating the cone stack of tropical maps, are delicate. These arguments are crucial in deducing structure results for the birational models of stable map spaces. Outside the maximal contact setup, a sequence of blowups exists but cannot be made explicit. The utility of a general systematic description is likely to be high. In particular, we are not aware of any other methods that calculate the set of invariants that our algorithm calculates.

Insights from this analysis lead to a new range of cases where the correspondence holds; see Section 5. These are not covered by the existing literature.

Theorem W. *Let* $X_1, \ldots, X_k$ *be smooth, equipped with smooth hyperplane sections* $D_1, \ldots, D_k$. *The local-logarithmic-naive correspondence holds for the pair* $(\prod X_i | \sum D_i)$ *with primary factorwise insertions.*

The condition 'primary factorwise insertions' is explained in Subsection 5.2. It includes in particular all primary invariants with three markings or fewer. These provide the first nontoric examples of the numerical correspondence in dimension larger than 2. Numerical consequences may again be extracted:

invariants of the pair $(\mathbb{P}^3 \times \mathbb{P}^2 | K3 + E)$ where $K3$ and E are a quartic and a cubic can be computed by [20, 17].

0.3. Rank reduction and further questions

The local-logarithmic correspondence is one among a number of beautiful results in the relative Gromov–Witten theory of a smooth pair, starting with Gathmann's striking work [16]. In simple normal crossings geometries, results are much harder to come by and the analogue of Gathmann's recursion is not known. The difficulty of working with the invariants is visible in the degeneration formalism [2, 26].

An approach to our results via degeneration appears to be difficult, at least to the authors. We chart a 'pure thought' alternative for reducing questions about the geometry of logarithmic stable map spaces to the case of smooth pairs and implement it completely in the maximal contacts case. The method is restricted to genus 0 invariants satisfying a positivity assumption, but even these invariants have not been computed by other methods, not even in principle. Moreover, many important phenomena, such as the failure of the local-logarithmic correspondence, appear already in this setting.

Our technique geometrises a categorical insight of Abramovich–Chen [1]. Given a logarithmic curve in $(X|D)$, one obtains a logarithmic curve in the smooth pairs $(X|D_i)$ by forgetting the logarithmic structure away from D_i. A naive expectation is that the intersection of these loci recovers the locus of logarithmic maps to $(X|D)$. This expectation fails but is corrected by blowing up the moduli of maps to X. The intersection of strict transforms recovers the space of logarithmic maps to $(X|D)$. Tropical geometry informs the blowups used to correct the intersection.

We open two directions for future work.

Problem 0.1 (Moduli factorisation). For fixed contact order data Γ, determine an efficient and explicit sequence of blowups at smooth centres $\mathsf{K}_{0,k}(\mathbb{P}, \beta)^\dagger \to \mathsf{K}_{0,k}(\mathbb{P}, \beta)$ such that the strict transform of $\mathsf{K}_\Gamma(\mathbb{P}|H) \to \mathsf{K}_{0,k}(\mathbb{P}, \beta)$ along the blowup is transverse to the strata.

A solution would generalise the combinatorics in this article. It dovetails with the following. For fixed contact orders Γ there is a cycle $\mathsf{K}_\Gamma(\mathbb{P}|H) \to \mathsf{K}_{0,k}(\mathbb{P}, \beta)$ in the space of stable maps. For any sufficiently fine logarithmic blowup of the codomain $\mathsf{K}_{0,k}(\mathbb{P}, \beta)^\dagger \to \mathsf{K}_{0,k}(\mathbb{P}, \beta)$, the strict transform of $\mathsf{K}_\Gamma(\mathbb{P}|H)$ is transverse to the boundary of $\mathsf{K}_{0,k}(\mathbb{P}, \beta)^\dagger$. We refer to this as the *transverse class*.

Problem 0.2 (The transverse class). Determine an expression, in terms of tautological classes, for the transverse relative Gromov–Witten class in any sufficiently fine blowup $\mathsf{K}_{0,k}(\mathbb{P}, \beta)^\dagger$ of the moduli space of stable maps.

We do not extract a closed form in the maximal contacts case; our expression is algorithmic, not closed. A solution to this question would determine the genus 0 Gromov–Witten theory of all section pairs, which is beyond the present state of the art and complete a parallel to [16].

To conclude this introduction, we note that an important recent development in the subject has been the development of an approach to Gromov–Witten theory relative to reducible divisors by means of orbifold structures by Tseng–You [29]. Our naive theory for section pairs coincides with this orbifold theory; see [7, Corollary 2.2]. Admitting this equality, our framework explains in simple geometric terms why the new orbifold theory does not coincide with the logarthmic theory.

Comparison with v1

An earlier version of this article incorrectly claimed a positive answer to the local-logarithmic conjecture for section pairs. The error was wrongly deducing that the strict transform of the relative Gromov–Witten class was equal to the total transform, which occurred via a misapplication of the vanishing results in [31]. The technique has been refined in this version, but the basic geometric strategy remains the same.

1. Counterexamples, conics, quartics

The counterexamples to the cycle-theoretic correspondence follow the same basic principle. The local theory of a split rank 2 vector bundle satisfies a simple product rule, coming from the Whitney sum formula for the obstruction bundle. The parallel splitting for the logarithmic theory fails. The analysis here is based on examples computed in the Ph.D. thesis of N.N. [23, §3].

1.1. Unpointed counterexample: plane quartics

Let $H_1, H_2 \subseteq \mathbb{P}^2$ be distinct lines. For $i \in \{1, 2\}$ we consider the moduli space

$$\mathsf{K}^{\max}_{0,1}(\mathbb{P}^2 | H_i, 4) \tag{1}$$

of logarithmic stable maps to $(\mathbb{P}^2 | H_i)$ with maximal tangency at a single marked point. The logarithmic Euler sequence shows that $T_{(\mathbb{P}^2 | H_i)}$ is convex, so the moduli space (1) is logarithmically smooth. As such, it contains the dimensionally transverse locus, consisting of maps with smooth domain whose image is not contained inside H_i, as a dense open. Using an explicit parametrisation of this open locus, we conclude that (1) is irreducible with dimension and expected dimension equal to 8.

Forgetting the logarithmic structures and the marking, we obtain a generically finite map:

$$\pi^i : \mathsf{K}^{\max}_{0,1}(\mathbb{P}^2 | H_i, 4) \rightarrow \mathsf{K}_{0,0}(\mathbb{P}^2, 4).$$

The target is a smooth Deligne–Mumford stack, of dimension 11.

Lemma 1.1. *There is an equality of classes in* $\mathsf{K}_{0,0}(\mathbb{P}^2, 4)$:

$$\pi^1_\star [\mathsf{K}^{\max}_{0,1}(\mathbb{P}^2 | H_1, 4)] \cdot \pi^2_\star [\mathsf{K}^{\max}_{0,1}(\mathbb{P}^2 | H_2, 4)] = 4^2 \cdot [\mathsf{K}_{0,0}(\mathcal{O}_{\mathbb{P}^2}(-H_1) \oplus \mathcal{O}_{\mathbb{P}^2}(-H_2), 4)]^{\mathrm{vir}}.$$

Proof. The rational Chow groups of $\mathsf{K}_{0,0}(\mathbb{P}^2, 4)$ possess an intersection product, as this space is smooth. The local class on the right-hand side is the product of the Euler classes of the two local classes associated to the Gromov–Witten theories of the bundles $\mathcal{O}_{\mathbb{P}^2}(-H_1)$ and $\mathcal{O}_{\mathbb{P}^2}(-H_2)$. By the local-logarithmic correspondence for smooth pairs [31], each Euler class is equal to the corresponding logarithmic term on the left-hand side. The result follows. □

Consider the moduli space

$$\mathsf{K}^{\max}_{0,2}(\mathbb{P}^2 | H_1 + H_2, 4)$$

of logarithmic stable maps with maximal contact to H_1 and H_2 at markings x_1 and x_2. As before, the logarithmic Euler sequence shows that this space is logarithmically smooth and contains the locus of maps from smooth domains not mapping into $H_1 \cup H_2$ as a dense open. It follows that it is irreducible, with dimension equal to the expected dimension 5. There is a forgetful morphism:

$$\pi : \mathsf{K}^{\max}_{0,2}(\mathbb{P}^2 | H_1 + H_2, 4) \rightarrow \mathsf{K}_{0,0}(\mathbb{P}^2, 4).$$

The remainder of this section will focus on the following result, which, combined with Lemma 1.1, demonstrates the failure of the local-logarithmic correspondence.

Proposition 1.2. *The following classes in* $\mathsf{K}_{0,0}(\mathbb{P}^2, 4)$ *are not equal:*

$$\pi^1_\star [\mathsf{K}^{\max}_{0,1}(\mathbb{P}^2 | H_1, 4)] \cdot \pi^2_\star [\mathsf{K}^{\max}_{0,1}(\mathbb{P}^2 | H_2, 4)] \neq \pi_\star [\mathsf{K}^{\max}_{0,2}(\mathbb{P}^2 | H_1 + H_2, 4)].$$

The forgetful morphisms are all generically injective, so the pushforward classes may be identified

with the fundamental classes of the images. Proposition 1.2 becomes

$$[\pi^1(\mathsf{K}_{0,1}^{\max}(\mathbb{P}^2|H_1, 4))] \cdot [\pi^2(\mathsf{K}_{0,1}^{\max}(\mathbb{P}^2|H_2, 4))] \neq [\pi(\mathsf{K}_{0,2}^{\max}(\mathbb{P}^2|H_1 + H_2, 4))]. \tag{2}$$

Lemma 1.3. *The intersection*

$$\pi^1(\mathsf{K}_{0,1}^{\max}(\mathbb{P}^2|H_1, 4)) \cap \pi^2(\mathsf{K}_{0,1}^{\max}(\mathbb{P}^2|H_2, 4)) \subseteq \mathsf{K}_{0,0}(\mathbb{P}^2, 4) \tag{3}$$

contains two irreducible components, each of dimension 5.

Proof. The first irreducible component is the main component, which is the closure of the locus of dimensionally transverse maps. This is contained in the intersection (3). As noted above, it coincides with the image of the moduli space of logarithmic stable maps to $(\mathbb{P}^2|H_1 + H_2)$:

$$\pi(\mathsf{K}_{0,2}^{\max}(\mathbb{P}^2|H_1 + H_2, 4)).$$

For the second irreducible component, consider the closure in $\mathsf{K}_{0,0}(\mathbb{P}^2, 4)$ of the locus parametrising maps from rational curves of the form

$$C = C_0 \cup C_1 \cup \cdots \cup C_4 \to \mathbb{P}^2$$

in which each C_i is smooth, the component C_0 is contracted to $H_1 \cap H_2$ and meets each of the components $C_1, \ldots, C_4$ in a node and the remaining components are mapped isomorphically onto lines. The image of such a map is a collection of four lines through the point $H_1 \cap H_2$ and there is an $\overline{\mathcal{M}}_{0,4}$ moduli for the internal component C_0; it follows that this locus is 5-dimensional. It remains to show that it is contained in each of the images $\pi^i(\mathsf{K}_{0,1}^{\max}(\mathbb{P}^2|H_i, 4))$. The image of

$$\mathsf{K}_{0,1}^{\max}(\mathbb{P}^2|H_i, 4) \to \mathsf{K}_{0,1}(\mathbb{P}^2, 4)$$

is the closure of its interior; the interior consists of maps from smooth domains which have maximal contact order to H_i but do not map inside H_i. The closure is identified by Gathmann's numerical balancing criterion [16, Remark 1.7(ii)]. Consider the locus of maps

$$C = C_0 \cup C_1 \cup \cdots \cup C_4 \to \mathbb{P}^2$$

as above, where C_0 bears the marked point. Each noncontracted component meets H_i with contact order 1, and by the numerical criterion we deduce that the locus is contained in the image of $\mathsf{K}_{0,1}^{\max}(\mathbb{P}^2|H_i, 4) \to \mathsf{K}_{0,1}(\mathbb{P}^2, 4)$. The claim follows by applying the forgetful morphism $\mathsf{K}_{0,1}(\mathbb{P}^2, 4) \to \mathsf{K}_{0,0}(\mathbb{P}^2, 4)$. $\qquad\square$

Proof of Proposition 1.2. By Lemma 1.3, the intersection contains at least two irreducible components, each of dimension 5. Any additional irreducible component must arise as the intersection of images of boundary strata in $\mathsf{K}_{0,1}^{\max}(\mathbb{P}^2|H_i, 4)$. We claim its dimension is at most 5. Consider $\mathsf{K}_{0,1}^{\max}(\mathbb{P}^2|H_1, 4)$. This has dimension 8 and any boundary stratum has dimension at most 7. Forgetting the marking reduces the dimension to 6 unless the marking lies on a contracted tail. In the former case, as in general, the resulting locus in $\mathsf{K}_{0,0}(\mathbb{P}^2, 4)$ does not generically satisfy the numerical criterion with respect to H_2, which cuts the dimension to at least 5. In the latter case, the component of four lines has been dealt with already. The remaining possibility comprises a conic tangent to H_1 and two lines through the tangency point. Elementary geometry again bounds the dimension at 5. It follows that every irreducible component of the intersection (3) has dimension 5.

The left-hand side of (2) is the sum of classes of the irreducible components, with positive multiplicities [15, Proposition 7.1]. The space $\mathsf{K}_{0,0}(\mathbb{P}^2, 4)$ is projective, and Bezout's theorem guarantees that the sum of classes of excess components is not zero in the Chow group of $\mathsf{K}_{0,0}(\mathbb{P}^2, 4)$. The component $[\pi(\mathsf{K}_{0,2}^{\max}(\mathbb{P}^2|H_1 + H_2, 4)]$ appears on both sides of (2), so the two sides cannot be equal. $\qquad\square$

Remark 1.4. The counterexample implies that there is *some* invariant for which the local and logarithmic theories differ. Indeed, Gromov–Witten theory includes all integrals against *tautological* classes on the moduli space of stable maps. Poincaré duality furnishes a cohomology class that distinguishes the two classes, but the cohomology of $K_{0,0}(\mathbb{P}^2, 4)$ is entirely tautological; see [24]. An explicit instance is calculated in Subsection 3.7.

Remark 1.5. The intersection in Lemma 1.3 is exactly the union of the two components described above; there are no additional components. This follows from the blowup analysis of Sections 2–3, which also provides a technique to calculate the excess components.

Remark 1.6 (Primary correspondence). In the degree 4 maximal contact geometry for $(\mathbb{P}^2 | H_1 + H_2)$, the excess component consists of four lines through a point. As a consequence, this component cannot contribute to a Gromov–Witten invariant with only primary insertions, as the cross ratio of the nodes on the contracted component cannot be fixed by primary insertions. In particular, the local-logarithmic correspondence holds in degrees up to 4 with primary insertions.

1.2. Pointed counterexample: plane conics

The strong form of the correspondence implies the original form, so the counterexample above also falsifies the strong form. We record a simpler failure of the strong form, which occurs in lower degree. We leave some verifications to the reader, since the analysis is simpler than the one above.

For $i \in \{1, 2\}$ consider the moduli space $K_{0,2}^{\max}(\mathbb{P}^2 | H_i, 2)$ with maximal contact order at the marking x_i and zero contact order at the marking $x_{\neq i}$. This is the universal curve over the moduli space $K_{0,1}^{\max}(\mathbb{P}^2 | H_i, 2)$. Proceeding as above, it suffices to show the following inequality:

$$[K_{0,2}^{\max}(\mathbb{P}^2 | H_1, 2)] \cdot [K_{0,2}^{\max}(\mathbb{P}^2 | H_2, 2)] \neq [K_{0,2}^{\max}(\mathbb{P}^2 | H_1 + H_2, 2)] \tag{4}$$

in $K_{0,2}(\mathbb{P}^2, 2)$, where we have suppressed pushforwards from the notation.

Proof of (4). We examine the intersection

$$K_{0,2}^{\max}(\mathbb{P}^2 | H_1, 2) \cap K_{0,2}^{\max}(\mathbb{P}^2 | H_2, 2) \subseteq K_{0,2}(\mathbb{P}^2, 2). \tag{5}$$

This has a main component: the closure of the space of maps intersecting each H_i in precisely one point. This component coincides with the locus $K_{0,2}^{\max}(\mathbb{P}^2 | H_1 + H_2, 2)$, which has dimension equal to the expected dimension 3.

A second 3-dimensional component of the intersection (5) parametrises maps of the form

$$C_0 \cup C_1 \cup C_2 \to \mathbb{P}^2$$

where C_0 bears the two marked points, is contracted to $H_1 \cap H_2$ and meets C_1 and C_2 at distinct points. The components C_1 and C_2 each map isomorphically onto lines. Elementary geometry shows that this locus has dimension 3: two dimensions for the two lines and one for the cross ratio of the four points. Direct analysis shows that there are no further irreducible components. This second irreducible component contributes with positive multiplicity. Therefore, the logarithmic class is not the product of the two classes associated to the relative theories of the smooth pairs. This yields a counterexample to the strong form of the conjecture. □

Remark 1.7. The examples are the lowest degree failures of the two conjectures. The degree 2 counterexample above does not yield a counterexample to the original form of the correspondence; the cross ratio of the points in the contracted component is lost, so the class vanishes in the pushforward. So the strong form of the conjecture is genuinely stronger than the original one.

2. Correcting the correspondence I: subdivisions and modifications

The failure of the local-logarithmic correspondence stems from the fact that moduli spaces of logarithmic maps do not satisfy a naive product formula over the space of ordinary maps:

$$\mathsf{K}(X|D_1) \times_{\mathsf{K}(X)} \mathsf{K}(X|D_2) \neq \mathsf{K}(X|D).$$

The left-hand side can include excess components, even in convex settings where the right-hand side is irreducible. The local and naive theories do satisfy a product formula, so the local-logarithmic correspondence cannot hold in generality. This observation led to the counterexamples of Section 1.

In the next two subsections, we establish a method for calculating the defect between the naive and logarithmic theories. We transversalise the naive intersection by performing blowups on $\mathsf{K}(X)$ and apply Fulton's blowup formula to quantify the difference between the theories.

2.1. Setup: target geometry and moduli spaces

Consider a target $(X|D)$ with $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ and $D = D_1 + D_2$ a divisor, with each smooth component D_i the pullback of a hyperplane in $\mathbb{P}^{n_i}$. Spaces of genus 0 logarithmic stable maps to X, $(X|D_1)$, $(X|D_2)$ and $(X|D)$ are logarithmically unobstructed; the discussion which follows applies to any target satisfying this.

We establish a corrected local-logarithmic correspondence in this setting; the case with more divisor components follows *mutatis mutandis* by replacing D_2 with $D_2 + \ldots + D_k$ and the case of hyperplane sections follows by virtual pullback (see Section 4).

We begin by establishing notation for the maximal contacts theory. We fix a curve class β and introduce markings x_1, x_2 which have maximal tangency with respect to D_1, D_2, respectively. We obtain a moduli space of logarithmic stable maps to $(X|D)$ with maximal contacts

$$\mathsf{K}_{0,2}^{\max}(X|D, \beta)$$

and for $i \in \{1, 2\}$ a moduli space of logarithmic stable maps $\mathsf{K}_{0,2}^{\max}(X|D_i, \beta)$ to $(X|D_i)$. The latter space is also two-pointed; the marking $x_{\neq i}$ carries no tangency condition. This is the universal curve over the one-pointed space $\mathsf{K}_{0,1}^{\max}(X|D_i, \beta)$.

The following target diagram is Cartesian in fine and saturated logarithmic schemes:

$$
\begin{array}{ccc}
(X|D) & \longrightarrow & (X|D_1) \\
\downarrow & \square & \downarrow \\
(X|D_2) & \longrightarrow & X.
\end{array}
$$

The moduli spaces of two-pointed logarithmic stable maps enjoy a similar relationship

$$
\begin{array}{ccc}
\mathsf{K}_{0,2}^{\max}(X|D, \beta) & \longrightarrow & \mathsf{K}_{0,2}^{\max}(X|D_1, \beta) \\
\downarrow & \square & \downarrow \\
\mathsf{K}_{0,2}^{\max}(X|D_2, \beta) & \longrightarrow & \mathsf{K}_{0,2}(X, \beta);
\end{array}
$$

see [1, Theorem 2.6]. This diagram is Cartesian in the category of fine and saturated logarithmic stacks but not typically Cartesian in the category of ordinary stacks (the Cartesian product in the category of ordinary stacks is instead the naive space). The failure is accounted for by the fact that neither of the morphisms $\mathsf{K}_{0,2}^{\max}(X|D_i, \beta) \to \mathsf{K}_{0,2}(X, \beta)$ is integral and saturated.

2.2. Semistable reduction

Our strategy is to correct this fibre product by replacing the morphism $\mathsf{K}^{\max}_{0,2}(X|D_1,\beta) \to \mathsf{K}_{0,2}(X,\beta)$ with an integral and saturated birational model. This will be constructed using weak semistable reduction [3, 22].

A toroidal morphism $X \to B$ of toroidal embeddings is logarithmically smooth with the divisorial structure. The morphism need not be equidimensional or have reduced fibres. In their work on weak semistable reduction, Abramovich–Karu [3] identified criteria for these properties.

Lemma 2.1 ([3, Lemma 4.1]). *Let $f : X \to B$ be a toroidal morphism of toroidal embeddings and let $\Sigma_X \to \Sigma_B$ be the morphism of cone complexes. Then f has equidimensional fibres if and only if every cone of Σ_X surjects onto a cone of Σ_B.*

Lemma 2.2 ([3, Lemma 5.2]). *Let $f : X \to B$ be a toroidal morphism with equidimensional fibres and let $\Sigma_X \to \Sigma_B$ be the morphism of cone complexes. Then f has reduced fibres if and only if for every cone σ with image cone τ, the image of the morphism on associated lattices is saturated.*

A toroidal morphism satisfying the conditions in the lemmas above is **weakly semistable**. Any toroidal morphism can be modified to a weakly semistable one.

Proposition 2.3 (Toroidal weak semistable reduction [3]). *Let $f : X \to B$ and $\Sigma_f : \Sigma_X \to \Sigma_B$ be as above. There exist subdivisions of the source and target $\Sigma^{\dagger}_X \to \Sigma^{\dagger}_B$, such that the resulting morphism $X^{\dagger} \to B^{\dagger}$ is equidimensional. By applying a sequence of root stack constructions (change of lattice) to $B^{\dagger}$, we obtain a Deligne–Mumford stack $\mathcal{B}^{\dagger}$ and a new morphism $X^{\dagger} \to \mathcal{B}^{\dagger}$ which is equidimensional with reduced fibres.*

The Abramovich–Karu construction is nonunique, depending on an auxiliary choice of piecewise-linear support functions. Later work of Molcho [22, Theorem 2.4.2] showed that if the morphism Σ_f is proper and surjective, there is a unique minimal choice. The construction declares the image of every cone to be a cone and subdivides the intersections as necessary.

2.3. The subdivision

We apply the construction in the previous section to spaces of logarithmic stable maps. The first step is to replace each of the moduli spaces $\mathsf{K}^{\max}_{0,2}(X|D_i,\beta)$ for $i = 1, 2$ with Kim's space of logarithmic stable maps to expansions [19]. As discussed in [6, Section 2.1], this is a logarithmic modification of the Abramovich–Chen–Gross–Siebert moduli space, representing the subfunctor of **image-ordered** tropical maps. The tropicalisation

$$\mathsf{T}^{\max}_{0,2}(X|D_i,\beta) = \mathsf{Trop}\,\mathsf{K}^{\max}_{0,2}(X|D_i,\beta)$$

is the cone complex parametrising image-ordered degree-weighted tropical stable maps to $\mathbb{R}_{\geq 0}$. The space $\mathsf{K}_{0,2}(X,\beta)$ has logarithmic structure induced by its normal crossings boundary (equivalently, by viewing it as a space of logarithmic stable maps to a trivial logarithmic scheme) and the tropicalisation

$$\mathsf{T}_{0,2}(X,\beta) = \mathsf{Trop}\,\mathsf{K}_{0,2}(X,\beta)$$

is the cone complex parametrising degree-weighted tropical stable curves. We apply weak semistable reduction to the morphism

$$\mathsf{K}^{\max}_{0,2}(X|D_1,\beta) \to \mathsf{K}_{0,2}(X,\beta).$$

This produces subdivisions of the associated cone complexes with an induced morphism

$$\mathsf{T}^{\max}_{0,2}(X|D_1,\beta)^{\dagger} \to \mathsf{T}_{0,2}(X,\beta)^{\dagger},$$

which is combinatorially equidimensional and reduced; it satisfies the polyhedral criteria for these conditions. On the associated logarithmic modifications, we obtain a morphism

$$\mathsf{K}_{0,2}^{\max}(X|D_1,\beta)^\dagger \to \mathsf{K}_{0,2}(X,\beta)^\dagger,$$

which is integral and saturated. We subdivide $\mathsf{T}_{0,2}(X|D_2,\beta)$ by pulling back the subdivision $\mathsf{T}_{0,2}(X,\beta)^\dagger$ of $\mathsf{T}_{0,2}(X,\beta)$ (note the asymmetry between D_1 and D_2 in this construction). We thus obtain a diagram

$$
\begin{array}{ccc}
\mathsf{K}_{0,2}^{\max}(X|D,\beta)^\dagger & \longrightarrow & \mathsf{K}_{0,2}^{\max}(X|D_1,\beta)^\dagger \\
\downarrow & \square & \downarrow{\scriptstyle g} \\
\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)^\dagger & \longrightarrow & \mathsf{K}_{0,2}(X,\beta)^\dagger
\end{array}
$$

which, since the morphism g is now integral and saturated, is Cartesian in both the category of fine and saturated logarithmic stacks and the category of ordinary stacks. The fibre product $\mathsf{K}_{0,2}^{\max}(X|D,\beta)^\dagger$ is a birational model for $\mathsf{K}_{0,2}^{\max}(X|D,\beta)$.

Remark 2.4. The construction is canonical, since the morphism of cone complexes

$$\mathsf{T}_{0,2}^{\max}(X|D_1,\beta) \to \mathsf{T}_{0,2}(X,\beta)$$

is surjective. This is shown at the start of the next section.

Remark 2.5. The preceding subdivisions do not require a change of lattice (saturation), as the minimal monoid associated to a tropical stable map is automatically saturated over the minimal monoid associated to the underlying tropical curve [18, §1.5]. However, the interpretation of our weighted blowups as stacky modifications will require a stacky change of lattice; see Subsection 3.6.

2.4. Modular description: image-ordering (left-to-right)

In order to access the intersection theory of these modifications, it is necessary to obtain a more explicit description of the subdivisions involved. We begin with a modular interpretation for the subdivision $\mathsf{T}_{0,2}(X,\beta)^\dagger$ in terms of order relations on the vertices of the tropical curve. A similar discussion can be found in [13], and additional examples are discussed there.

Remark 2.6. Both the results and the arguments of this section apply beyond the maximal contacts setting to any moduli space of genus 0 logarithmic stable maps relative to a smooth divisor.

Given a two-pointed, degree-weighted stable tropical curve Γ over a base cone σ, we may assign the **formal expansion factor** $D_1 \cdot \beta$ to the semi-infinite leg corresponding to x_1 and the formal expansion factor 0 to the semi-infinite leg corresponding to x_2. Having done this, there is then a unique way to assign a formal expansion factor m_e to each (directed) edge $e \in \Gamma$, in such a way that the resulting tropical curve is balanced; this is a consequence of the genus 0 hypothesis. From this, we obtain a tropical map

$$f : \Gamma \to \mathrm{Hom}(\sigma, \mathbb{R}),$$

well-defined up to overall translation in $\mathbb{R}$. For vertices $v_1, v_2 \in \Gamma$ we declare

$$f(v_1) \leq f(v_2) \quad \text{if and only if} \quad f(v_2) - f(v_1) \in \mathrm{Hom}(\sigma, \mathbb{R}_{\geq 0})$$

and observe that this defines a partial ordering on the vertices of Γ.

Proposition 2.7. $\mathsf{T}_{0,2}(X,\beta)^\dagger$ *is the space of degree-weighted stable tropical curves, such that the $f(v)$ are totally ordered. The cones of this subdivision are the images of cones of* $\mathsf{T}_{0,2}^{\max}(X|D_1,\beta)$.

We refer to $\mathsf{T}_{0,2}(X,\beta)^\dagger$ as the moduli space of **image-ordered** tropical curves. A similar construction was outlined in [6, Section 2.1].

Proof. Temporarily denote the moduli space of image-ordered tropical curves by $\mathsf{T}_{0,2}(X,\beta)^\ddagger$; it is clear that this is a subdivision of $\mathsf{T}_{0,2}(X,\beta)$.

Consider a cone $\tau \in \mathsf{T}_{0,2}^{\max}(X|D_1,\beta)$. This corresponds to a combinatorial type of tropical stable map to $\mathbb{R}_{\geq 0}$, and if we consider the image $\bar\tau \subseteq |\mathsf{T}_{0,2}(X,\beta)|$ and restrict the universal curve C to $\bar\tau$, we obtain a tropical curve whose $f(v)$ are totally ordered (we obtain a total ordering because we work with Kim's space). This total ordering determines a combinatorial type of image-ordered curve, corresponding to a cone $\rho \in \mathsf{T}_{0,2}(X,\beta)^\ddagger$ such that $\bar\tau \subseteq \rho$. We need to show that in fact $\bar\tau = \rho$.

The cone τ is simplicial, with coordinates over $\mathbb{Q}$ given by the target edge lengths $l_1,\ldots,l_k$. We may assume that over τ there is at least one vertex $v_0 \in \mathsf{C}$ mapping to $0 \in \mathbb{R}_{\geq 0}$ (if not, replace τ with the subcone defined by $l_1 = 0$ and note that this does not alter $\bar\tau$ or ρ).

Choosing for each i a stable vertex $v_i \in \mathsf{C}$ mapping to the ith target vertex, we have $f(v_0) < f(v_1) < \ldots < f(v_k)$ on ρ and every other vertex satisfies $f(v) = f(v_i)$ for some i. Thus, we see that ρ is also a simplicial cone, with coordinates over $\mathbb{Q}$ given by

$$f(v_1) - f(v_0), f(v_2) - f(v_1), \ldots, f(v_k) - f(v_{k-1}).$$

The map $\tau \to \rho$ is given by $l_i \mapsto f(v_i) - f(v_{i-1})$, which is clearly surjective, so $\bar\tau = \rho$ as required.

On the other hand, given a cone $\rho \in \mathsf{T}_{0,2}(X,\beta)^\ddagger$ corresponding to a combinatorial type of image-ordered tropical curve, we obtain a unique minimal combinatorial type for a stable tropical map $\mathsf{C} \to \mathbb{R}_{\geq 0}$, by forcing vertices in C with minimal $f(v)$ to map to $0 \in \mathbb{R}_{\geq 0}$; there are no further edge length relations as C has genus 0. This corresponds to a cone $\tau \in \mathsf{T}_{0,2}^{\max}(X|D_1,\beta)$, and it follows from the discussion above that $\bar\tau = \rho$. $\qquad\square$

Corollary 2.8. *The subdivision procedure does not modify the source cone complex:*

$$\mathsf{T}_{0,2}^{\max}(X|D_1,\beta)^\dagger = \mathsf{T}_{0,2}^{\max}(X|D_1,\beta).$$

Proof. The image $\bar\tau$ of every cone of $\mathsf{T}_{0,2}^{\max}(X|D_1,\beta)$ is a cone in the image-ordered subdivision, so the images of two cones cannot intersect away from the image of a common face. $\qquad\square$

Remark 2.9. The previous result helps in understanding the subdivision procedure. It is essentially unimportant for our later arguments.

Corollary 2.10. *The subdivision* $\mathsf{T}_{0,2}(X,\beta)^\dagger$ *is simplicial and* $\mathsf{K}_{0,2}(X,\beta)^\dagger$ *is a smooth orbifold.*

Proof. We saw in the proof of Proposition 2.7 that image-ordered cones are simplicial. The fact that $\mathsf{K}_{0,2}(X,\beta)^\dagger$ is smooth follows immediately, interpreting the logarithmic modification as a nonrepresentable orbitoroidal embedding (see Subsection 3.6 for details). $\qquad\square$

2.5. Modular description: alignment (right-to-left)

The results of the previous subsection are general, applying to moduli spaces with arbitrary tangency orders. When the contact order is maximal, we exhibit a combinatorial factorisation of the subdivision, describing it as a sequence of weighted stellar subdivisions along smooth cones. The description resembles the radial alignments in [27].

Observe that if we let $v_0 \in \mathsf{C}$ denote the vertex containing the marking x_1, the balancing condition implies that $f(v_0)$ must be maximal amongst the $f(v)$. If we therefore let

$$\varphi(v) = f(v_0) - f(v) \in \mathrm{Hom}(\sigma, \mathbb{R}_{\geq 0}),$$

then we see that totally ordering the $f(v)$ is equivalent to totally ordering the $\varphi(v)$. We think of $\varphi(v)$ as the **distance from the root** v_0: it is the expansion factor–weighted sum of the edge lengths along the unique path connecting v_0 to v. We obtain the following.

Proposition 2.11. $\mathsf{T}_{0,2}(X,\beta)^\dagger$ *is the moduli space of degree-weighted stable tropical curves, such that the distances $\varphi(v)$ from the root v_0 are totally ordered.*

We call such a tropical curve **radially aligned**, or **aligned**, with respect to v_0.

2.6. Iterative description

The modular interpretation via alignments gives a very concrete iterative description of $\mathsf{T}_{0,2}(X,\beta)^\dagger \to \mathsf{T}_{0,2}(X,\beta)$ and therefore of the logarithmic modification $\mathsf{K}_{0,2}(X,\beta)^\dagger \to \mathsf{K}_{0,2}(X,\beta)$. This description, inspired by results in [27, 30], will be crucial in Section 3.

Definition 2.12. A **floral cone** $\sigma \in \mathsf{T}_{0,2}(X,\beta)$ is a cone indexed by a type of the following form:
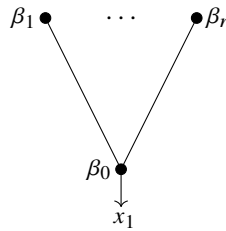
The vertex supporting the marking x_2 is allowed to be arbitrary and is denoted $v(x_2)$. We impose a partial ordering on the floral cones as follows:

(6)
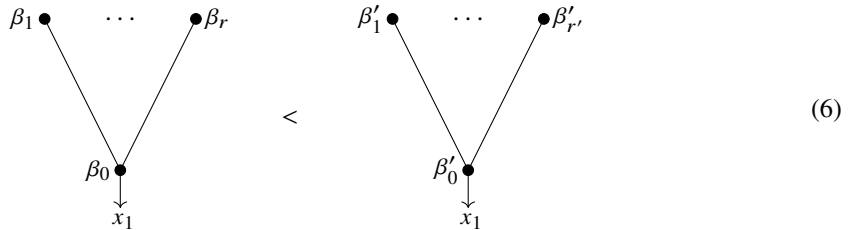
if and only if

(1) $\beta_0 < \beta_0'$; or
(2) $\beta_0 = \beta_0'$ and $r < r'$; or
(3) $\beta_0 = \beta_0'$, $r = r'$ and $v(x_2) \neq v_0$ but $v'(x_2) = v_0$.

Remark 2.13. Given floral cones $\sigma, \sigma' \in \mathsf{T}_{0,2}(X,\beta)$ with $\sigma' \in \mathrm{Star}(\sigma)$, stability ensures that $\sigma' < \sigma$. Equivalently,

$$\sigma' \not< \sigma \Rightarrow \sigma' \notin \mathrm{Star}(\sigma).$$

Therefore, σ' will be unaffected by taking a weighted stellar subdivision along σ; that is, it will remain a cone in the subdivided cone complex.

Given this setup, we have the following strong combinatorial structure result.

Theorem 2.14. *The morphism $\mathsf{T}_{0,2}(X,\beta)^\dagger \to \mathsf{T}_{0,2}(X,\beta)$ is an iterated weighted stellar subdivision of $\mathsf{T}_{0,2}(X,\beta)$ along floral cones, in an order extending the partial order (6).*

A **floral stratum** is a closed boundary stratum $Z(\sigma) \subseteq \mathsf{K}_{0,2}(X, \beta)$ corresponding to a floral cone $\sigma \in \mathsf{T}_{0,2}(X, \beta)$.

Corollary 2.15. *The morphism $\mathsf{K}_{0,2}(X, \beta)^{\dagger} \to \mathsf{K}_{0,2}(X, \beta)$ is an iterated weighted blowup of $\mathsf{K}_{0,2}(X, \beta)$ along strict transforms of floral strata, in an order extending the partial order (6).*

Remark 2.16. The statement of Remark 2.13 asserts that $Z(\sigma')$ is not contained in the blowup centre $Z(\sigma)$ and therefore that its strict transform under the blowup – indexed by the same cone σ' in the subdivided complex – is nonempty.

Proof of Theorem 2.14. Fix a cone $\sigma \in \mathsf{T}_{0,2}(X, \beta)$ corresponding to a combinatorial type of a two-pointed, degree-weighted tropical curve C. As before, let $v_0 \in \mathsf{C}$ denote the root vertex containing the marking x_1 and use the balancing condition to assign formal expansion factors to every edge.

We construct the radially aligned subdivision $\sigma^{\dagger} \to \sigma$ inductively. The idea is as follows: in order to choose a total ordering of the distances $\varphi(v)$ of the vertices from the root, we first must decide which vertex has smallest $\varphi(v)$. Having done this, we then need to decide which vertex is the next smallest; that is, the smallest amongst the remaining vertices and so on. Each step is a weighted stellar subdivision of a floral cone, consistent with the ordering (6).

Edges with zero expansion factor play no role in the subdivision, since their length parameters do not appear in the $\varphi(v)$. Therefore, we formally contract all such edges for this discussion. Orient the graph C in such a way that every edge points away from the root. The first step is to decide which v has minimal $\varphi(v)$; the candidate vertices are the immediate descendants of v_0:
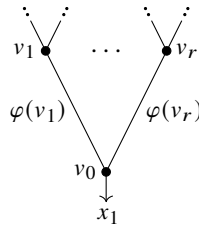
Setting all coordinates other than $\varphi(v_1), \ldots, \varphi(v_r)$ to zero, we obtain the floral subcone
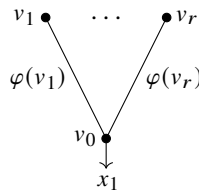
The weighted stellar subdivision of σ along this floral subcone subdivides σ into cones, on each of which we have

$$\min(\varphi(v_1), \ldots, \varphi(v_n)) = \varphi(v_i)$$

for some i. The weights are determined by the edge expansion factors, noting that the $\varphi(v_i)$ may not be primitive in σ. On each cone of the subdivision there is a minimal vertex of C. This forms the base of the induction.

For the induction step, choose a cone ρ of the subdivision constructed so far; to simplify notation, we assume that ρ is maximal. On this cone we have a total ordering of a subset $\{u_1, \ldots, u_k\}$ of the vertices of C:

$$\varphi(u_1) < \ldots < \varphi(u_k) < \varphi(v) \quad \text{for} \quad v \notin \{u_1, \ldots, u_k, v_0\}. \tag{7}$$

Suppose that at the previous step we had taken a weighted stellar subdivision along a floral cone

$$(8)$$

and that (without loss of generality) ρ is the cone of this subdivision on which $\varphi(v_1)$ is minimal (that is, $v_1 = u_k$ in (7)).

The candidates for the next-smallest vertex of $\sqsubset$ comprise the vertices $v_2, \ldots, v_r$ along with any immediate descendants of v_1; denote these by $w_1, \ldots, w_s$. The following picture describes $\sqsubset$:
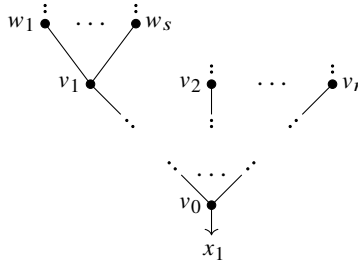
The following functions then form part of a coordinate system for the cone ρ:

$$\varphi(v_2) - \varphi(v_1), \ldots, \varphi(v_r) - \varphi(v_1), \varphi(w_1) - \varphi(v_1), \ldots, \varphi(w_s) - \varphi(v_1) \qquad (9)$$

(geometrically, the curve is destabilised by slicing it with the circle of radius $\varphi(v_1)$ and the parameter $\varphi(v_i) - \varphi(v_1)$ is the length of the final edge segment preceding the vertex v_i). Set all parameters other than (9) to zero to obtain the following floral subcone:

$$(10)$$
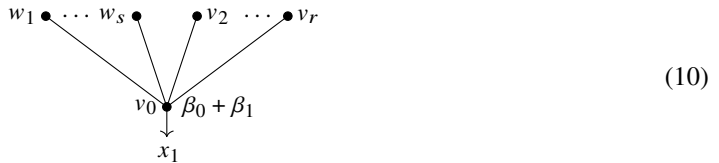
Note that, since $\varphi(v_1) = 0$ on this cone, the degree of the root changes from β_0 in (8) to $\beta_0 + \beta_1$ in (10). Taking the weighted stellar subdivision along this cone corresponds to choosing a minimum amongst the parameters (9). But of course this is equivalent to choosing a minimum amongst

$$\varphi(v_2), \ldots, \varphi(v_r), \varphi(w_1), \ldots, \varphi(w_s).$$

We have completed the induction step of the construction. Either $\beta_1 > 0$, in which case $\beta_0 < \beta_0 + \beta_1$, or $\beta_1 = 0$ and so by stability we have either $s \geq 2$ and so $r < s + r - 1$, or $s = 1$, in which case v_1 must contain the marking x_2, which lies on the vertex v_0 once we set $\varphi(v_1) = 0$. In every case, we see that the floral locus (10) appears strictly later than (8) in our ordering (6). $\qquad \square$

Remark 2.17. We note $\mathsf{T}_{0,2}(X, \beta)^\dagger \to \mathsf{T}_{0,2}(X, \beta)$ is not a weighted stellar subdivision along *every* floral cone. It is obtained by subdividing along those floral cones for which each edge of the tropical curve is assigned a nonzero formal expansion factor with respect to D_1; equivalently, those floral cones

for which $D_1 \cdot \beta_i > 0$ for $i \in \{1, \ldots, r\}$. This follows from the formal contraction of edges with zero expansion factor, carried out in the proof above.

Example 2.18. Take $X = \mathbb{P}^n$ with $D_1 = H_1$ a hyperplane and consider the 4-dimensional cone $\rho \in \mathsf{T}_{0,2}(\mathbb{P}^n, 3)$ indexed by the following combinatorial type:
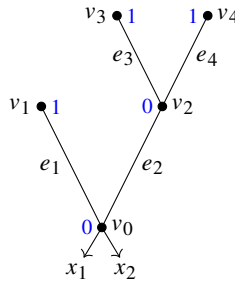

where the degree data are given in blue. We show how the above procedure produces the radial alignment subdivision of ρ. We assign formal expansion factors to the edges, which in this case gives
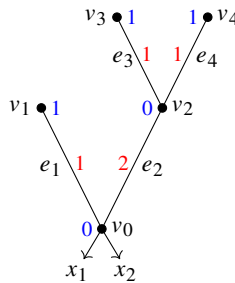
The distances from the root vertex are then given by

$$\varphi(v_1) = e_1, \; \varphi(v_2) = 2e_2, \; \varphi(v_3) = 2e_2 + e_3, \; \varphi(v_4) = 2e_2 + e_4.$$

The radial alignment construction subdivides ρ into cones on which these quantities are totally ordered. Following the process outlined in the proof of Theorem 2.14, the first step is to compare $\varphi(v_1)$ and $\varphi(v_2)$. This amounts to taking a weighted stellar subdivision along the floral subcone
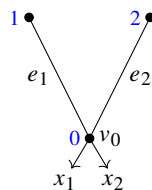
obtained inside ρ by setting $e_3 = e_4 = 0$. The maximal cones of this first subdivision are $\rho_1 = \{e_1 < 2e_2\}$ and $\rho_2 = \{2e_2 < e_1\}$. We focus on the latter (similar arguments apply to the former). On ρ_2 we have $\varphi(v_2) < \varphi(v_1)$, and the next step is to select a minimum amongst $\varphi(v_1), \varphi(v_3)$ and $\varphi(v_4)$. This amounts to subdividing along the floral subcone obtained inside ρ_2 by setting $e_2 = 0$:
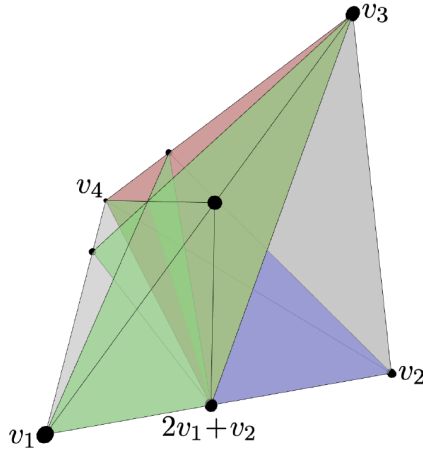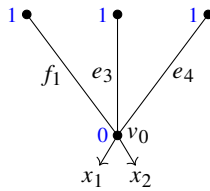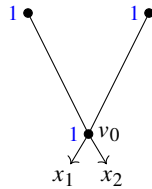
Figure 1. *The subdivision of ρ described in Example 2.18. The v_i are dual to the parameters e_i.*

Here $f_1 = e_1 - 2e_2$ forms part of the natural coordinate system on ρ_2. This second subdivision produces three maximal cones inside ρ_2, and restricting to any one of these we see that the third and final step is to subdivide along a floral subcone of type

Note that the ordering (6) of floral cones is respected. The height- 1 slice is shown in Figure 1.

A note on monodromy. Floral strata typically have self-intersections. However, the iterative process described above only involves blowups along strata with no self-intersections. This is the content of the following lemma; the key observation is that the self-intersection of each floral stratum is separated by taking the strict transforms along the blowups appearing earlier in the process.

Lemma 2.19. *Let $\sigma \in \mathsf{T}_{0,2}(X, \beta)$ be a floral cone and $\mathsf{T}_{0,2}(X, \beta)^{\ddagger}$ be the partial subdivision of $\mathsf{T}_{0,2}(X, \beta)$, such that σ is the next floral cone to be subdivided. The stratum $Z(\sigma) \subseteq \mathsf{K}_{0,2}(X, \beta)^{\ddagger}$ has empty self-intersection.*

Proof. Combinatorially, this says that there is no cone $\rho \in \mathsf{T}_{0,2}(X, \beta)^{\ddagger}$ which contains σ as a face in two different positions. Suppose that such a cone exists. As in the proof of Theorem 2.14, ρ is indexed by a combinatorial type of tropical curve, with a partial ordering of a subset of its vertices (which without loss of generality are chosen to be strict):

$$\varphi(u_1) < \ldots < \varphi(u_k) < \varphi(v) \quad \text{for} \quad v \notin \{u_1, \ldots, u_k, v_0\}.$$

Let $v_1, \ldots, v_r$ denote the vertices of $\sqsubset$ lying immediately outside the circle of radius $\varphi(u_k)$ around v_0. Then a $\mathbb{Q}$-coordinate system for the simplicial cone ρ is given by

$$\varphi(u_1), \; \varphi(u_2) - \varphi(u_1), \; \ldots, \; \varphi(u_k) - \varphi(u_{k-1}), \tag{11}$$

$$\varphi(v_1) - \varphi(u_k), \; \varphi(v_2) - \varphi(u_k), \; \ldots, \; \varphi(v_r) - \varphi(u_k), \tag{12}$$

$$f_1, \; \ldots, \; f_l, \tag{13}$$

where the functions $f_1, \ldots, f_l$ are the lengths of all edges of $\sqsubset$ which descend from $v_1, \ldots, v_r$.

The above parameters are presented in increasing order of distance from v_0. The ordering of the $\varphi(u_i)$ defines a collection of concentric circles around v_0, and the coordinates (11) give the widths of the corresponding annuli. Since σ is the next cone to subdivide, the functions (12) form a coordinate system for σ, cut out inside ρ by setting the coordinates (11) and (13) to zero.

Recall that we assume that ρ contains σ as a face in another position. Hence, there is a different collection of coordinates on ρ whose common vanishing locus also gives σ. We claim that this collection must also contain all of the coordinates (11); otherwise, the resulting combinatorial type involves a nontrivial order relation between the edge lengths and hence cannot give σ since σ was not altered by the previous subdivisions; see Remark 2.13.

Therefore, all of the coordinates (11) must be set to zero. For dimension reasons, at least one of the coordinates (12) must also be set to zero. As in the proof of Theorem 2.14, it follows from stability that the resulting combinatorial type cannot be σ, since the discrete data associated to v_0 (curve class, number of adjacent edges, markings) must increase. □

3. Correcting the correspondence II: blowup formula analysis

The previous section furnishes a sequence of birational modifications of the space of maps, resulting in a completely explicit intersection problem. We now unwind this problem and explain how it corrects the local-logarithmic correspondence.

3.1. Corrected products

Consider again the diagram of subdivided moduli spaces from Subsection 2.3:

$$
\begin{array}{ccc}
\mathsf{K}_{0,2}^{\max}(X|D,\beta)^\dagger & \longrightarrow & \mathsf{K}_{0,2}^{\max}(X|D_1,\beta)^\dagger \\
\downarrow & \square & \downarrow \\
\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)^\dagger & \longrightarrow & \mathsf{K}_{0,2}(X,\beta)^\dagger.
\end{array}
$$

The entries in this diagram are all logarithmically smooth and irreducible. The fibre product is transverse over the dense locus where the logarithmic structure is trivial, yielding an equality

$$[\mathsf{K}_{0,2}^{\max}(X|D,\beta)^\dagger] = [\mathsf{K}_{0,2}^{\max}(X|D_1,\beta)^\dagger] \cdot [\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)^\dagger] \qquad \text{in } \mathsf{K}_{0,2}(X,\beta)^\dagger$$

(pushforwards have been suppressed from the notation). Pushing down along the modification $\rho \colon \mathsf{K}_{0,2}(X,\beta)^\dagger \to \mathsf{K}_{0,2}(X,\beta)$, we obtain

$$[\mathsf{K}_{0,2}^{\max}(X|D,\beta)] = \rho_\star \left([\mathsf{K}_{0,2}^{\max}(X|D_1,\beta)^\dagger] \cdot [\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)^\dagger] \right) \qquad \text{in } \mathsf{K}_{0,2}(X,\beta). \tag{14}$$

It is natural to ask how this class relates to the naive intersection class:

$$[\mathsf{K}_{0,2}^{\max}(X|D_1,\beta)] \cdot [\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)] \qquad \text{in } \mathsf{K}_{0,2}(X,\beta). \tag{15}$$

We probe the geometry of the logarithmic modifications in order to explicitly describe the difference between these classes. The main result is Theorem 3.4, which expresses this difference as a sum of correction terms supported on excess loci.

This result compares the logarithmic and naive theories; see [23, §3]. We view this comparison as the more fundamental result; the local-logarithmic comparison arises as a consequence.

3.2. Corrected local-logarithmic correspondence

Let $F\colon \mathsf{K}_{0,2}(X,\beta) \to \mathsf{K}_{0,0}(X,\beta)$ denote the forgetful morphism. The corrected local-logarithmic correspondence is obtained by pushing forward Theorem 3.4 along F. The key is the following simple result concerning the local class.

Lemma 3.1. *The following relation holds in* $\mathsf{K}_{0,0}(X,\beta)$:

$$F_\star \left([\mathsf{K}_{0,2}^{\max}(X|D_1,\beta)] \cdot [\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)] \right) = \alpha \cdot [\mathsf{K}_{0,0}(\mathcal{O}_X(-D_1) \oplus \mathcal{O}_X(-D_2),\beta)]^{\mathrm{vir}} \qquad (16)$$

where $\alpha = (-1)^{(D_1+D_2)\cdot\beta}(D_1 \cdot \beta)(D_2 \cdot \beta)$.

Proof. This follows from a comparison of diagonals. For $i \in \{1,2\}$ consider the morphism

$$G_i \colon \mathsf{K}_{0,2}(X,\beta) \to \mathsf{K}_{0,1}(X,\beta),$$

forgetting the marking $x_{\neq i}$, and let $H\colon \mathsf{K}_{0,1}(X,\beta) \to \mathsf{K}_{0,0}(X,\beta)$ be the morphism forgetting the remaining marking. Consider the tower

$$\mathsf{K}_{0,2}(X,\beta) \times \mathsf{K}_{0,2}(X,\beta) \xrightarrow{G_1 \times G_2} \mathsf{K}_{0,1}(X,\beta) \times \mathsf{K}_{0,1}(X,\beta) \xrightarrow{H \times H} \mathsf{K}_{0,0}(X,\beta) \times \mathsf{K}_{0,0}(X,\beta).$$
$$\underbrace{\phantom{\mathsf{K}_{0,2}(X,\beta) \times \mathsf{K}_{0,2}(X,\beta) \xrightarrow{G_1 \times G_2} \mathsf{K}_{0,1}(X,\beta) \times \mathsf{K}_{0,1}(X,\beta)}}_{F \times F}$$

For $n \in \{0,1,2\}$ we denote the inclusion and fundamental class of the diagonal by

$$\iota_n \colon \mathsf{K}_{0,n}(X,\beta) \hookrightarrow \mathsf{K}_{0,n}(X,\beta) \times \mathsf{K}_{0,n}(X,\beta), \qquad \Delta_{0,n} = (\iota_n)_\star [\mathsf{K}_{0,n}(X,\beta)].$$

These moduli spaces are unobstructed and we have

$$(G_1 \times G_2)_\star(\Delta_{0,2}) = (H \times H)^\star(\Delta_{0,0}) \qquad (17)$$

since this equality holds on the dense open locus where the source curve is smooth. There is also an equality

$$[\mathsf{K}_{0,2}^{\max}(X|D_i,\beta)] = (G_i)^\star [\mathsf{K}_{0,1}^{\max}(X|D_i,\beta)].$$

From these we obtain

$$
\begin{aligned}
(\iota_0)_\star\big(F_\star\big(&[\mathsf{K}_{0,2}^{\max}(X|D_1,\beta)]\cdot[\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)]\big)\big) \\
&= (F\times F)_\star(\iota_2)_\star\left([\mathsf{K}_{0,2}^{\max}(X|D_1,\beta)]\cdot[\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)]\right) \\
&= (F\times F)_\star\left(\left([\mathsf{K}_{0,2}^{\max}(X|D_1,\beta)]\times[\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)]\right)\cdot\Delta_{0,2}\right) \\
&= (H\times H)_\star(G_1\times G_2)_\star\left(\left(G_1^\star[\mathsf{K}_{0,1}^{\max}(X|D_1,\beta)]\times G_2^\star[\mathsf{K}_{0,1}^{\max}(X|D_2,\beta)]\right)\cdot\Delta_{0,2}\right) \\
&= (H\times H)_\star\left(\left([\mathsf{K}_{0,1}^{\max}(X|D_1,\beta)]\times[\mathsf{K}_{0,1}^{\max}(X|D_2,\beta)]\right)\cdot(G_1\times G_2)_\star(\Delta_{0,2})\right) \\
&= (H\times H)_\star\left(\left([\mathsf{K}_{0,1}^{\max}(X|D_1,\beta)]\times[\mathsf{K}_{0,1}^{\max}(X|D_2,\beta)]\right)\cdot(H\times H)^\star(\Delta_{0,0})\right) \\
&= \left(H_\star[\mathsf{K}_{0,1}^{\max}(X|D_1,\beta)]\times H_\star[\mathsf{K}_{0,1}^{\max}(X|D_2,\beta)]\right)\cdot\Delta_{0,0} \\
&= (\iota_0)_\star\left(H_\star[\mathsf{K}_{0,1}^{\max}(X|D_1,\beta)]\cdot H_\star[\mathsf{K}_{0,1}^{\max}(X|D_2,\beta)]\right).
\end{aligned}
$$

Letting $p\colon \mathsf{K}_{0,0}(X,\beta)\times\mathsf{K}_{0,0}(X,\beta)\to\mathsf{K}_{0,0}(X,\beta)$ be the first projection, we have $p\circ\iota_0=\mathrm{Id}$ and so applying $p_\star$ gives

$$
F_\star\big([\mathsf{K}_{0,2}^{\max}(X|D_1,\beta)]\cdot[\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)]\big) = H_\star[\mathsf{K}_{0,1}^{\max}(X|D_1,\beta)]\cdot H_\star[\mathsf{K}_{0,1}^{\max}(X|D_2,\beta)].
$$

The claim then follows from the local-logarithmic correspondence for smooth divisors [31] applied to the targets $(X|D_1)$ and $(X|D_2)$ and the product formula for the local class (a consequence of the splitting of the obstruction bundle). $\qquad\square$

Given this result, if we can relate the classes (14) and (15) on $\mathsf{K}_{0,2}(X,\beta)$, then pushing forward along F will relate the logarithmic class to the local class on $\mathsf{K}_{0,0}(X,\beta)$.

3.3. *Iterated blowups: conventions and notation*

The iterative description of the modification ρ given in Subsection 2.6 will allow us to access this intersection theory. The key tool is Fulton's blowup formula comparing the strict transform to the refined total transform [15, §6.7].

By Corollary 2.15 we may factor the modification ρ as a tower of weighted blowups along strict transforms of floral strata:

$$
\mathsf{K}_{0,2}(X,\beta)^\dagger = \mathsf{K}_{0,2}(X,\beta)_m \to \mathsf{K}_{0,2}(X,\beta)_{m-1} \to \cdots \to \mathsf{K}_{0,2}(X,\beta)_0 = \mathsf{K}_{0,2}(X,\beta). \tag{18}
$$

For $j\in\{1,\ldots,m\}$ we let σ_j denote the floral cone whose weighted blowup produces $\mathsf{K}_{0,2}(X,\beta)_j \to \mathsf{K}_{0,2}(X,\beta)_{j-1}$. Notice that σ_j is a cone in both $\mathsf{T}_{0,2}(X,\beta)$ and in the subdivided cone complex $\mathsf{T}_{0,2}(X,\beta)_{j-1}$ and, as such, represents strata in both $\mathsf{K}_{0,2}(X,\beta)$ and $\mathsf{K}_{0,2}(X,\beta)_{j-1}$. We denote these respectively by

$$
Z(\sigma_j) = Z(\sigma_j)_0 \subseteq \mathsf{K}_{0,2}(X,\beta), \qquad Z(\sigma_j)_{j-1} \subseteq \mathsf{K}_{0,2}(X,\beta)_{j-1}.
$$

Of course, $Z(\sigma_j)_{j-1}$ is nothing but the strict transform of $Z(\sigma_j)$ under the preceding weighted blowups $\mathsf{K}_{0,2}(X,\beta)_{j-1}\to\mathsf{K}_{0,2}(X,\beta)$ (which is well-defined because of the ordering: see Remark 2.16).

For $0\le j<k\le m$ we let $\rho_{k,j}$ denote the birational morphism

$$
\rho_{k,j}\colon \mathsf{K}_{0,2}(X,\beta)_k \to \mathsf{K}_{0,2}(X,\beta)_j.
$$

3.4. The blowup formula: strict and total transforms

For $i \in \{1, 2\}$ there is a corresponding tower of strict transforms:

$$\mathsf{K}_{0,2}^{\max}(X|D_i, \beta)^{\dagger} = \mathsf{K}_{0,2}^{\max}(X|D_i, \beta)_m \to \mathsf{K}_{0,2}^{\max}(X|D_i, \beta)_{m-1} \to \cdots \to \mathsf{K}_{0,2}^{\max}(X|D_i, \beta)_0 = \mathsf{K}_{0,2}^{\max}(X|D_i, \beta).$$

(Note that by Corollary 2.8 these strict transforms are all isomorphic for $i = 1$; this does not hold for $i = 2$ and in any case is not important for the arguments which follow.)

Fulton's blowup formula compares the fundamental class of the strict transform to the refined fundamental class of the total transform, the latter being defined by Gysin pullback [15, Proposition 6.7 and Example 6.7.1]. In Subsection 3.6 we explain how to extend this to weighted blowups. For each $j \in \{1, \ldots, m\}$ we obtain the following relation in $\mathsf{K}_{0,2}(X, \beta)_j$:

$$\rho_{j,j-1}^{\star}[\mathsf{K}_{0,2}^{\max}(X|D_i, \beta)_{j-1}] = [\mathsf{K}_{0,2}^{\max}(X|D_i, \beta)_j] + A_j^i \tag{19}$$

where A_j^i is a correction term supported on the excess locus of the total transform

$$
\begin{array}{ccccc}
\mathsf{K}_{0,2}^{\max}(X|D_i, \beta)_j & \hookrightarrow & \mathsf{K}_{0,2}^{\max}(X|D_i, \beta)_j^{\mathrm{tot}} & \longrightarrow & \mathsf{K}_{0,2}(X, \beta)_j \\
\downarrow & & \square & & \downarrow{\scriptstyle \rho_{j,j-1}} \\
& & \mathsf{K}_{0,2}^{\max}(X|D_i, \beta)_{j-1} & \longrightarrow & \mathsf{K}_{0,2}(X, \beta)_{j-1}.
\end{array}
$$

We describe the terms A_j^i in detail. The weighted blowup of the moduli space of ordinary stable maps has an exceptional divisor lying over the blowup centre

$$
\begin{array}{ccc}
E_j & \hookrightarrow & \mathsf{K}_{0,2}(X, \beta)_j \\
\downarrow & \square & \downarrow \\
Z(\sigma_j)_{j-1} & \hookrightarrow & \mathsf{K}_{0,2}(X, \beta)_{j-1}
\end{array}
$$

and the excess locus F_j^i in the total transform of the logarithmic moduli space is obtained by fibring over this blowup centre:

$$
\begin{array}{ccc}
F_j^i & \longrightarrow & E_j \\
\downarrow & \square & \downarrow \\
\mathsf{K}_{0,2}^{\max}(X|D_i, \beta)_{j-1} & \longrightarrow & \mathsf{K}_{0,2}(X, \beta)_{j-1}.
\end{array}
$$

To describe F_j^i we must identify the maximal strata in $\mathsf{K}_{0,2}^{\max}(X|D_i, \beta)_{j-1}$ which map to $Z(\sigma_j)_{j-1}$. This is equivalent to identifying the minimal cones in $\mathsf{T}_{0,2}^{\max}(X|D_i, \beta)_{j-1}$ that map to the interior of $\mathrm{Star}(\sigma_j) \subseteq \mathsf{T}_{0,2}(X, \beta)_{j-1}$.

3.4.1. Excess locus for D_1

For $i = 1$ these cones can be identified very explicitly. We expect the following result to be extremely useful for future applications of the rank reduction technique. Recall that associated to the floral cone $\sigma_j \in \mathsf{T}_{0,2}(X, \beta)$ there is a unique cone

$$\tau_j \in \mathsf{T}_{0,2}^{\max}(X|D_1, \beta)$$

obtained by assigning formal expansion factors to the edges of the tropical curve and requiring all nonroot vertices to map to $0 \in \mathbb{R}_{\geq 0}$:
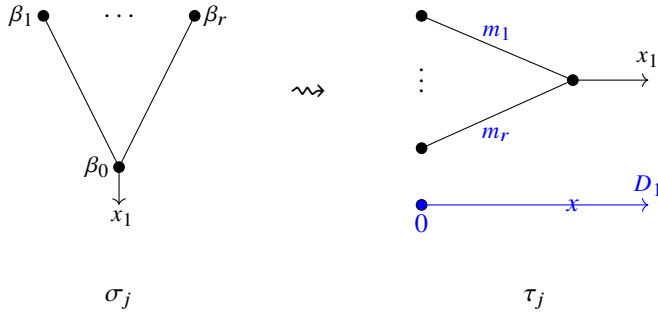
Here each expansion factor $m_i = D_1 \cdot \beta_i$ is nonzero (see Remark 2.17) and so $\dim \tau_j = 1$. We refer to τ_j as a **comb cone**. Clearly, we have $\tau_j \to \sigma_j$ and both τ_j and σ_j are unaffected by the first $j - 1$ weighted stellar subdivisions.

Theorem 3.2. *The cone τ_j in $\mathsf{T}_{0,2}^{\max}(X|D_1, \beta)_{j-1}$ is the unique minimal cone mapping into $\mathrm{Star}(\sigma_j) \subseteq \mathsf{T}_{0,2}(X, \beta)_{j-1}$; that is, every cone θ in $\mathsf{T}_{0,2}^{\max}(X|D_1, \beta)_{j-1}$ mapping into $\mathrm{Star}(\sigma_j)$ contains τ_j as a face.*

Remark 3.3. The analogous statement on the initial moduli space $\mathsf{T}_{0,2}(X, \beta)$ fails to hold; a simple counterexample with $(X|D_1) = (\mathbb{P}^n|H)$ is given by
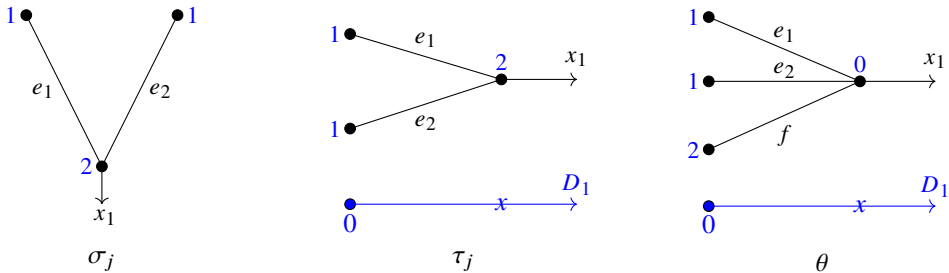
where the curve classes are indicated in blue and the tropical edge lengths in black. Here θ maps into $\mathrm{Star}(\sigma_j)$ but does not contain τ_j as a face. The issue is that when we try to specialise θ by setting $f = 0$, the edge length relations $e_1 = e_2 = 2f$ force $e_1 = e_2 = 0$ as well.

Nevertheless, we claim that after performing the first $j - 1$ subdivisions, the statement holds. The reason for this is that these subdivisions cause the star of σ_j to become smaller, so eventually every such minimal cone θ maps outside the star; this occurs because the tropical edge length relations for continuity of the tropical map to $\mathbb{R}_{\geq 0}$ become incompatible with the radial alignment inequalities. Geometrically, this means that every such maximal stratum $Z(\theta)$ becomes separated from $Z(\sigma_j)$ by the process of blowing up and taking strict transforms.

This process is visible in the above example; the first subdivision reduces the star of σ_j to the locus where $2f < e_1, e_2$, and this is incompatible with the continuity relations on θ.

Proof. Suppose that we are given a cone ρ in $\mathrm{Star}(\sigma_j)$:

$$\sigma_j \subseteq \rho \in \mathsf{T}_{0,2}(X, \beta)_{j-1}.$$

The cone ρ is indexed by the combinatorial type of a more degenerate tropical curve, together with a partial radial alignment of its vertices. Crucially, this partial alignment is such that, in the iterative

subdivision process described in the proof of Theorem 2.14, σ_j is the next floral cone at which we must subdivide. Now consider a cone

$$\theta \in \mathsf{T}_{0,2}^{\max}(X|D_1, \beta)_{j-1} = \mathsf{T}_{0,2}^{\max}(X|D_1, \beta)$$

which maps into ρ. We wish to prove that θ contains the comb cone τ_j as a face.

In order for $\theta \to \rho$, their combinatorial types must have the same source curve, after possibly removing some 2-valent vertices. The vertex alignments/orderings induced by the combinatorial types must be compatible. This means that, if the combinatorial type of θ has k target expansion levels, then ρ aligns precisely those vertices mapped to the final l levels (for some $l < k$), in the same order as prescribed by θ.

As noted, the partial ordering on ρ is such that in the iterative subdivision process, σ_j is the next cone at which we must subdivide. Following the procedure in the proof of Theorem 2.14, this means the following: after we set $\varphi(u_i) = 0$ for every vertex u_i appearing in the partial alignment, we obtain a tropical curve such that the immediate descendants of v_0 give the type of σ_j, in the sense that setting all further edge lengths to zero specialises to the cone σ_j.

On θ, the tropical parameters are given by the target edge lengths and the above description of the shape of the tropical curve implies that when we set every target edge length except the $(k - l - 1)$st to zero, we specialise to the type of τ_j. This shows τ_j is a face of θ, as claimed.   □

By Theorem 3.2, the excess locus F_j^1 in the total transform of $\mathsf{K}_{0,2}^{\max}(X|D_1, \beta)_{j-1}$ is given by

$$F_j^1 = Z(\tau_j)_{j-1} \times_{Z(\sigma_j)_{j-1}} E_j.$$

This is an explicit weighted projective bundle of dimension $r-1$ over the comb stratum $Z(\tau_j)_{j-1} = Z(\tau_j)$ (where r is the number of leaves of the source curve in σ_j). It has excess dimension $r-2$ and the blowup formula takes the form

$$\rho_{j,j-1}^\star [\mathsf{K}_{0,2}^{\max}(X|D_1, \beta)_{j-1}] = [\mathsf{K}_{0,2}^{\max}(X|D_1, \beta)_j] + \gamma_j^1 \cap [F_j^1] \tag{20}$$

where γ_j^1 is an excess class of codimension $r-2$. This is obtained from Chern and Segre classes of strata and hence can be described algorithmically in terms of tautological classes; see Remark 3.5. We defer this computation to future work.

We may now pass up the tower (18) of weighted blowups, applying the formula (20) at each level. At the top we obtain

$$\rho^\star [\mathsf{K}_{0,2}^{\max}(X|D_1, \beta)] = [\mathsf{K}_{0,2}^{\max}(X|D_1, \beta)^\dagger] + \sum_{j=1}^m \rho_{m,j}^\star \left(\gamma_j^1 \cap [F_j^1] \right) \tag{21}$$

in $\mathsf{K}_{0,2}(X, \beta)^\dagger = \mathsf{K}_{0,2}(X, \beta)_m$ (where $\rho = \rho_{m,0}$). We note that the pullback $\rho_{m,j}^\star(\gamma_j^1 \cap [F_j^1])$ is typically nontransverse and so further excess classes will enter into the formula; however, since we will push back down along ρ, there is no need to describe these. This formula provides a quantitative relationship between the strict transform and the refined total transform of $\mathsf{K}_{0,2}^{\max}(X|D_1, \beta)$.

3.4.2. Excess locus for D_2
For $i = 2$ there is no result analogous to Theorem 3.2 and there are typically several minimal cones

$$\theta \in \mathsf{T}_{0,2}^{\max}(X|D_2, \beta)_{j-1}$$

which map into $\mathrm{Star}(\sigma_j)$. It is possible to construct examples where these minimal cones have different dimensions. This is not surprising; the subdivision of $\mathsf{T}_{0,2}(X, \beta)$ is constructed with reference to D_1 and so is typically insensitive to the geometry of D_2.

Let us denote the minimal cones of $\mathsf{T}_{0,2}^{\max}(X|D_2,\beta)_{j-1}$ mapping into $\mathrm{Star}(\sigma_j)$ by

$$\theta_j(1),\ldots,\theta_j(l_j).$$

We note that some of these cones may be exceptional; that is, the corresponding strata

$$Z(\theta_j(k))_{j-1} \subseteq \mathsf{K}_{0,2}^{\max}(X|D_2,\beta)_{j-1}$$

may have positive dimension over $\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)$. The irreducible components of the excess locus F_j^2 are the fibre products

$$F_j^2(k) = Z(\theta_j(k))_{j-1} \times_{Z(\sigma_j)_{j-1}} E_j.$$

As before, each such component is an explicit weighted projective bundle of dimension $r-1$ over the stratum $Z(\theta_j(k))_{j-1}$. Its excess dimension is determined by the dimension of the cone $\theta_j(k)$ and is at most $r-2$. The excess class γ_j^2 may be written as a sum of classes pushed forward from the irreducible components (such an expression is necessarily nonunique, but see Remark 3.5). We arrive at the correction term:

$$\rho_{j,j-1}^{\star}[\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)_{j-1}] = [\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)_j] + \Sigma_{k=1}^{l_j} \gamma_j^2(k) \cap [F_j^2(k)]. \tag{22}$$

Applying this iteratively, at each level of the tower (18), we obtain

$$\rho^{\star}[\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)] = [\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)^{\dagger}] + \sum_{j=1}^{m}\sum_{k=1}^{l_j} \rho_{m,j}^{\star}\left(\gamma_j^2(k) \cap [F_j^2(k)]\right) \tag{23}$$

in $\mathsf{K}_{0,2}(X,\beta)^{\dagger} = \mathsf{K}_{0,2}(X,\beta)_m$.

3.5. *Corrected product formula and local-logarithmic correspondence*

Expressions (21) and (23) allow us to compare the intersections before and after blowing up. We obtain

$$\begin{aligned}
\rho^{\star}&\left([\mathsf{K}_{0,2}^{\max}(X|D_1,\beta)] \cdot [\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)]\right)\\
&= [\mathsf{K}_{0,2}^{\max}(X|D_1,\beta)^{\dagger}] \cdot [\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)^{\dagger}]\\
&\quad + \rho^{\star}[\mathsf{K}_{0,2}^{\max}(X|D_1,\beta)] \cdot \sum_{j=1}^{m}\sum_{k=1}^{l_j} \rho_{m,j}^{\star}\left(\gamma_j^2(k) \cap [F_j^2(k)]\right)\\
&\quad + \rho^{\star}[\mathsf{K}_{0,2}^{\max}(X|D_2,\beta)] \cdot \sum_{j=1}^{m} \rho_{m,j}^{\star}\left(\gamma_j^1 \cap [F_j^1]\right)\\
&\quad - \sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\left(\rho_{m,j_1}^{\star}\left(\gamma_{j_1}^1 \cap [F_{j_1}^1]\right) \cdot \sum_{k=1}^{l_{j_2}} \rho_{m,j_2}^{\star}\left(\gamma_{j_2}^2(k) \cap [F_{j_2}^2(k)]\right)\right).
\end{aligned}$$

We now apply $\rho_{\star}$ to this. The first term on the right-hand side pushes forward to

$$[\mathsf{K}_{0,2}^{\max}(X|D,\beta)]$$

by (14). On the other hand, the second and third terms push forward to zero by the projection formula, since the individual correction terms vanish under pushforward:

$$(\rho_{j,j-1})_\star \left(\gamma_j^1 \cap [F_j^1]\right) = 0, \qquad (\rho_{j,j-1})_\star \left(\gamma_j^2(k) \cap [F_j^2(k)]\right) = 0.$$

Similarly, in the final term only products of correction terms with $j_1 = j_2$ can survive. We obtain the comparison of logarithmic and naive virtual classes.

Theorem 3.4. *The following relation holds in* $\mathsf{K}_{0,2}(X, \beta)$:

$$[\mathsf{K}_{0,2}^{\max}(X|D_1, \beta)] \cdot [\mathsf{K}_{0,2}^{\max}(X|D_2, \beta)] = [\mathsf{K}_{0,2}^{\max}(X|D, \beta)]$$
$$- \sum_{j=1}^m (\rho_{j,0})_\star \left(\sum_{k=1}^{l_j} (\gamma_j^1 \cap [F_j^1]) \cdot (\gamma_j^2(k) \cap [F_j^2(k)]) \right).$$

Pushforward along the morphism F produces the corrected local-logarithmic correspondence by Lemma 3.1.

Remark 3.5 (Implementation). The correction terms are well-understood as tautological classes. Excess loci are always weighted projective bundles over logarithmic strata, obtained by imposing edge length equalities in the tropical moduli. Excess *classes* are handled as follows. (1) Chern classes of normal bundles to strata of the space of absolute maps are easily calculated. (2) The classes of the exceptional divisors are handled directly. (3) Segre classes of strata of (strict transforms of) the relative spaces; these are the most complex and arise as follows. The blowup centre is cut out by a monoidal ideal; the intersection with the relative space is therefore cut out by the pullback of this ideal. The Segre class is computed by Aluffi's formula for Segre classes [5, Theorem 1.1]. The output is an explicit weighted linear combination of smooth strata with tautological excess classes. The correction is therefore determined.

In ongoing work, we calculate the terms explicitly and identify situations in which they vanish, establishing new cases of the numerical local-logarithmic correspondence.

Remark 3.6. Theorem 3.4 holds with the same proof with additional markings without tangency.

3.6. Stacky subdivisions, roots and weighted blowups

The key formula (19) above (along with its more specific counterparts (20) and (22)) requires a generalisation of Fulton's blowup formula to weighted blowups. For this, it is more convenient to interpret each weighted blowup as a smooth orbitoroidal embedding, rather than a logarithmically smooth toroidal embedding. We now explain this process. The basic idea is that a weighted blowup factors uniquely as a root stack followed by an ordinary blowup. We assume familiarity with the basics of toric orbifolds [9, 14].

For $j \in \{1, \dots, m\}$ the weighted stellar subdivision $\mathsf{T}_{0,2}(X, \beta)_j \to \mathsf{T}_{0,2}(X, \beta)_{j-1}$ is induced by a primitive weight vector $w_j \in \sigma_j \cap N_{\sigma_j}$ where N_{σ_j} is the lattice of integral points. The lattice has a natural basis $v_1, \dots, v_r$ dual to the edge lengths of the associated tropical curve, and we can write $w_j = m_1 v_1 + \dots + m_r v_r$ with each $m_i > 0$. Let

$$N_{\sigma_j}^\diamond \subseteq N_{\sigma_j} \qquad (24)$$

be the finite-index sublattice generated by $m_1 v_1, \dots, m_r v_r$. The triple $(\sigma_j, N_{\sigma_j}, N_{\sigma_j}^\diamond)$ constitutes a stacky cone and hence corresponds to an affine toric orbifold, with isotropy given by the cokernel of the

lattice inclusion (24). This globalises uniquely to a family of compatible sublattices, producing a stacky modification:

$$\mathsf{T}_{0,2}(X,\beta)^{\diamond}_{j-1} \to \mathsf{T}_{0,2}(X,\beta)_{j-1}.$$

This is a stacky cone complex; that is, a complex of stacky cones. By [10, §4] the stacky modification induces a nonrepresentable toroidal modification:

$$\mathsf{K}_{0,2}(X,\beta)^{\diamond}_{j-1} \to \mathsf{K}_{0,2}(X,\beta)_{j-1}.$$

This is an iterated root stack, with rooting index m_i along the divisor Z_i corresponding to the ray v_i. Let

$$Z_i/m_i \subseteq \mathsf{K}_{0,2}(X,\beta)^{\diamond}_{j-1}$$

denote the gerby divisor in the root stack; note that $m_i \cdot (Z_i/m_i) = Z_i$, which justifies the notation. The resulting space $\mathsf{K}_{0,2}(X,\beta)^{\diamond}_{j-1}$ is an **orbitoroidal embedding**; that is, a pair which is locally isomorphic to a toric orbifold.

The weight vector w_j has co-ordinates $(1,\ldots,1)$ in the lattice $N^{\diamond}_{\sigma_j}$. We denote the stellar subdivision at w_j by

$$\mathsf{T}_{0,2}(X,\beta)_j \to \mathsf{T}_{0,2}(X,\beta)^{\diamond}_{j-1}.$$

This produces an associated toroidal modification

$$\mathsf{K}_{0,2}(X,\beta)_j \to \mathsf{K}_{0,2}(X,\beta)^{\diamond}_{j-1}$$

which is the blowup of the intersection of the gerby divisors Z_i/m_i. The composite

$$\mathsf{K}_{0,2}(X,\beta)_j \to \mathsf{K}_{0,2}(X,\beta)^{\diamond}_{j-1} \to \mathsf{K}_{0,2}(X,\beta)_{j-1} \tag{25}$$

is a stacky toroidal modification in the sense of [22, §3.1], with relative coarse moduli space given by the ordinary weighted blowup. Locally, $\mathsf{K}_{0,2}(X,\beta)_j$ is the toric stack canonically associated to the simplicial toric variety obtained via the ordinary weighted blowup [14, Theorem 4.11].

Strictly speaking, the above construction differs from the output of the weak semistable reduction algorithm, the latter being the relative coarse moduli space of the former. Crucially, however, the above construction still results in an integral and saturated morphism $\mathsf{K}_{0,2}(X|D_1,\beta)^{\dagger} \to \mathsf{K}_{0,2}(X,\beta)^{\dagger}$, which is all we require.

The composition (25) shows that this stacky toroidal modification is the composition of a root stack and an ordinary blowup. We obtain the weighted blowup formula by pulling back to the root stack and applying the ordinary blowup formula. Pullbacks of characteristic classes to the root stack are well-understood. Since $\mathsf{K}_{0,2}(X,\beta)$ is a smooth Deligne–Mumford stack, it follows inductively that each $\mathsf{K}_{0,2}(X,\beta)_j$ is also a smooth Deligne–Mumford stack (see Corollary 2.10).

3.7. Example

We now apply the iterated blowup procedure to the plane conic example of Subsection 1.2, calculating the defect between local/naive and logarithmic theories. To indicate how the discussion generalises, we employ the same notation as used in Subsections 3.3–3.5.

Consider degree 2 logarithmic stable maps to $(\mathbb{P}^2|H_1 + H_2)$ with maximal tangency at two distinct markings. By Subsection 2.6 the relevant floral cones in $\mathsf{T}_{0,2}(\mathbb{P}^2, 2)$ are
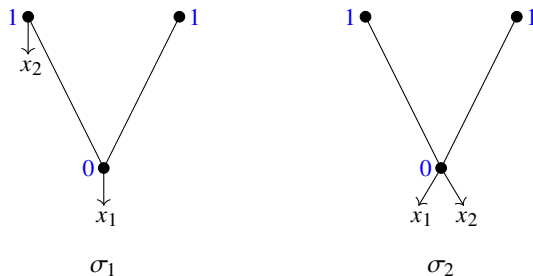
The iterated blowup is obtained by first blowing up $Z(\sigma_1)$ and then blowing up the strict transform of $Z(\sigma_2)$. The blowup weights are trivial because all edges have expansion factor 1. In the notation of Subsection 3.3 we have

$$\mathsf{K}_{0,2}(\mathbb{P}^2, 2)_0 = \mathsf{K}_{0,2}(\mathbb{P}^2, 2)$$
$$\mathsf{K}_{0,2}(\mathbb{P}^2, 2)_1 = \mathrm{Bl}_{Z(\sigma_1)} \mathsf{K}_{0,2}(\mathbb{P}^2, 2)$$
$$\mathsf{K}_{0,2}(\mathbb{P}^2, 2)^\dagger = \mathsf{K}_{0,2}(\mathbb{P}^2, 2)_2 = \mathrm{Bl}_{Z(\sigma_2)_1} \mathrm{Bl}_{Z(\sigma_1)} \mathsf{K}_{0,2}(\mathbb{P}^2, 2)$$

where $Z(\sigma_2)_1 \subseteq \mathsf{K}_{0,2}(\mathbb{P}^2, 2)_1$ is the strict transform of $Z(\sigma_2) \subseteq \mathsf{K}_{0,2}(\mathbb{P}^2, 2)$. Following Subsection 3.4, we calculate the excess loci and correction terms at each step. For $Z(\sigma_1)$, the minimal cones mapping to $\mathrm{Star}(\sigma_1) \subseteq \mathsf{T}_{0,2}(\mathbb{P}^2, 2)$ are
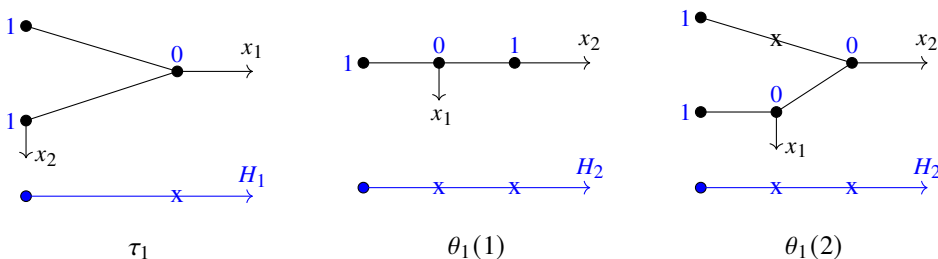
As guaranteed by Theorem 3.2, there is a unique minimal cone $\tau_1 \leq \mathsf{T}_{0,2}^{\max}(\mathbb{P}^2|H_1, 2)$ mapping to $\mathrm{Star}(\sigma_1)$. On the other hand, we see that in this case there are two minimal cones $\theta_1(1), \theta_1(2) \leq \mathsf{T}_{0,2}^{\max}(\mathbb{P}^2|H_2, 2)$ mapping to $\mathrm{Star}(\sigma_1)$. Note in particular that $\theta_1(2)$ maps to $\mathrm{Star}(\sigma_1)$ but not to σ_1 itself.

The exceptional divisor $E_1 \subseteq \mathsf{K}_{0,2}(\mathbb{P}^2, 2)_1 = \mathrm{Bl}_{Z(\sigma_1)} \mathsf{K}_{0,2}(\mathbb{P}^2, 2)$ is a $\mathbb{P}^1$ bundle over the stratum $Z(\sigma_1)$. Correspondingly, the excess loci

$$F_1^1 \subseteq \mathsf{K}_{0,2}^{\max}(\mathbb{P}^2|H_1, 2)_1^{\mathrm{tot}}, \quad F_1^2(1), F_1^2(2) \subseteq \mathsf{K}_{0,2}^{\max}(\mathbb{P}^2|H_2, 2)_1^{\mathrm{tot}}$$

are $\mathbb{P}^1$ bundles over the maximal strata mapping to $Z(\sigma_1)$:

$$Z(\tau_1) \subseteq \mathsf{K}_{0,2}^{\max}(\mathbb{P}^2|H_1, 2), \quad Z(\theta_1(1)), Z(\theta_1(2)) \subseteq \mathsf{K}_{0,2}^{\max}(\mathbb{P}^2|H_2, 2).$$

Since $Z(\theta_1(1))$ and $Z(\theta_1(2))$ are strata of codimension 2, $F_1^2(1)$ and $F_1^2(2)$ have excess dimension $-2 + 1 = -1$; that is, they do not carry an excess class. We conclude that the correction term arising from the first blowup vanishes.

We now blow up $Z(\sigma_2)_1$, the strict transform of $Z(\sigma_2)$. The minimal cones mapping to $\mathrm{Star}(\sigma_2) \subseteq \mathsf{T}_{0,2}(\mathbb{P}^2, 2)_1$ are

$$\tau_2 \qquad\qquad\qquad \theta_2(1)$$
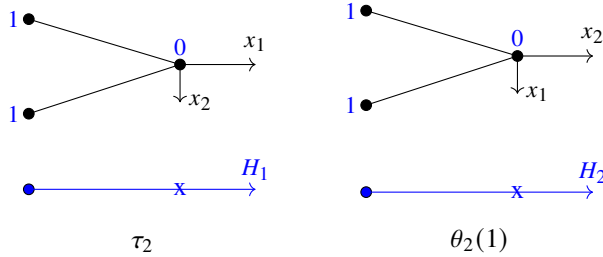
The exceptional divisor E_2 in $\mathsf{K}_{0,2}(\mathbb{P}^2, 2)_2$ is again a $\mathbb{P}^1$ bundle over $Z(\sigma_2)_1$ and so the excess loci

$$F_2^1 \subseteq \mathsf{K}_{0,2}^{\max}(\mathbb{P}^2|H_1, 2)_2^{\mathrm{tot}}, \quad F_2^2(1) \subseteq \mathsf{K}_{0,2}^{\max}(\mathbb{P}^2|H_2, 2)_2^{\mathrm{tot}}$$

are $\mathbb{P}^1$ bundles over $Z(\tau_2)_1$ and $Z(\theta_2(1))_1$. Both of these strata have codimension 1, so both excess loci have excess dimension 0 and in each case the excess class is simply the fundamental class of the excess locus. We conclude that the correction term in $\mathsf{K}_{0,2}(\mathbb{P}^2, 2)_2$ is the product

$$[F_2^1] \cdot [F_2^2(1)].$$

Denoting the blowup morphisms by

$$\mathsf{K}_{0,2}(\mathbb{P}^2, 2)_2 \xrightarrow{\rho_{2,1}} \mathsf{K}_{0,2}(\mathbb{P}^2, 2)_1 \xrightarrow{\rho_{1,0}} \mathsf{K}_{0,2}(\mathbb{P}^2, 2),$$

we have from Theorem 3.4:

$$[\mathsf{K}_{0,2}^{\max}(\mathbb{P}^2|H_1, 2)] \cdot [\mathsf{K}_{0,2}^{\max}(\mathbb{P}^2|H_2, 2)] = [\mathsf{K}_{0,2}^{\max}(\mathbb{P}^2|H_1 + H_2, 2)] - (\rho_{1,0})_\star (\rho_{2,1})_\star \left([F_2^1] \cdot [F_2^2(1)] \right).$$

It remains to calculate the final term. Let $i \colon E_2 \hookrightarrow \mathsf{K}_{0,2}(\mathbb{P}^2, 2)_2$ denote the inclusion and let $\pi \colon E_2 \to Z(\sigma_2)_1$ denote the bundle projection. We have

$$[F_2^1] = i_\star \pi^\star [Z(\tau_2)_1], \quad [F_2^2(1)] = i_\star \pi^\star [Z(\theta_2(1))_1],$$

from which we obtain

$$[F_2^1] \cdot [F_2^2(1)] = i_\star \left(-H \cap \pi^\star ([Z(\tau_2)_1] \cdot [Z(\theta_2(1))_1]) \right)$$

where $H = -c_1(N_{E_2})$ is the fibrewise hyperplane class of the projective bundle. Using $\pi_\star H = 1$ and the projection formula, we obtain

$$(\rho_{2,1})_\star \left([F_2^1] \cdot [F_2^2(1)] \right) = -j_\star \left([Z(\tau_2)_1] \cdot [Z(\theta_2(1))_1] \right)$$

where $j \colon Z(\sigma_2)_1 \hookrightarrow \mathsf{K}_{0,2}(\mathbb{P}^2, 2)_1$ is the inclusion. The intersection $Z(\sigma_1) \cap Z(\sigma_2)$ is a divisor in $Z(\sigma_2)$ and, consequently, the strict transform $Z(\sigma_2)_1 \to Z(\sigma_2)$ is an isomorphism. We thus have

$$(\rho_{0,1})_\star (\rho_{2,1})_\star \left([F_2^1] \cdot [F_2^2(1)] \right) = -k_\star \left([Z(\tau_2)] \cdot [Z(\theta_2(1))] \right)$$

where $k \colon Z(\sigma_2) \hookrightarrow \mathsf{K}_{0,2}(\mathbb{P}^2, 2)$ is the inclusion. The intersection of $Z(\tau_2)$ and $Z(\theta_2(1))$ inside $Z(\sigma_2)$

is transverse. We denote this locus by

$$W = Z(\tau_2) \cap Z(\theta_2(1)) \subseteq Z(\sigma_2) \subseteq \mathsf{K}_{0,2}(\mathbb{P}^2, 2).$$

Geometrically, it parametrises pairs of lines through the point $H_1 \cap H_2$ joined to a contracted component of the source curve containing both markings. It has dimension 3 and we conclude

$$[\mathsf{K}^{\max}_{0,2}(\mathbb{P}^2 | H_1, 2)] \cdot [\mathsf{K}^{\max}_{0,2}(\mathbb{P}^2 | H_2, 2)] = [\mathsf{K}^{\max}_{0,2}(\mathbb{P}^2 | H_1 + H_2, 2)] + [W]. \tag{26}$$

This precisely quantifies the difference between the local/naive and logarithmic theories. In this case, the formula reflects the geography of the naive space, which consists of two irreducible components corresponding to the two terms on the right-hand side. On the other hand, the iterated blowup procedure gives a general-purpose algorithm which does not rely on ad hoc descriptions of the naive space.

It is easy to find insertions which pair nontrivially with $[W]$. Introduce two additional markings with no tangency conditions and consider the forgetful morphism $F \colon \mathsf{K}_{0,4}(\mathbb{P}^2, 2) \to \mathsf{K}_{0,2}(\mathbb{P}^2, 2)$. Applying $F^\star$ to (26) gives

$$[\mathsf{K}^{\max}_{0,4}(\mathbb{P}^2 | H_1, 2)] \cdot [\mathsf{K}^{\max}_{0,4}(\mathbb{P}^2 | H_2, 2)] = [\mathsf{K}^{\max}_{0,4}(\mathbb{P}^2 | H_1 + H_2, 2)] + F^\star[W].$$

We cap this with the insertion $\gamma = \psi_1 \operatorname{ev}_3^\star(\mathrm{pt}) \operatorname{ev}_4^\star(\mathrm{pt})$ on $\mathsf{K}_{0,4}(\mathbb{P}^2, 2)$. The correction term is

$$\gamma \cap F^\star[W] = \psi_1 \cap [\overline{\mathcal{M}}_{0,4}] = 1 \tag{27}$$

as the point constraints fix the moduli of the two lines. On the other hand, the strong form of the local-logarithmic correspondence for smooth pairs gives

$$\begin{aligned}
\gamma \cap [\mathsf{K}^{\max}_{0,4}(\mathbb{P}^2 | H_1, 2)] \cdot [\mathsf{K}^{\max}_{0,4}(\mathbb{P}^2 | H_2, 2)] &= \gamma \cdot \operatorname{ev}_1^\star(H) \operatorname{ev}_2^\star(H) \cap [\mathsf{K}_{0,4}(\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}, 2)]^{\mathrm{vir}} \\
&= \operatorname{ev}_1^\star(H) \psi_1 \operatorname{ev}_2^\star(H) \operatorname{ev}_3^\star(\mathrm{pt}) \operatorname{ev}_4^\star(\mathrm{pt}) \, \mathrm{e}(R^1 \pi_\star f^\star \mathcal{O}_{\mathbb{P}^2}(-1))^2 \\
&\quad \cap [\mathsf{K}_{0,4}(\mathbb{P}, 2)].
\end{aligned}$$

We compute this by torus localisation. Let $H_0, H_1, H_2 \subseteq \mathbb{P}^2$ be the coordinate hyperplanes and $p_0, p_1, p_2 \in \mathbb{P}^2$ be the coordinate points. We choose the following equivariant lifts of the insertions:

$$\operatorname{ev}_1^\star(H_0) \psi_1 \operatorname{ev}_2^\star(H_1) \operatorname{ev}_3^\star(p_0) \operatorname{ev}_4^\star(p_1).$$

We equip the first copy of $\mathcal{O}_{\mathbb{P}^2}(-1)$ with the torus action which has weight zero at p_0 and equip the second copy of $\mathcal{O}_{\mathbb{P}^2}(-1)$ with the torus action which has weight zero at p_1. Under these choices of weights and equivariant insertions, the only graph contributing to the localised integral is
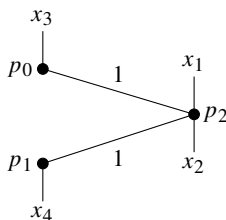
and a direct calculation of its contribution gives:

$$\gamma \cap [\mathsf{K}^{\max}_{0,4}(\mathbb{P}^2 | H_1, 2)] \cdot [\mathsf{K}^{\max}_{0,4}(\mathbb{P}^2 | H_2, 2)] = 1. \tag{28}$$

Combining (28) and (27) with (26), we obtain the logarithmic invariant

$$\gamma \cap [\mathsf{K}_{0,4}^{\max}(\mathbb{P}^2|H_1 + H_2, 2)] = 1 - 1 = 0.$$

This value can be independently verified using heuristic arguments relating the logarithmic invariant to tropical curve counts.

4. Virtual pullback

We employ virtual pullback techniques to extend Theorem 3.4 to general hyperplane sections.

4.1. Setup

Consider a pair $(Z|E)$ with $E = E_1 + E_2$ and each E_i a hyperplane section. We have embeddings

$$(Z|E_i) \hookrightarrow (\mathbb{P}^{n_i}|H_i)$$

where H_i is a hyperplane. Let $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ and $D = D_1 + D_2$ be the simple normal crossings divisor induced by the H_i. There is a closed embedding $Z \hookrightarrow X$ with $E_i = Z \cap D_i$.

Lemma 4.1. *The following morphisms of moduli spaces are strict:*

$$\mathsf{K}_{0,2}(Z, \beta) \to \mathsf{K}_{0,2}(X, \beta),$$
$$\mathsf{K}_{0,2}^{\max}(Z|E_i, \beta) \to \mathsf{K}_{0,2}^{\max}(X|D_i, \beta), \ \ for \ i \in \{1, 2\},$$
$$\mathsf{K}_{0,2}^{\max}(Z|E, \beta) \to \mathsf{K}_{0,2}^{\max}(X|D, \beta).$$

They carry relative perfect obstruction theories given (in every case) by

$$(\pi_\star f^\star \mathsf{N}_{Z|X})^\vee[1],$$

and the induced virtual pullback morphism identifies the virtual fundamental classes.

Proof. The maps $(Z|E_i) \to (X|D_i)$ and $(Z|E) \to (X|D)$ are strict closed embeddings, so the logarithmic normal bundle coincides with the ordinary normal bundle. The obstruction theories are both relative to the moduli space of maps to the Artin fan [4], so the claim follows from functoriality of virtual pullbacks [21]. The obstruction theory is perfect due to the convexity of X. □

Lemma 4.2. *The following square is Cartesian:*

$$
\begin{array}{ccc}
\mathsf{K}_{0,2}^{\max}(Z|E, \beta) & \xrightarrow{\ j\ } & \mathsf{K}_{0,2}^{\max}(X|D, \beta) \\
\downarrow & \square & \downarrow \\
\mathsf{K}_{0,2}(Z, \beta) & \xrightarrow{\ i\ } & \mathsf{K}_{0,2}(X, \beta)
\end{array}
$$

and satisfies

$$[\mathsf{K}_{0,2}^{\max}(Z|E, \beta)]^{\mathrm{vir}} = i^![\mathsf{K}_{0,2}^{\max}(X|D, \beta)].$$

The analogous statements hold for $(Z|E_i) \to (X|D_i)$.

Proof. Since i is strict, the square is a Cartesian diagram of ordinary stacks. Equality of virtual classes holds as the relative perfect obstruction theory for i pulls back to give the relative perfect obstruction theory for j. □

4.2. Virtual birational models

The blowups in Section 3 may now be pulled back. For $j \in \{1, \ldots, m\}$ we obtain virtual birational models:

$$
\begin{array}{ccc}
\mathsf{K}_{0,2}(Z, \beta)_j & \longrightarrow & \mathsf{K}_{0,2}(X, \beta)_j \\
\downarrow & \square & \downarrow \\
\mathsf{K}_{0,2}(Z, \beta) & \xrightarrow{\ i\ } & \mathsf{K}_{0,2}(X, \beta).
\end{array}
$$

Since $i \colon \mathsf{K}_{0,2}(Z, \beta) \to \mathsf{K}_{0,2}(X, \beta)$ is strict, the morphism

$$
\mathsf{K}_{0,2}(Z, \beta)_j \to \mathsf{K}_{0,2}(Z, \beta) \tag{29}
$$

is also a logarithmic modification. The space $\mathsf{K}_{0,2}(Z, \beta)_j$ carries a natural perfect obstruction theory, and the pushforward morphism (29) identifies the virtual classes [26, §3.5].

We similarly obtain virtual strict transforms

$$
\mathsf{K}_{0,2}^{\max}(Z|E_i, \beta)_m \to \mathsf{K}_{0,2}^{\max}(Z|E_i, \beta)_{m-1} \to \cdots \to \mathsf{K}_{0,2}^{\max}(Z|E_i, \beta)_0 = \mathsf{K}_{0,2}^{\max}(Z|E_i, \beta)
$$

for $i \in \{1, 2\}$, and a comparison of obstruction theories gives

$$
[\mathsf{K}_{0,2}^{\max}(Z|E_i, \beta)_j]^{\mathrm{vir}} = i^! [\mathsf{K}_{0,2}^{\max}(X|D_i, \beta)_j].
$$

4.3. Corrected product formula

We first restrict to the case where Z is convex, so that products in $\mathsf{K}_{0,2}(Z, \beta)_j$ are well-defined. Applying the virtual pullback $i^!$ to the blowup formulae (20) and (22) results in the following relations in $\mathsf{K}_{0,2}(Z, \beta)_j$:

$$
\rho_{j,j-1}^\star [\mathsf{K}_{0,2}^{\max}(Z|E_1, \beta)_{j-1}]^{\mathrm{vir}} = [\mathsf{K}_{0,2}^{\max}(Z|E_1, \beta)_j]^{\mathrm{vir}} + \gamma_j^1 \cap [F_j^1]^{\mathrm{vir}}, \tag{30}
$$

$$
\rho_{j,j-1}^\star [\mathsf{K}_{0,2}^{\max}(Z|E_2, \beta)_{j-1}]^{\mathrm{vir}} = [\mathsf{K}_{0,2}^{\max}(Z|E_2, \beta)_j]^{\mathrm{vir}} + \Sigma_{k=1}^{l_j} \gamma_j^2(k) \cap [F_j^2(k)]^{\mathrm{vir}}. \tag{31}
$$

As in Subsections 3.4 and 3.5, we now pass up the tower of logarithmic blowups, take the product and then push back down to $\mathsf{K}_{0,2}(Z, \beta)$, obtaining the following.

Theorem 4.3. *The following relation holds in* $\mathsf{K}_{0,2}(Z, \beta)$:

$$
[\mathsf{K}_{0,2}^{\max}(Z|E_1, \beta)]^{\mathrm{vir}} \cdot [\mathsf{K}_{0,2}^{\max}(Z|E_2, \beta)]^{\mathrm{vir}} = [\mathsf{K}_{0,2}^{\max}(Z|E, \beta)]^{\mathrm{vir}}
$$
$$
- \sum_{j=1}^{m} (\rho_{j,0})_\star \left(\sum_{k=1}^{l_j} (\gamma_j^1 \cap [F_j^1]^{\mathrm{vir}}) \cdot (\gamma_j^2(k) \cap [F_j^2(k)]^{\mathrm{vir}}) \right).
$$

Applying $F_\star$ *gives the corrected local-logarithmic correspondence.*

If Z is not convex, the Chow groups of $\mathsf{K}_{0,2}(Z, \beta)$ need not admit a product and the corrected product formula cannot even be formulated. Instead, we apply $i^!$ to Theorem 3.4 to obtain

$$
i^! \left([\mathsf{K}_{0,2}^{\max}(X|D_1, \beta)] \cdot [\mathsf{K}_{0,2}^{\max}(X|D_2, \beta)] \right) = [\mathsf{K}_{0,2}^{\max}(Z|E, \beta)]^{\mathrm{vir}}
$$
$$
- i^! \sum_{j=1}^{m} (\rho_{j,0})_\star \left(\sum_{k=1}^{l_j} (\gamma_j^1 \cap [F_j^1]) \cdot (\gamma_j^2(k) \cap [F_j^2(k)]) \right).
$$

The strong form of the local-logarithmic correspondence for $(X|D_1)$ and $(X|D_2)$ can be used to identify the left-hand side with the local theory of $\mathcal{O}_Z(E_1) \oplus \mathcal{O}_Z(E_2)$ capped with $\mathrm{ev}_1^\star E_1 \cdot \mathrm{ev}_2^\star E_2$. Applying $F_\star$ we again obtain the corrected local-logarithmic correspondence. A difference with Theorem 4.3 is that the correction terms are calculated in $\mathsf{K}_{0,2}(X,\beta)$ and then pulled back.

5. Local-logarithmic on product geometries

We establish an instance of the numerical local-logarithmic correspondence for products. The argument here is elementary and independent of the more technical blowup arguments elsewhere in the article. However, the understanding of the blowup geometry leads very naturally to the proof; we simply look for situations where the correction terms can be shown to vanish.

5.1. Setup: unobstructed case

Let $X_1, \ldots X_k$ be smooth projective varieties equipped with smooth hyperplane sections $D_1, \ldots, D_k$. As before, we first specialise to the situation where each X_i is a projective space $\mathbb{P}^{n_i}$ and each $D_i = H_i$ is a hyperplane. Let

$$\mathbb{P} := \prod_{i=1}^k \mathbb{P}^{n_i}$$

be the target, H the union of pullbacks of hyperplanes H_i from the factors and β the curve class.

We work with the space of $k + 3$ pointed maps. The final three points will be taken to have zero contact order and each of the first k points will have maximal contact order with the corresponding divisor. We have the following composition of forgetful morphisms:

$$\mathsf{K}_{0,k+3}^{\max}(\mathbb{P}|H,\beta) \to \mathsf{K}_{0,k+3}^{\max}(\mathbb{P}^{n_i}|H_i,\beta_i) \to \mathsf{K}_{0,4}^{\max}(\mathbb{P}^{n_i}|H_i,\beta_i).$$

The first arrow projects onto the appropriate factor and stabilises the map; the second arrow forgets all marked points except x_i and the three markings with zero contact order. These give rise to a morphism

$$\rho \colon \mathsf{K}_{0,k+3}^{\max}(\mathbb{P}|H,\beta) \to \prod_{i=1}^k \mathsf{K}_{0,4}^{\max}(\mathbb{P}^{n_i}|H_i,\beta_i).$$

Proposition 5.1. *The morphism ρ is proper and birational.*

Proof. Arguments as in Section 1 show that the source and target of ρ are irreducible. Examine the locus in the source comprising maps from smooth domains, dimensionally transverse to H. Notice that the source curve is a parametrised $\mathbb{P}^1$ with the parametrisation given by the three markings with zero contact order; this locus is dense and ρ has an inverse on this locus. □

5.2. Primary theory with factorwise insertions

Consider the morphism

$$\nu \colon \mathsf{K}_{0,3}(\mathbb{P},\beta) \to \prod_{i=1}^k \mathsf{K}_{0,3}(\mathbb{P}^{n_i},\beta_i)$$

of spaces of ordinary stable maps. It is clear that ν is proper and birational. We assemble primary insertions on $\mathsf{K}_{0,3}(\mathbb{P},\beta)$ without appealing to the existence of marked points. This is likely well-known

to experts. Consider the universal family:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\;f\;} & \mathbb{P} \\
\downarrow{\scriptstyle \pi} & & \\
\mathsf{K}_{0,3}(\mathbb{P}, \beta). & &
\end{array}
$$

Given a cohomology class γ in the target $\mathbb{P}$, we obtain a cycle class $\pi_\star f^\star \gamma$ on $\mathsf{K}_{0,3}(\mathbb{P}, \beta)$. Primary invariants are defined by integrating products of such classes. The comparison of diagonals (see the proof of Lemma 3.1) equates these integrals with the ordinary Gromov–Witten invariants:

$$
\Pi_{j=1}^r \pi_\star f^\star \gamma_j \cap [\mathsf{K}_{0,3}(\mathbb{P}, \beta)] = \Pi_{j=1}^r \mathrm{ev}_j^\star \gamma_j \cap [\mathsf{K}_{0,r+3}(\mathbb{P}, \beta)].
$$

The three auxiliary marked points can be removed by attaching divisorial insertions and appealing to the divisor axiom; alternatively, they may be equipped with arbitrary insertions.

We now restrict to a particular class of insertions: we require that each class γ_j is equal to the pullback of a class in $\mathbb{P}^{n_i}$ along one of the projections $\mathbb{P} \to \mathbb{P}^{n_i}$. We refer to this as the **primary theory with factorwise insertions**. The three auxiliary markings are allowed to carry arbitrary classes; the factorwise constraint only applies to additional markings introduced via the above procedure. We assemble these insertions into a single class on $\mathsf{K}_{0,3}(\mathbb{P}, \beta)$:

$$
\gamma = \prod_{i=1}^{3} \mathrm{ev}_i^\star \delta_i \cdot \prod_{j=1}^{r} \pi_\star f^\star \gamma_j. \tag{32}
$$

Theorem 5.2. (Local-logarithmic correspondence with primary factorwise insertions). *If γ is the class (32) with factorwise insertions, then there is an equality*

$$
\gamma \cap \psi_\star [\mathsf{K}_{0,k+3}^{\max}(\mathbb{P}|H, \beta)] = \alpha \cdot \gamma \cap [\mathsf{K}_{0,3}(\oplus_{i=1}^k \mathcal{O}_\mathbb{P}(-H_i), \beta)]^{\mathrm{vir}}
$$

where $\psi \colon \mathsf{K}_{0,k+3}^{\max}(\mathbb{P}|H, \beta) \to \mathsf{K}_{0,3}(\mathbb{P}, \beta)$ *and* $\alpha = \Pi_{i=1}^k (-1)^{d_i+1} d_i$.

Proof. There is a commutative diagram

$$
\begin{array}{ccc}
\mathsf{K}_{0,k+3}^{\max}(\mathbb{P}|H, \beta) & \xrightarrow{\;\rho\;} & \Pi_{i=1}^k \mathsf{K}_{0,4}^{\max}(\mathbb{P}^{n_i}|H_i, \beta_i) \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} \\
\mathsf{K}_{0,3}(\mathbb{P}, \beta) & \xrightarrow{\;\nu\;} & \Pi_{i=1}^k \mathsf{K}_{0,3}(\mathbb{P}^{n_i}, \beta_i)
\end{array}
$$

with ρ and ν birational. Because γ is assembled from primary factorise insertions, it follows that $\gamma = \nu^\star \delta$ for some class δ. This can be seen by comparing the universal curve over $\mathsf{K}_{0,3}(\mathbb{P}, \beta)$ to the pullback of the universal curve over $\mathsf{K}_{0,3}(\mathbb{P}^{n_i}, \beta_i)$.

The product formula [8] applied to the total spaces of $\mathcal{O}_{\mathbb{P}^{n_i}}(-H_i)$ shows that the local class associated to $\oplus_{i=1}^k \mathcal{O}_\mathbb{P}(-H_i)$ pushes forward along ν to the product of the local classes associated to $\mathcal{O}_{\mathbb{P}^{n_i}}(-H_i)$. Combining with the local-logarithmic correspondence for the smooth pairs $(\mathbb{P}^{n_i}, H_i)$ gives

$$
\varphi_\star \left(\Pi_{i=1}^k [\mathsf{K}_{0,4}^{\max}(\mathbb{P}^{n_i}|H_i, \beta_i)] \right) = \alpha \cdot \Pi_{i=1}^k [\mathsf{K}_{0,3}(\mathcal{O}_{\mathbb{P}^{n_i}}(-H_i), \beta_i)]^{\mathrm{vir}} = \alpha \cdot \nu_\star [\mathsf{K}_{0,3}(\oplus_{i=1}^k \mathcal{O}_\mathbb{P}(-H_i), \beta)]^{\mathrm{vir}}.
$$

From this and the projection formula applied to ν, we conclude

$$\gamma \cap \psi_\star [\mathsf{K}_{0,k+3}^{\max}(\mathbb{P}|H,\beta)] = \delta \cap \varphi_\star \rho_\star [\mathsf{K}_{0,k+3}^{\max}(\mathbb{P}|H,\beta)]$$
$$= \delta \cap \varphi_\star \left(\Pi_{i=1}^k [\mathsf{K}_{0,4}^{\max}(\mathbb{P}^{n_i}|H_i,\beta_i)] \right)$$
$$= \alpha \cdot \delta \cap \nu_\star [\mathsf{K}_{0,3}(\oplus_{i=1}^k \mathcal{O}_\mathbb{P}(-H_i),\beta)]^{\text{vir}}$$
$$= \alpha \cdot \gamma \cap [\mathsf{K}_{0,3}(\oplus_{i=1}^k \mathcal{O}_\mathbb{P}(-H_i),\beta)]^{\text{vir}}. here$$

$\square$

Remark 5.3. The above result does not contradict Theorem 3.4. Rather, the factorwise insertions kill the correction terms in this setting. The same phenomenon explains other cases where numerical forms of the local-logarithmic correspondence are known to hold [11, 12].

5.3. *Virtual pullback*

In light of the preceding result, note that for an arbitrary section pair $(X|D)$ of product type, an identical construction produces a Cartesian diagram:

$$
\begin{array}{ccc}
\mathsf{K}_{0,k+3}^{\max}(X|D,\beta) & \longrightarrow & \mathsf{K}_{0,k+3}^{\max}(\mathbb{P}|H,\beta) \\
\downarrow & \square & \downarrow \\
\prod_{i=1}^k \mathsf{K}_{0,4}^{\max}(X_i|D_i,\beta_i) & \longrightarrow & \prod_{i=1}^k \mathsf{K}_{0,4}^{\max}(\mathbb{P}^{n_i}|H_i,\beta_i).
\end{array}
$$

The horizontal arrows possess compatible perfect obstruction theories, as in Section 4. The right vertical map is birational; therefore, we conclude that the left vertical arrow identifies virtual classes. The proof of Theorem 5.2 then applies verbatim, extending the correspondence to section pairs:

$$\gamma \cap \psi_\star [\mathsf{K}_{0,k+3}^{\max}(X|D,\beta)]^{\text{vir}} = \alpha \cdot \gamma \cap [\mathsf{K}_{0,3}(\oplus_{i=1}^k \mathcal{O}_X(-D_i),\beta)]^{\text{vir}}.$$

Remark 5.4. We do not believe that the product structure is the true reason for the result; products only produce a birational morphism that kills the corrections. The morphism

$$\mathsf{K}_{0,k+3}^{\max}(\mathbb{P}|H,\beta) \rightarrow \prod_{i=1}^k \mathsf{K}_{0,4}^{\max}(\mathbb{P}^{n_i},\beta_i)$$

is a contraction. On the right-hand side there is not necessarily a universal map to $\mathbb{P}$. This is analogous to the quasimap moduli. A study of naive and logarithmic quasimap theory may be worthwhile.

Conflict of Interest: None.

References

[1] D. Abramovich and Q. Chen, 'Stable logarithmic maps to Deligne-Faltings pairs II', *Asian J. Math.* **18** (2014), 465–488.
[2] D. Abramovich, Q. Chen, M. Gross and B. Siebert, '*Punctured logarithmic maps*', Preprint, 2020, arXiv:2009.07720.
[3] D. Abramovich and K. Karu, 'Weak semistable reduction in characteristic 0', *Invent. Math.* **139** (2000), 241–273.
[4] D. Abramovich and J. Wise, 'Birational invariance in logarithmic Gromov–Witten theory', *Comp. Math.* **154** (2018), 595–620.

[5] P. Aluffi, 'Segre classes as integrals over polytopes', *J. Eur. Math. Soc.* **18** (2016), 2849–2863.

[6] L. Battistella, N. Nabijou and D. Ranganathan, 'Curve counting in genus one: elliptic singularities & relative geometry', *Algebr. Geom.* **8** (2021), no. 6, 637–679.

[7] L. Battistella, N. Nabijou, H.-H. Tseng and F. You, 'The local-orbifold correspondence for simple normal crossings pairs', Preprint, 2021, arXiv:2011.08830).

[8] K. Behrend, 'The product formula for Gromov–Witten invariants', *J. Algebr. Geom.* **3** (1999), 529–541.

[9] L. A. Borisov, L. Chen and G. G. Smith, 'The orbifold Chow ring of toric Deligne-Mumford stacks', *J. Amer. Math. Soc.* **18** (2005), 193–215.

[10] N. Borne and A. Vistoli, 'Parabolic sheaves on logarithmic schemes', *Adv. Math.* **231** (2012), 1327–1363.

[11] P. Bousseau, A. Brini and M. van Garrel, 'On the local-log principle for the toric boundary', Preprint, 2019, arXiv:1908.04371).

[12] P. Bousseau, A. Brini and M. van Garrel, *'Stable maps to Looijenga pairs'*, Preprint, 2020, arXiv:2011.08830).

[13] R. Cavalieri, H. Markwig and D. Ranganathan, 'Tropical compactification and the Gromov–Witten theory of $\mathbf{P}^1$', *Selecta Math.* **23** (2017), 1027–1060.

[14] B. Fantechi, E. Mann and F. Nironi, 'Smooth toric Deligne-Mumford stacks', *J. Reine Angew. Math.* **648** (2010), 201–244.

[15] W. Fulton, *Intersection Theory*, Vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics* [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics] second ed. (Springer, Berlin, 1998).

[16] A. Gathmann, 'Absolute and relative Gromov-Witten invariants of very ample hypersurfaces', *Duke Math. J.* **115** (2002), 171–203.

[17] A. Gathmann, 'Relative Gromov–Witten invariants and the mirror formula', *Math. Ann.* **325** (2003), 393–412.

[18] M. Gross and B. Siebert, 'Logarithmic Gromov–Witten invariants', *J. Amer. Math. Soc.* **26** (2013), 451–510.

[19] B. Kim, 'Logarithmic stable maps', in *New Developments in Algebraic Geometry, Integrable Systems and Mirror Symmetry, Adv. Stud. Pure Math.* (Math. Soc. Japan, Tokyo, 2010), 167–200.

[20] A. Klemm and R. Pandharipande, 'Enumerative geometry of Calabi–Yau 4-folds', *Comm. Math. Phys.* **281** (2008), 621–653.

[21] C. Manolache, 'Virtual pull-backs', *J. Algebr. Geom.* **21** (2012), 201–245.

[22] S. Molcho, 'Universal stacky semistable reduction', *Isr. J. Math.* **242** (2021), no. 1, 55–82.

[23] N. Nabijou, *Recursion formulae in logarithmic Gromov–Witten theory and quasimap theory*, PhD thesis (Imperial College London, 2018). https://www.dpmms.cam.ac.uk/\~nn333/NabijouThesis.pdf.

[24] D. Oprea, 'Tautological classes on the moduli spaces of stable maps to $\mathbb{P}^r$ via torus actions', *Adv. Math.* **207** (2006), 661–690.

[25] D. Ranganathan, 'A *note on cycles of curves in a product of pairs*', Preprint, 2019 arXiv:1910.00239.

[26] D. Ranganathan, *'Logarithmic Gromov–Witten theory with expansions'*, Preprint, 2019, arXiv:1903.09006.

[27] D. Ranganathan, K. Santos-Parker and J. Wise, 'Moduli of stable maps in genus one and logarithmic geometry I', *Geom. Topol.* **23** (2019), no. 7, 3315–3366.

[28] N. Takahashi, 'Log mirror symmetry and local mirror symmetry', *Commun. Math. Phys.* **220** (2001), 293–299.

[29] H.-H. Tseng and F. You, *'A mirror theorem for multi-root stacks and applications'*, Preprint, 2020, arXiv:2006.08991.

[30] R. Vakil and A. Zinger, 'A desingularization of the main component of the moduli space of genus-one stable maps into $\mathbb{P}^n$', *Geom. Topol.* **12** (2008), 1–95.

[31] M. van Garrel, T. Graber and H. Ruddat, 'Local Gromov–Witten invariants are log invariants', *Adv. Math.*, **350** (2019), 860–876.