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A NOTE ON BERNSTEIN'S BIVARIATE INEQUALITY

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SUMMARY. An upper bound for $P[\sum X_i \geq t\sigma, \sum Y_i \geq t\sigma]$, where (X_i, Y_i), $i = 1, 2, \ldots, n$ are bounded independent random variables, was given by Mullen (1973). An improvement to the bound is possible without further assumptions.

Discussion. If $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ are n independent random variables for which $E(X_i) = E(Y_i) = 0$, $\text{Var}(X_i) = \text{Var}(Y_i) = \sigma_i^2$, $\text{Cov}(X_i, Y_i) = \rho\sigma_i^2$ and for which $|X_i| \leq R, |Y_i| \leq R$, and if a, b, t are positive real numbers, then Mullen (1973) has shown that

$$P\left[\sum_i X_i \geq t\sigma, \sum_i Y_i \geq t\sigma\right] < \exp\left\{\frac{-t^2(2-|\rho|)}{2\left(1+\dfrac{tR}{3\sigma}\right)}\right\}$$

A brief sketch of the proof is as follows. Since

$$Ee^{aX_i+bY_i} = 1 + ab\rho\sigma_i^2 + \frac{a^2\sigma_i^2 A_i}{2} + \frac{b^2\sigma_i^2 B_i}{2} + ab\sigma_i^2(C_i + D_i + E_i)$$

where

$$A_i = \sum_{r=2}^{\infty} \frac{a^{r-2}EX_i^r}{r!\,\sigma_i^2\,\frac{1}{2}}, \qquad B_i = \sum_{s=2}^{\infty} \frac{b^{s-2}EY_i^s}{s!\,\sigma_i^2\,\frac{1}{2}}$$

$$C_i = \sum_{r=2}^{\infty} \frac{a^{r-1}EY_iX_i^r}{r!\,\sigma_i^2}, \qquad D_i = \sum_{s=2}^{\infty} \frac{b^{s-1}EX_iY_i^s}{s!\,\sigma_i^2}, \qquad E_i = \sum_{r=2}^{\infty}\sum_{s=2}^{\infty} \frac{a^{r-1}b^{s-1}EX_i^rY_i^s}{r!\,s!\,\sigma_i^2}$$

then

$$Ee^{aX_i+bY_i} < \exp\left\{\left(\frac{a^2\sigma_i^2}{2} + \frac{b^2\sigma_i^2}{2}\right)M_i + ab\sigma_i^2 G_i\right\}$$

where

$$M_i = \text{Max}(A_i, B_i), \qquad G_i = \rho + C_i + D_i + E_i$$

and

$$Ee^{a\sum X_i + b\sum Y_i} < \exp\left\{\left(\frac{a^2\sigma^2}{2} + \frac{b^2\sigma^2}{2}\right)M + ab\sigma G\right\}$$

where

$$M = \underset{i}{\text{Max}}\, M_i, \qquad G = \underset{i}{\text{Max}}\, G_i, \qquad \sigma^2 = \sum_i \sigma_i^2$$

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Extending the idea of Chebyshev's inequality, we can write

$$P\left[\sum X_i \geq t\sigma, \sum Y_i \geq t\sigma\right] \leq \exp\left\{\left(\frac{a^2\sigma^2}{2}+\frac{b^2\sigma^2}{2}\right)M + ab\sigma^2 G - t\sigma(a+b)\right\}$$

which is minimized (with respect to a and b) for $M = G = t/(\sigma(a+b))$, so that

$$P\left[\sum X_i \geq t\sigma,\ \sum Y_i \geq t\sigma\right] < \exp\left\{\frac{-t\sigma(a+b)}{2}\right\} \tag{1}$$

Now if

$$\left.\begin{aligned} E\,|X_i|^r &\leq \tfrac{1}{2}\,\sigma_i^2 r!\ W^{r-2} \\ E\,|Y_i|^s &\leq \tfrac{1}{2}\,\sigma_i^2 s!\ W^{s-2} \end{aligned}\right\} \quad \begin{aligned} &r, s \geq 2 \\ &W \text{ a constant} \end{aligned}$$

then

$$A_i \leq \frac{1}{1-aw}, \qquad B_i \leq \frac{1}{1-bW}$$

and

$$(\text{for } a > b), \quad M \leq \frac{1}{1-aw}$$

so that

$$a \geq \frac{t-\sigma b}{\sigma + tw}$$

which when substituted into (1) gives

$$P\left[\sum X_i \geq t\sigma,\ \sum Y_i \geq t\sigma\right] < \exp\left\{\frac{-t^2\sigma}{2}\frac{(1+bW)}{\sigma+tW}\right\}$$

Now choosing b so that $1-bW \leq |\rho|$ we have

$$P\left[\sum X_i \geq t\sigma,\ \sum Y_i \geq t\sigma\right] < \exp\left\{\frac{-t^2(2-|\rho|)}{2\left(1+t\frac{W}{\sigma}\right)}\right\}$$

Of particular interest are variables satisfying $|X_i| \leq R$, $|Y_i| \leq R$, for which

$$E\,|X_i|^r \leq \sigma_i^2 R^{r-2}, \qquad E\,|Y_i|^s \leq \sigma_i^2 R^{s-2}, \qquad E\,|Y_i|\,|X_i|^r \leq |\rho|\,\sigma_i^2 R^{r-1},$$
$$E\,|X_i|\,|Y_i|^s \leq |\rho|\,\sigma_i^2 R^{s-1}, \qquad E\,|X_i|^r\,|Y_i|^s \leq |\rho|\,\sigma_i^2 R^{r-1} R^{s-1}$$

so that $R = 3W$ and

$$P\left[\sum X_i \geq t\sigma,\ \sum Y_i \geq t\sigma\right] < \exp\left\{\frac{-t^2(2-|\rho|)}{2\left(1+\frac{tR}{3\sigma}\right)}\right\}$$

This Bernstein-type inequality, after Bernstein (1924), who developed the univariate case, may be improved upon without further assumptions.

Writing $S(X)$ and $S(Y)$ for $\sum_i X_i$ and $\sum_i Y_i$ respectively, then (from Mullen with $aR = x$, $bR = y$, $\sigma/R = s$, we have

$$P[S(X) \geq t\sigma,\ S(Y) \geq t\sigma] < \exp\left\{\frac{x^2s^2A}{2} + \frac{y^2s^2B}{2} + xys^2G - ts(x+y)\right\} \tag{2}$$

where from the same reference we let A, B and G correspond to $\max_i A_i$, $\max_i B_i$ and $\max G_i$ respectively, so that

$$A, B \leq \frac{9\,|\rho|}{(3-x)(3-y)}$$

$$G \leq \frac{9\,|\rho|}{(3-x)(3-y)}$$

we may write the exponent of (2) as

$$\frac{9(x+y)^2s^2\,|\rho|}{2(3-x)(3-y)} - ts(x+y)$$

which is minimized (with respect to x and y) for

$$\frac{9s\,|\rho|}{2}\,\frac{y^2 - x^2 + 6x + 6y}{(3-x)^2(3-y)} - t = 0$$

$$\frac{9s\,|\rho|}{2}\,\frac{x^2 - y^2 + 6x + 6y}{(3-x)(3-y)^2} - t = 0$$

which imply that $x = y$ is the single real zero of

$$x^3 - 9x^2 + x\left(27 + \frac{54\,|\rho|\,s}{t}\right) - 27 = 0 \tag{3}$$

The results of substituting the roots of (3) into (2) were evaluated for a wide range of $|\rho|$, s and t, and below we give an abbreviated table for $|\rho| = 0.2(0.3)0.8$, $s = 0.5(1)4.5$, $t = 1(1)7$. A comparison with Mullen's expression (8) is also given.

The improvement discussed here is analogous to the 'First Improvement' given by Bennett (1962). This leads one to ask if further improvement can be made to the bound, using the ideas of Bennett's "Second Improvement." The author took this approach, but found that for a similar range of $|\rho|$, s and t that no general improvement could be affected.

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TABLE. Comparing the Old Bound (Mullen's expression (8)) and the Improved Bound (expression (1)).

$\lvert\rho\rvert$	s	t	Old Bound	Improved Bound
0.20	0.50	0.50	0.84472	0.75207
0.20	0.50	1.00	0.58275	0.44830
0.20	0.50	1.50	0.36331	0.24066
0.20	0.50	2.00	0.21377	0.12083
0.20	0.50	2.50	0.12131	0.05780
0.20	0.50	3.00	0.06721	0.02663
0.20	0.50	3.50	0.03661	0.01190
0.20	1.50	0.50	0.81669	0.65291
0.20	1.50	1.00	0.47885	0.25862
0.20	1.50	1.50	0.21899	0.07697
0.20	1.50	2.00	0.08272	0.01878
0.20	1.50	2.50	0.02689	0.00394
0.20	1.50	3.00	0.00775	0.00073
0.20	1.50	3.50	0.00203	0.00012
0.20	2.50	0.50	0.80982	0.61718
0.20	2.50	1.00	0.45198	0.19866
0.20	2.50	1.50	0.18498	0.04211
0.20	2.50	2.00	0.05830	0.00657
0.20	2.50	2.50	0.01472	0.00081
0.20	2.50	3.00	0.00307	0.00008
0.20	2.50	3.50	0.00054	0.00001
0.20	3.50	0.50	0.80672	0.59831
0.20	3.50	1.00	0.43967	0.16909
0.20	3.50	1.50	0.17001	0.02848
0.20	3.50	2.00	0.04861	0.00324
0.20	3.50	2.50	0.01064	0.00027
0.20	3.50	3.00	0.00184	0.00002
0.20	3.50	3.50	0.00026	0.00000
0.20	4.50	0.50	0.80496	0.58657
0.20	4.50	1.00	0.43260	0.15148
0.20	4.50	1.50	0.16162	0.02156
0.20	4.50	2.00	0.04348	0.00193
0.20	4.50	2.50	0.00869	0.00012
0.20	4.50	3.00	0.00132	0.00001
0.20	4.50	3.50	0.00016	0.00000
0.50	0.50	0.50	0.86882	0.85077
0.50	0.50	1.00	0.63763	0.60501
0.50	0.50	1.50	0.43009	0.38992
0.50	0.50	2.00	0.27645	0.23494
0.50	0.50	2.50	0.17242	0.13456
0.50	0.50	3.00	0.10540	0.07402
0.50	0.50	3.50	0.06353	0.03938
0.50	1.50	0.50	0.84472	0.81241
0.50	1.50	1.00	0.54138	0.48510
0.50	1.50	1.50	0.28206	0.23350
0.50	1.50	2.00	0.12532	0.09528
0.50	1.50	2.50	0.04912	0.03401
0.50	1.50	3.00	0.01742	0.01086
0.50	1.50	3.50	0.00570	0.00315
0.50	2.50	0.50	0.83880	0.80087
0.50	2.50	1.00	0.51594	0.44664
0.50	2.50	1.50	0.24506	0.18839
0.50	2.50	2.00	0.09363	0.06333
0.50	2.50	2.50	0.02973	0.01759
0.50	2.50	3.00	0.00806	0.00415
0.50	2.50	3.50	0.00190	0.00085
0.50	3.50	0.50	0.83612	0.79525
0.50	3.50	1.00	0.50420	0.42732
0.50	3.50	1.50	0.22842	0.16682
0.50	3.50	2.00	0.08046	0.04975
0.50	3.50	2.50	0.02268	0.01176
0.50	3.50	3.00	0.00525	0.00226
0.50	3.50	3.50	0.00102	0.00036
0.50	4.50	0.50	0.83460	0.79192
0.50	4.50	1.00	0.49744	0.41566
0.50	4.50	1.50	0.21899	0.15416
0.50	4.50	2.00	0.07332	0.04235
0.50	4.50	2.50	0.01916	0.00893
0.50	4.50	3.00	0.00399	0.00149
0.50	4.50	3.50	0.00068	0.00020
0.80	0.50	0.50	0.89360	0.89153
0.80	0.50	1.00	0.69768	0.68681
0.80	0.50	1.50	0.50916	0.48394
0.80	0.50	2.00	0.35752	0.31991
0.80	0.50	2.50	0.24506	0.20128
0.80	0.50	3.00	0.16530	0.12167
0.80	0.50	3.50	0.11025	0.07112
0.80	1.50	0.50	0.87372	0.87095
0.80	1.50	1.00	0.61207	0.60650
0.80	1.50	1.50	0.36331	0.35583
0.80	1.50	2.00	0.18985	0.18173
0.80	1.50	2.50	0.08975	0.08263
0.80	1.50	3.00	0.03916	0.03400
0.80	1.50	3.50	0.01601	0.01282

$\|\rho\|$	s	t	Old Bound	Improved Bound
0.80	2.50	0.50	0.86882	0.86533
0.80	2.50	1.00	0.58895	0.58221
0.80	2.50	1.50	0.32465	0.31782
0.80	2.50	2.00	0.15036	0.14510
0.80	2.50	2.50	0.06005	0.05668
0.80	2.50	3.00	0.02113	0.01929
0.80	2.50	3.50	0.00666	0.00580
0.80	3.50	0.50	0.86660	0.86269
0.80	3.50	1.00	0.57821	0.57033
0.80	3.50	1.50	0.30689	0.29933
0.80	3.50	2.00	0.13319	0.12815
0.80	3.50	2.50	0.04837	0.04572
0.80	3.50	3.00	0.01500	0.01384
0.80	3.50	3.50	0.00404	0.00360
0.80	4.50	0.50	0.86533	0.86116
0.80	4.50	1.00	0.57200	0.56326
0.80	4.50	1.50	0.29671	0.28836
0.80	4.50	2.00	0.12365	0.11838
0.80	4.50	2.50	0.04225	0.03975
0.80	4.50	3.00	0.01206	0.01110
0.80	4.50	3.50	0.00292	0.00262

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