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Genericity of Representations of p-Adic Sp_{2n} and Local Langlands Parameters

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Abstract. Let G be the F-rational points of the symplectic group Sp_{2n}, where F is a non-Archimedean local field of characteristic 0. Cogdell, Kim, Piatetski-Shapiro, and Shahidi constructed local Langlands functorial lifting from irreducible generic representations of G to irreducible representations of $GL_{2n+1}(F)$. Jiang and Soudry constructed the descent map from irreducible supercuspidal representations of $GL_{2n+1}(F)$ to those of G, showing that the local Langlands functorial lifting from the irreducible supercuspidal generic representations is surjective. In this paper, based on above results, using the same descent method of studying SO_{2n+1} as Jiang and Soudry, we will show the rest of local Langlands functorial lifting is also surjective, and for any local Langlands parameter $\phi \in \Phi(G)$, we construct a representation σ such that ϕ and σ have the same twisted local factors. As one application, we prove the G-case of a conjecture of Gross-Prasad and Rallis, that is, a local Langlands parameter $\phi \in \Phi(G)$ is generic, *i.e.,* the representation attached to ϕ is generic, if and only if the adjoint L-function of ϕ is holomorphic at $s = 1$. As another application, we prove for each Arthur parameter ψ, and the corresponding local Langlands parameter ϕ_ψ, the representation attached to ϕ_ψ is generic if and only if ϕ_ψ is tempered.

1 Introduction

Let $\mathbf{G}$ be a connected reductive algebraic group split over F, where F is a non-Archimedean local field of characteristic 0, and let $G = \mathbf{G}(F)$. Let $\Pi(G)$ be the set of all equivalence classes of irreducible admissible representations of G.

Since $Sp_{2n}^\vee(\mathbb{C}) = SO_{2n+1}(\mathbb{C})$, $GL_{2n+1}^\vee(\mathbb{C}) = GL_{2n+1}(\mathbb{C})$, and there is a natural embedding $i\colon SO_{2n+1}(\mathbb{C}) \to GL_{2n+1}(\mathbb{C})$, by the local Langlands functoriality conjecture, there would have a local functorial map $l\colon \Pi(Sp_{2n}) \to \Pi(GL_{2n+1})$.

Let $\Pi^{(g)}(Sp_{2n})$ be the subset of $\Pi(Sp_{2n})$ consisting of irreducible generic representations of Sp_{2n}. In [CKP-SS], Cogdell, Kim, Piatetski-Shapiro, and Shahidi constructed local Langlands functorial lifting l from this subset to $\Pi^{(g)}(GL_{2n+1})$, which is a subset of $\Pi(GL_{2n+1})$ (explicit definition will be given Section 4.5), such that

$$L(\sigma \times \pi, s) = L(l(\sigma) \times \pi, s), \quad \epsilon(\sigma \times \pi, s, \psi) = \epsilon(l(\sigma) \times \pi, s, \psi),$$

for any irreducible generic representation π of $GL_k(F)$, with $k \in \mathbb{Z}_{>0}$, where ψ is a fixed nontrivial character of F. The left-hand side are the local factors defined by Shahidi [S1], and the right-hand side are the local factors defined by Jacquet, Piatetski-Shapiro, and Shalika [JP-SS]; both sides are the Langlands local factors with respect to the standard representations, called standard local factors.

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In [JngS3], Jiang and Soudry constructed the descent map from supercuspidal representations of $GL_{2n+1}(F)$ to irreducible supercuspidal generic representations of $Sp_{2n}(F)$, showing that the local Langlands functorial lifting from irreducible supercuspidal generic representations of $Sp_{2n}(F)$ is surjective.

The first aim of this paper is to prove in Section 4 that the rest of local Langlands functorial lifting is also surjective. Note that for SO_{2n+1}, in [JngS2], Jiang and Soudry have already constructed corresponding local Langlands functorial lifting, and proved that it is actually bijective. To prove the surjectivity in the Sp_{2n}-case, we use the same descent method as in [JngS2]. Since the Jiang's conjecture (a refinement of local converse theorem conjecture; see [Jng] and Section 3) of this case has not been proved, while the SO_{2n+1}-case was proved in [JngS1], the local functorial lifting here may not be injective. We will discuss Jiang's conjecture further in Section 3. Using the same method as in [JngS1], we can show that the equality of local γ-factors of generic representations of $Sp_{2n}(F)$ twisted by any irreducible supercuspidal representation τ of $GL_l(F)$ with $l = 1, 2, \dots, 2n$, can be reduced to supercuspidal generic representations, see Theorem 3.5.

Let W_F be the Weil group associated with F. Let $W_F \times SL_2(\mathbb{C})$ be the Weil–Deligne group, see [Kn, Ku, GR], and let $G^\vee(\mathbb{C})$ be the Langlands dual group of G. A homomorphism ϕ from $W_F \times SL_2(\mathbb{C})$ to $G^\vee(\mathbb{C})$ is called admissible if it can be decomposed into a direct sum of irreducible representations of $W_F \times SL_2(\mathbb{C})$

$$\phi = \bigoplus_i \phi_i \otimes S_{w_i},$$

which satisfy the following conditions:

(i) the representations ϕ_i are continuous complex representations of W_F;
(ii) $\phi_i(W_F)$ consists of semi-simple elements;
(iii) S_{w_i} is the unique irreducible algebraic complex representations of $SL_2(\mathbb{C})$ of dimension w_i.

Let $\Phi(G)$ be the set of conjugacy classes of such admissible homomorphisms. The elements in the set $\Phi(G)$ are called the *local Langlands parameters* for G. Then the local Langlands reciprocity conjecture that associates a local L-packet $\Pi(\phi)$ with each $\phi \in \Phi(G)$ implies the parametrization relation between $\Phi(G)$ and $\Pi(G)$. This conjecture implies the arithmetic aspects of representations of p-adic groups.

For $\mathbf{G} = GL_n$, Zelevinsky [Z] reduced this conjecture to the supercuspidal case, which was proved by Harris and Taylor [HT] and by Henniart [H1]. Note that for each $\phi \in \Phi(GL_n)$, there is only one element in $\Pi(\phi)$. See [Ku] for more discussion in this case.

For $\mathbf{G} = SO_{2n+1}$, this conjecture was studied by Jiang and Soudry [JngS2]. For each local Langlands parameter ϕ, they associated an irreducible admissible representation σ of G, such that ϕ and σ have the same twisted local factors.

The second aim of this paper is to construct in Section 5 an irreducible admissible representation σ of $Sp_{2n}(F)$, for any local Langlands parameter $\phi \in \Phi(Sp_{2n})$ such that ϕ and σ have the same twisted local factors, as in [JngS2]. We state the result.

Let $\Pi'(Sp_{2n})$ be the set of equivalence classes of irreducible admissible representations of $Sp_{2n}(F)$, which are the Langlands quotients of induced representations

$$\delta(\Sigma_1) \times \delta(\Sigma_2) \times \cdots \times \delta(\Sigma_f) \rtimes \sigma^{(t)},$$

where $\sigma^{(t)}$ is an irreducible generic tempered representation of $Sp_{2n^*}(F)$, and $\Sigma_1, \Sigma_2, \dots, \Sigma_f$ are imbalanced segments, whose exponents are positive and in non-increasing order, and $\delta(\Sigma_i)$ is the essentially square-integrable representation of $GL_{n_i}(F)$ associated with Σ_i for $i = 1, 2, \dots, f$ $(n = n^* + \sum_{i=1}^{n} n_i)$.

Theorem 1.1 *There is a surjective map ι from $\Pi'(Sp_{2n})$ to the set $\Phi(Sp_{2n})$. Moreover, the map ι preserves the local factors*

$$L(\sigma \times \tau, s) = L(\iota(\sigma) \otimes r^{-1}(\tau), s) \quad \text{and} \quad \epsilon(\sigma \times \tau, s, \psi) = \epsilon(\iota(\sigma) \otimes r^{-1}(\tau), s, \psi),$$

for all $\sigma \in \Pi'(Sp_{2n})$ and all irreducible admissible representations τ of $GL_k(F)$, with all $k \in \mathbb{Z}_{>0}$. Here $r^{-1}(\tau) \in \Phi(GL_k)$, corresponding to τ by the local Langlands reciprocity map for GL_k as in [HT, H1].

In [JngS2], as an application of their result, Jiang and Soudry proved the SO_{2n+1}-case of a conjecture of Gross-Prasad and Rallis, that is, a local Langlands parameter $\phi \in \Phi(SO_{2n+1})$ is generic, *i.e.*, there is a generic representation attached to ϕ if and only if the adjoint L-function of ϕ is holomorphic at $s = 1$. As an application, we prove the Sp_{2n}-case of this conjecture. This is the third part of this paper (Section 6). Note that this gives a criterion for determining the genericity of the representation attached to ϕ in Section 5.

Theorem 1.2 *For each local Langlands parameter $\phi \in \Phi(Sp_{2n})$, the representation σ attached to ϕ in Theorem* 1.1 *is generic if and only if the local adjoint L-function $L(Ad_{Sp_{2n}} \circ \phi, s)$ is regular at $s = 1$.*

Recently, Gross and Reeder [GR] proved this conjecture for general connected reductive groups and for discrete parameters.

As another application of this paper, for each Arthur parameter (A-parameter) ψ, and the corresponding local Langlands parameter (L-parameter) ϕ_ψ, we give another criterion for determining the genericity of the representation attached to ϕ_ψ in Section 5, besides the criterion in Theorem 1.2. Explicit definitions will be given in Section 7. This is the fourth main part of this paper (Section 7).

Theorem 1.3 *For each A-parameter ψ and the corresponding L-parameter ϕ_ψ, the representation attached to ϕ_ψ in Section 5 is generic if and only if ϕ_ψ is tempered.*

First, for each $\sigma \in \Pi^{(g)}(Sp_{2n})$ with L-parameter ϕ_σ, we will compute its Aubert involution $\widehat{\sigma}$ and the corresponding L-parameter $\phi_{\widehat{\sigma}}$. Then we will prove Theorem 1.3. And if ϕ_ψ is tempered, then there is also an A-parameter $\widehat{\psi}$ such that $\phi_{\widehat{\psi}} = \phi_{\widehat{\sigma}}$, and ψ and $\widehat{\psi}$ are symmetric. The SO_{2n+1}-case of these results were proved By Ban [Ban2]; the same method is used here.

There are two main ingredients for this paper, the descent map given by Jiang and Soudry [JngS3] (see Theorem 4.2) and the classification theory of generic representations of $Sp_{2n}(F)$ given by Muic [M2].

Finally, we describe the structure of this paper. In Section 2, we give some notation and preliminaries. In Section 3, we discuss Jiang's conjecture for $Sp_{2n}(F)$. In Section 4, we prove the surjectivity of the rest of functorial lifting, and write down the corresponding local Langlands parameters for each case of representations. In Section 5, we discuss the structure of local Langlands parameters and prove Theorem 1.1. In Section 6, we prove Theorem 1.2. In Section 7, we prove Theorem 1.3.

2 Notation and Preliminaries

In this paper, we mainly follow the notation in [JngS2, Jng, Z].

Let F be a non-archimedean local field of characteristic zero. We fix a non-trivial character ψ of F. $Sp_{2n}(F)$ denotes the group of F-rational points of the split group Sp_{2n}. From now on, let $G = Sp_{2n}(F)$.

Note that here we use $St(\tau, 2m+1)$ to denote $\delta[\nu^{-m}\tau, \nu^{m}\tau]$ in [JngS2].

In this paper, all representations in the lifting images are required to have trivial central characters.

3 The $Sp(2n)$-case of Jiang's Conjecture

In this section, we discuss the Sp_{2n}-case of a conjecture of Jiang, see [Jng].

The notation in this section follows [Jng]. Given an irreducible admissible representation σ of G, define the *set of generic characters attached to* σ to be

$$\mathcal{F}(\pi) = \{\psi_U | \sigma \text{ is } \psi_U\text{-generic}\},$$

where ψ_U's are characters of U.

Conjecture 3.1 (Jiang) *For any irreducible admissible generic representation σ and σ' of G, the following two conditions hold:*

(i) *the intersection of $\mathcal{F}(\sigma)$ and $\mathcal{F}(\sigma')$ is not empty, and*

(ii) *the twisted local γ-factors are equal,* i.e. $\gamma(s, \sigma \times \tau, \psi) = \gamma(s, \sigma' \times \tau, \psi)$ *holds for all irreducible supercuspidal representation τ of $GL_l(F)$ with $l = 1, 2, \dots, n$, where r is F-rank of G.*

Then $\sigma \cong \sigma'$.

Remark 3.2 For SO_{2n+1}, in each conjectural local L-packet, there exists at most one generic member. The main reason for this is the proved local converse theorem for $SO_{2n+1}(F)$ in [JngS1], *i.e.,* the SO_{2n+1}-case in Jiang's conjecture (Conjecture 3.1). But for Sp_{2n}, Jiang's conjecture has not been proved, so different generic representations may share the same twisted γ-factors, *i.e.,* twisted γ-factors may not be able to distinguish two generic representations. This is the key point of G that is different from $GL_n(F)$ and $SO_{2n+1}(F)$, which we have to always keep in mind. This is also the reason that the local Langlands functorial lifting from irreducible generic representations of G to irreducible representations of $GL_{2n+1}(F)$ may not be injective.

This conjecture is a refinement of the local converse theorem conjecture. For the rest of this section, using same method as in [JngS1], *i.e.*, using information of poles of local γ-factors, we will show that the equality of local γ-factors of generic representations twisted by any irreducible supercuspidal representation τ of $GL_l(F)$ with $l = 1, 2, \ldots, 2n$, can be reduced to supercuspidal representations.

By the classification of the irreducible generic representations of $GL_n(F)$ and G (see [BZ, Z, M2] and Section 4.5), for any irreducible generic representation σ of G, there exists a standard parabolic subgroup P whose Levi part M is isomorphic to

$$GL_{m_1}(F) \times \cdots \times GL_{m_r}(F) \times Sp_{2m_0}(F), \quad n = m_0 + \cdots + m_r,$$

irreducible unitary supercuspidal representations τ_i of $Gl_{m_i}(F)$, $1 \leq i \leq r$, an irreducible supercuspidal generic representation $\sigma^{(0)}$ of $Sp_{m_0}(F)$, and real numbers $z_1 \geq z_2 \geq \cdots \geq z_r \geq 0$, such that σ is a subquotient of the following induced representation

$$\nu^{z_1}\tau_1 \times \cdots \times \nu^{z_r}\tau_r \rtimes \sigma^{(0)}.$$

Then we say σ has supercuspidal support $(P; \tau_1, \tau_2, \ldots, \tau_r; \sigma^{(0)})$ and exponents $(z_1, z_2, \ldots, z_r)$.

Lemma 3.3 *If an irreducible generic representation σ of G has supercuspidal support $(P; \tau_1, \tau_2, \ldots, \tau_r; \sigma^{(0)})$ and exponents $(z_1, z_2, \ldots, z_r)$, then $s = 1 + z_1$ is the rightmost possible real pole of $\gamma(\sigma \times \rho, s, \psi)$, where ρ is any irreducible unitary supercuspidal representation of $GL_{k_\rho}(F)$, with $k_\rho \in \mathbb{Z}_{>0}$. If it really is a pole for ρ, then $\rho \cong \tau_{i_0}$, where $1 \leq i_0 \leq r$, with $z_{i_0} = z_1$.*

Proof By the multiplicativity of local γ-factors (see [S2]),

$$\gamma(\sigma \times \rho, s, \psi) = \Big[\prod_{i=1}^{r} \gamma(\tau_i \times \rho, s + z_i, \psi)\gamma(\widetilde{\tau}_i \times \rho, s - z_i, \psi)\Big]\, \gamma(\sigma^{(0)} \times \rho, s, \psi)$$

for any irreducible unitary supercuspidal representation ρ of $GL_{k_\rho}(F)$, with $1 \leq k_\rho \leq 2n$.

We have

$$\gamma(\tau_i \times \rho, s + z_i, \psi) = \epsilon(\tau_i \times \rho, s + z_i, \psi)\frac{L(\widetilde{\tau}_i \times \widetilde{\rho}, 1 - (s + z_i))}{L(\tau_i \times \rho, s + z_i)},$$

and if $\rho \cong \widetilde{\tau}_i$, then $L(\widetilde{\tau}_i \times \widetilde{\rho}, 1 - (s + z_i))$ has a simple pole at $s = 1 - z_i$, and $L(\tau_i \times \rho, s + z_i)$ has a simple pole at $s = -z_i$. Therefore by [JP-SS, Proposition 8.1] $\gamma(\tau_i \times \rho, s + z_i, \psi)$ has a simple pole at $s = 1 - z_i$ and a simple zero at $s = -z_i$ when and only when $\rho \cong \widetilde{\tau}_i$.

Similarly, $\gamma(\widetilde{\tau}_i \times \rho, s - z_i, \psi)$ has a simple pole at $s = 1 + z_i$, and a simple zero at $s = z_i$ when and only when $\rho \cong \tau_i$.

By [CKP-SS, Theorem 7.3], $\gamma(\sigma^{(0)} \times \rho, s, \psi)$ has a possible zero at $s = 0$, does not have zero at $s > 0$, has a possible simple pole at $s = 1$, and if the pole occurs, $\rho \cong \widetilde{\rho}$, and $L(\rho, \mathrm{Sym}^2, s)$ has a pole at $s = 0$.

Hence, $s = 1 + z_i$ is the rightmost possible real pole of $\gamma(\sigma \times \rho, s, \psi)$, and if it is indeed a pole, it cannot be cancelled by any possible zeros of other factors and $\rho \cong \tau_{i_0}$, where $1 \leq i_0 \leq r$, with $z_{i_0} = z_1$. This proves the lemma. ■

Corollary 3.4 *If σ and σ' both are irreducible generic representations of G, with supercuspidal support $(P; \tau_1, \tau_2, \ldots, \tau_r; \sigma^{(0)})$ and $(P'; \tau_1', \tau_2', \ldots, \tau_{r'}'; \sigma'^{(0)})$, exponents $(z_1, z_2, \ldots, z_r)$ and exponents $(z_1', z_2', \ldots, z_{r'}')$, respectively, and*

$$\gamma(\sigma \times \rho, s, \psi) = \gamma(\sigma' \times \rho, s, \psi)$$

for any irreducible supercuspidal representation ρ of $GL_{k_\rho}(F)$, with $1 \leq k_\rho \leq 2n$. Then $z_1 = z_1'$.

Theorem 3.5 *If σ and σ' both are irreducible generic representations of G, with supercuspidal support $(P; \tau_1, \tau_2, \ldots, \tau_r; \sigma^{(0)})$ and $(P'; \tau_1', \tau_2', \ldots, \tau_{r'}'; \sigma'^{(0)})$, exponents $(z_1, z_2, \ldots, z_r)$ and exponents $(z_1', z_2', \ldots, z_{r'}')$, respectively, and*

$$\gamma(\sigma \times \rho, s, \psi) = \gamma(\sigma' \times \rho, s, \psi)$$

for any irreducible supercuspidal representation ρ of $GL_{k_\rho}(F)$, with $1 \leq k_\rho \leq 2n$. Then after a possible rearrangement of $(\tau_1', z_1'; \ldots; \tau_{r'}', z_{r'}')$, without affecting the decreasing order of $z_1', z_2', \ldots, z_{r'}'$,

(i) *$r = r'$, and $m_i = m_i'$, for $0 \leq i \leq r$,*
(ii) *$z_i = z_i'$, and $\tau_i \cong \tau_i'$, for $0 \leq i \leq r$,*
(iii) *$\gamma(\sigma^{(0)} \times \rho, s, \psi) = \gamma(\sigma'^{(0)} \times \rho, s, \psi)$ for any irreducible supercuspidal representation ρ of $GL_{k_\rho}(F)$, with $1 \leq k_\rho \leq 2n$.*

Note that the proof is same as that of [JngS1, Theorem 5.1], we omit it here.

4 Surjectivity of local Langlands Functorial Lifting

In this section, first we will summarize the results on the local Langlands functorial lifting from $\Pi^{(sg)}(Sp_{2n})$ to $\Pi^{(sg)}(GL_{2n+1})$, then using the same descent method as in [JngS2], we will prove that the rest of local Langlands functorial lifting given by Cogdell, Kim, Piatetski-Shapiro, Shahidi [CKP-SS] is also surjective. In each case, we will write down the corresponding local Langlands parameters. Note that some proofs will be omited, due to the similarity between the cases of Sp_{2n} here and SO_{2n+1} in [JngS2].

4.1 Supercuspidal Generic Representations

Let $\Pi^{(sg)}(Sp_{2n})$ be the set of all equivalence classes of irreducible supercuspidal generic representations of G. Let $\Pi^{(sg)}(GL_{2n+1})$ be the set of all equivalence classes of irreducible tempered representations of $GL_{2n+1}(F)$ of the form

$$\tau_1 \times \tau_2 \times \cdots \times \tau_r = \mathrm{Ind}_Q^{GL_{2n+1}(F)}(\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_r),$$

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where Q is a standard parabolic subgroup of $GL_{2n+1}(F)$ of type $(n_1, \dots, n_r)$ with $2n + 1 = \sum_{i=1}^{r} n_i$, and for each $1 \le i \le r$, τ_i is an irreducible supercuspidal self-dual representation of $GL_{n_i}(F)$ such that $L(\tau_i, \mathrm{Sym}^2, s)$ has a pole at $s = 0$ and for $i \ne j$, $\tau_i \ncong \tau_j$.

Cogdell, Kim, Piatetski-Shapiro, and Shahidi [CKP-SS] gave the following local Langlands functorial lifting.

Theorem 4.1 (Cogdell–Kim–Piatetski-Shapiro–Shahidi) *There is a map l from $\Pi^{(sg)}(Sp_{2n})$ to $\Pi^{(sg)}(GL_{2n+1})$. Moreover, the map l preserves local L and ϵ factors with GL-twists, namely,*

$$L(\sigma \times \pi, s) = L(l(\sigma) \times \pi, s) \quad \epsilon(\sigma \times \pi, s, \psi) = \epsilon(l(\sigma) \times \pi, s, \psi)$$

for any $\sigma \in \Pi^{(sg)}(Sp_{2n})$ and any irreducible generic representation π of $GL_k(F)$ (k is any positive integer).

Jiang and Soudry [JngS3] constructed the descent map from supercuspidal representations of GL_{2n+1} to irreducible supercuspidal representations of Sp_{2n}, implying the following theorem, which is one of the main ingredients of this paper, as we mentioned in the introduction.

Theorem 4.2 (Jiang–Soudry) *The map l in Theorem* 4.1 *is surjective.*

Next, let us figure out the corresponding parameters of irreducible supercuspidal generic representations of G.

The following is a result of Henniart [H2].

Theorem 4.3 (Henniart) *The local Langlands reciprocity map for $GL_n(F)$ has the following property: the gamma factor $\gamma(\phi, \mathrm{Sym}^2, s, \psi)$ ($\gamma(\phi, \wedge^2, s, \psi)$, respectively) has the same poles as the local gamma factor $\gamma(r(\phi), \mathrm{Sym}^2, s, \psi)$ ($\gamma(r(\phi), \wedge^2, s, \psi)$, respectively) for any irreducible ϕ (*i.e.*, $r(\phi)$ supercuspidal), where r is the local Langlands reciprocity map for GL_n.*

As in [JngS1], using Henniart's result, we have the following proposition.

Proposition 4.4 (i) *Assume τ is an irreducible supercuspidal self-dual representation of $GL_m(F)$, having the local Langlands parameter ϕ that is an irreducible admissible m-dimensional complex representation of W_F, and the local symmetric square L-function $L(\tau, \mathrm{Sym}^2, s)$ has a pole at $s = 0$. Then ϕ is orthogonal,* i.e.,

$$\phi(W_F) \subset SO_m(\mathbb{C}).$$

(ii) *Let $\phi = \phi_1 \oplus \phi_2$ be an admissible, completely irreducible, complex representation of W_F with the following property:* $\mathrm{Hom}_{W_F}(\phi_1 \otimes \phi_2, 1) = 0$. *Then ϕ is orthogonal of and only if ϕ_1 and ϕ_2 are both orthogonal.*

Proof For part (i), by definition

$$\gamma(\widetilde{\tau}, \mathrm{Sym}^2, s, \psi) = \epsilon(\widetilde{\tau}, \mathrm{Sym}^2, s, \psi) \cdot \frac{L(\tau, \mathrm{Sym}^2, 1 - s)}{L(\widetilde{\tau}, \mathrm{Sym}^2, s)}.$$

Since by assumption, the local symmetric square L-function $L(\tau, \mathrm{Sym}^2, s)$ has a pole at $s = 0$, so the gamma factor $\gamma(\widetilde{\tau}, \mathrm{Sym}^2, s, \psi)$ has a pole at $s = 1$. Hence by Theorem 4.3, the gamma factor $\gamma(\widetilde{\phi}, \mathrm{Sym}^2, s, \psi)$ also has a pole at $s = 1$. Since we also have

$$\gamma(\widetilde{\phi}, \mathrm{Sym}^2, s, \psi) = \epsilon(\widetilde{\phi}, \mathrm{Sym}^2, s, \psi)\frac{L(\phi, \mathrm{Sym}^2, 1-s)}{L(\widetilde{\phi}, \mathrm{Sym}^2, s)},$$

so the L-function $L(\phi, \mathrm{Sym}^2, s)$ has a pole at $s = 0$. Therefore, by definition of local Artin L-functions, we can see that $\phi(W_F) \subset SO_m(\mathbb{C})$, *i.e.,* the parameter is orthogonal; see [JngS1, p. 796] and [Ban2, p. 7].

For part (ii), it is easy to figure out that if both ϕ_1 and ϕ_2 are orthogonal, then so is ϕ. Conversely, first we know that ϕ is orthogonal if and only if $\mathrm{Sym}^2(\phi)$ has W_F-invariant functionals. Since

$$\mathrm{Sym}^2(\phi) = \mathrm{Sym}^2(\phi_1) \oplus \mathrm{Sym}^2(\phi_2) \oplus [\phi_1 \otimes \phi_2],$$

and by assumption $\mathrm{Hom}_{W_F}(\phi_1 \otimes \phi_2, 1) = 0$, so the W_F-invariant functionals will be nonzero on at least one of $\mathrm{Sym}^2(\phi_1)$, $\mathrm{Sym}^2(\phi_2)$. Without loss of generality, we assume that there exists a nonzero W_F-invariant functional that does not vanish on $\mathrm{Sym}^2(\phi_1)$. Hence ϕ_1 is orthogonal. Since ϕ is non-degenerate and ϕ_2 is the complement of ϕ_1, we conclude that ϕ_2 is also orthogonal. This completes the proof. ■

Let $\Phi^{(sg)}(Sp_{2n})$ be the subset of $\Phi(Sp_{2n})$ consisting of all parameters of type $\phi = \bigoplus_i \phi_i$ with the following properties:

(i) $\phi_i \ncong \phi_j$, if $i \neq j$;

(ii) for each i, ϕ_i is an irreducible element in $\Phi(Sp_{2n_i})$ (or $\Phi(SO_{2n_i})$) for some non-negative integer n_i, *i.e.,* ϕ_i is orthogonal.

Following from Theorem 4.3 and Proposition 4.4, we have the following result for irreducible, generic, supercuspidal representations of G.

Theorem 4.5 *There is a surjective map ι from $\Pi^{(sg)}(Sp_{2n})$ to the set $\Phi^{(sg)}(Sp_{2n})$. The map ι preserves the local factors as follows:*

$$L(\sigma \times \tau, s) = L(\iota(\sigma) \otimes r^{-1}(\tau), s),$$

$$\epsilon(\sigma \times \tau, s, \psi) = \epsilon(\iota(\sigma) \otimes r^{-1}(\tau), s, \psi)$$

for any $\sigma \in \Pi^{(sg)}(Sp_{2n})$ and any irreducible generic representations τ of $GL_{k_\tau}(F)$, with all $k_\tau \in \mathbb{Z}_{>0}$. Here $r^{-1}(\tau)$ is the irreducible admissible representation of of $W_F \times SL_2(\mathbb{C})$ of dimension k_τ, corresponding to τ by the local Langlands reciprocity map for GL_{k_τ}.

4.2 Discrete Series Generic Representations

First, we recall the description of the structure of square-integrable generic representations of G given in [Td, M2].

Let P' be a finite set of irreducible, supercuspidal, self-dual (unitary) representations τ of $GL_{k_\tau}(F)$. Assume that for each $\tau \in P'$, there is a sequence of segments

$$D_i(\tau) = [\nu^{-a_i(\tau)}\tau, \nu^{b_i(\tau)}\tau], \quad i = 1, 2, \ldots, e_\tau,$$

satisfying

$$2a_i(\tau) \in \mathbb{Z} \quad \text{and} \quad 2b_i(\tau) \in \mathbb{Z}_{\geq 0}, \tag{4.1}$$

and

$$a_1(\tau) < b_1(\tau) < a_2(\tau) < b_2(\tau) < \cdots < a_{e_\tau}(\tau) < b_{e_\tau}(\tau). \tag{4.2}$$

Let $\sigma^{(0)}$ be an irreducible supercuspidal generic representation of $Sp_{2n'}(F)$. Assume the following hold:

(DS1) (C1) if $L(\sigma^{(0)} \times \tau, s)$ has a pole at $s = 0$, then $-1 \leq a_i(\tau) \in \mathbb{Z} \setminus \{0\}$, for $1 \leq i \leq e_\tau$;

(DS2) (C0) if $L(\tau, \mathrm{Sym}^2, s)$ has a pole at $s = 0$, but $L(\sigma^{(0)} \times \tau, s)$ is holomorphic at $s = 0$, then $a_i(\tau) \in \mathbb{Z}_{\geq 0}$, for $1 \leq i \leq e_\tau$;

(DS3) (C$\frac{1}{2}$) if $L(\tau, \wedge^2, s)$ has a pole at $s = 0$, then $a_i(\tau) \in -\frac{1}{2} + \mathbb{Z}_{\geq 0}$, for $1 \leq i \leq e_\tau$.

Then the unique generic constituent of

$$\left(\times_{\tau \in P'} \times_{i=1}^{e_\tau} \delta(D_i(\tau)) \right) \rtimes \sigma^{(0)} \tag{4.3}$$

is square-integrable ([Td]). Assume that the element in (4.3) is in G. Then every square-integrable generic representation of G is obtained in this way for a unique set consisting of a finite set P', segments $\{D_i(\tau) | 1 \leq i \leq e_\tau, \tau \in P'\}$ and a unique generic supercuspidal representation $\sigma^{(0)}$ [M2, Proposition 2.1], satisfying conditions (4.1), (4.2), and (DS1)–(DS3).

Note that we say $(\tau, \sigma^{(0)})$ satisfies $(C\alpha)$, where $\alpha \in \{0, \frac{1}{2}, 1\}$, if $\nu^{\pm\alpha}\tau \rtimes \sigma^{(0)}$ reduces, and $\nu^{\pm\beta}\tau \rtimes \sigma^{(0)}$ is irreducible for all $|\beta| \neq \alpha$. And from [M1, Lemma 1.3], we know that our $(\tau, \sigma^{(0)})$ must satisfy one of $(C\alpha)$.

Remark 4.6 If $L(\sigma^{(0)} \times \tau, s)$ has a pole at $s = 0$ (case C1), then $L(\tau, \mathrm{Sym}^2, s)$ has a pole at $s = 0$. We can see this from Theorem 4.5 and [M1, Proposition 3.1]. So, we can see that (DS1) and (DS2) cover all possible cases, where $L(\tau, \mathrm{Sym}^2, s)$ has a pole at $s = 0$.

Let $\Pi^{(dg)}(Sp_{2n})$ be the set of all equivalence classes of irreducible discrete series generic representations of G. Let $\Pi^{(dg)}(GL_{2n+1})$ be the set of all equivalence classes of irreducible tempered representations of $GL_{2n+1}(F)$ of the form

$$St(\tau_1, 2m_1 + 1) \times St(\tau_2, 2m_2 + 1) \times \cdots \times St(\tau_r, 2m_r + 1), \tag{4.4}$$

where the balanced segments $[\nu^{-m_i}\tau_i, \nu^{m_i}\tau_i]$ are pairwise distinct self-dual (*i.e.,* $\tau_i \cong \widetilde{\tau_i}$) and satisfy the following properties. For each i,

(i) if $L(\tau_i, \wedge^2, s)$ has a pole at $s = 0$, then $m_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$, or

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(ii) if $L(\tau_i, \mathrm{Sym}^2, s)$ has a pole at $s = 0$, then $m_i \in \mathbb{Z}_{\geq 0}$.

Remark 4.7 An irreducible admissible representation ρ of $GL_{2n+1}(F)$ lies in $\Pi^{(dg)}(GL_{2n+1})$ if and only if ρ is tempered and satisfies the following properties. For any irreducible unitary supercuspidal representation τ of $GL_k(F)$ with $k = 1, 2, \ldots, 2n+1$,

(i) if $\tau \ncong \widetilde{\tau}$, then $L(\rho \times \tau, s)$ has no poles on the real line;
(ii) if $\tau \cong \widetilde{\tau}$ and $L(\rho \times \tau, s)$ is not holomorphic, then
  (a) if $L(\tau, \wedge^2, s)$ has a pole at $s = 0$, then $L(\rho \times \tau, s)$ has only simple poles, whose real parts lie inside $-\frac{1}{2} + \mathbb{Z}_{\leq 0}$,
  (b) if $L(\tau, \mathrm{Sym}^2, s)$ has a pole at $s = 0$, then $L(\rho \times \tau, s)$ has only simple poles, whose real parts lie inside $\mathbb{Z}_{\leq 0}$.

Then we have the following theorem.

Theorem 4.8 *There is a surjective map l (which extends the one in Theorem 4.2) from $\Pi^{(dg)}(Sp_{2n})$ to $\Pi^{(dg)}(GL_{2n+1})$. Moreover, l preserves local factors*

$$L(\sigma \times \pi, s) = L(l(\sigma) \times \pi, s), \quad \epsilon(\sigma \times \pi, s, \psi) = \epsilon(l(\sigma) \times \pi, s, \psi),$$

for any $\sigma \in \Pi^{(dg)}(Sp_{2n})$ and any irreducible generic representation π of $GL_k(F)$ with all $k \in \mathbb{Z}_{>0}$.

Proof Let $\rho \in \Pi^{(dg)}(GL_{2n+1})$. As in the proof in [JngS2], the idea is to use the poles on the real line of the local L-functions $L(\rho \times \tau, s)$, for all τ in the set $\Pi^{(ss)}(GL_k)$ of equivalence classes of irreducible self-dual supercuspidal of $GL_k(F)$ (with k being any positive integers), to determine the structure of the tempered representation ρ.

Let

$$P(\rho) := \{\tau \in \Pi^{(ss)}(GL_k) | L(\rho \times \tau, s) \text{ has a pole in } \mathbb{R}, k \in \mathbb{Z}_{>0}\}.$$

Then $P(\rho)$ is finite. For $\tau \in P(\rho)$, we list the real poles of $L(\rho \times \tau, s)$ as follows:

$$-m_{d_\tau}(\tau) < \cdots < -m_2(\tau) < -m_1(\tau) \leq 0.$$

Put $d_\tau = 0$ if $L(\rho \times \tau, s)$ is holomorphic for τ irreducible supercuspidal (self-dual or not). Let us consider the following subset of $P(\rho)$:

$$A(\rho) = \{\tau \in P(\rho) | L(\tau_i, \mathrm{Sym}^2, s) \text{ has a pole at } s = 0, \text{ and } d_\tau \text{ is odd}\},$$

$$B(\rho) = \{\tau \in P(\rho) | L(\tau_i, \mathrm{Sym}^2, s) \text{ has a pole at } s = 0, \text{ and } d_\tau \text{ is even}\},$$

$$C(\rho) = \{\tau \in P(\rho) | L(\tau_i, \wedge^2, s) \text{ has a pole at } s = 0\}.$$

By Remark 4.7, we have $P(\rho) = A(\rho) \cup B(\rho) \cup C(\rho)$. And if $\tau \in A(\rho) \cup B(\rho)$, then $\{m_i(\tau)\}_{i=1}^{d_\tau} \subset \mathbb{Z}_{\geq 0}$; if $\tau \in C(\rho)$, then $\{m_i(\tau)\}_{i=1}^{d_\tau} \subset \frac{1}{2} + \mathbb{Z}_{\geq 0}$.

Since for $\tau \in A(\rho)$, d_τ is odd, for $\tau \in B(\rho)$, d_τ is even, and for $\tau \in C(\rho)$, we have $\{m_i(\tau)\}_{i=1}^{d_\tau} \subset \frac{1}{2} + \mathbb{Z}_{\geq 0}$, the representation $\times_{\tau \in A(\rho)} \tau$ is a representation of GL_{2k+1}, k is

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an integer, $2k+1 = \sum_{\tau \in A(\rho)} k_\tau$. Since for $\tau \in A(\rho)$, $L(\tau_i, \mathrm{Sym}^2, s)$ has a pole at s=0, by Theorem 4.2, there exists an irreducible supercuspidal generic representation $\sigma^{(0)}$ (not necessarily unique up to equivalence) of $Sp_{2k}(F)$ such that

$$l(\sigma^{(0)}) = \times_{\tau \in A(\rho)} \tau \tag{4.5}$$

on $GL_{2k+1}(F)$. Let

$$\begin{aligned} A_0(\rho)7 &= \{\tau \in A(\rho) \mid d_\tau = 1 \text{ and } m_1(\tau) = 0\}, \\ A_1(\rho) &= \{\tau \in A(\rho) \mid d_\tau \geq 3 \text{ and } m_1(\tau) = 0\}, \\ A_2(\rho) &= \{\tau \in A(\rho) \mid m_1(\tau) \geq 1\}. \end{aligned}$$

Then they form a partition of $A(\rho)$. For $\tau \in A_1(\rho)$, let

$$\Delta_i(\tau) = \delta[\nu^{-m_{2i}(\tau)}\tau, \nu^{m_{2i+1}(\tau)}\tau], \quad i = 1, 2, \ldots, \frac{d_\tau - 1}{2}, \tag{4.6}$$

for $\tau \in A_2(\rho)$, let

$$\begin{aligned} \Delta_0(\tau) = \delta[\nu\tau, \nu^{m_1(\tau)}\tau], \Delta_i(\tau) = \delta[\nu^{-m_{2i}(\tau)}\tau, \nu^{m_{2i+1}(\tau)}\tau], \\ i = 1, 2, \ldots, \frac{d_\tau - 1}{2}. \end{aligned} \tag{4.7}$$

For $\tau \in B(\rho)$, let

$$\Delta_i(\tau) = \delta[\nu^{-m_{2i-1}(\tau)}\tau, \nu^{m_{2i}(\tau)}\tau], i = 1, 2, \ldots, \tfrac{d_\tau}{2}. \tag{4.8}$$

Similarly, for $\tau \in C(\rho)$, if d_τ is odd, let

$$\begin{aligned} \Delta_0(\tau) = \delta[\nu^{\frac{1}{2}}\tau, \nu^{m_1(\tau)}\tau], \Delta_i(\tau) = \delta[\nu^{-m_{2i}(\tau)}\tau, \nu^{m_{2i+1}(\tau)}\tau], \\ i = 1, 2, \ldots, \frac{d_\tau - 1}{2}. \end{aligned} \tag{4.9}$$

And for $\tau \in C(\rho)$, if d_τ is even, let

$$\Delta_i(\tau) = \delta[\nu^{-m_{2i-1}(\tau)}\tau, \nu^{m_{2i}(\tau)}\tau], i = 1, 2, \ldots, \tfrac{d_\tau}{2}. \tag{4.10}$$

Then, define

$$J_\tau = \begin{cases} \{1, 2, \ldots, \frac{d_\tau - 1}{2}\}, & \text{in case (4.6);} \\ \{0, 1, 2, \ldots, \frac{d_\tau - 1}{2}\}, & \text{in cases (4.7) and (4.9);} \\ \{1, 2, \ldots, \frac{d_\tau}{2}\}, & \text{in cases (4.8) and (4.10).} \end{cases}$$

And let σ_ρ be the unique irreducible generic subrepresentation of

$$\big(\times_{\tau \in P(\rho) \smallsetminus A_0(\rho)} \times_{j \in J_\tau} \Delta_j(\tau)\big) \rtimes \sigma^{(0)}.$$

Actually, σ_ρ is a representation of G. It is now easy to see that the sequence of segments in (4.6)–(4.10), together with $\sigma^{(0)}$ satisfy (4.1), (4.2), (DS1)–(DS3), hence σ_ρ is square-integrable.

As in [JngS2], we can see that the local factors are preserved. ■

Next we generalize Theorem 4.5 to $\Pi^{(dg)}(Sp_{2n})$.

Let $\Phi^{(d)}(Sp_{2n})$ be the subset of $\Phi(Sp_{2n})$ consisting of all the local Langlands parameters of type $\phi = \bigoplus_i \phi_i \otimes S_{2m_i+1}$, where ϕ_i's are irreducible self-dual representation of W_F of dimension k_{ϕ_i}, and S_{2m_i+1}'s are irreducible representations of $SL_2(\mathbb{C})$ of dimension $2m_i + 1$, satisfying the following conditions:

(i) the tensor products $\phi_i \otimes S_{2m_i+1}$ are irreducible and orthogonal;
(ii) $\phi_i \otimes S_{2m_i+1}$ and $\phi_j \otimes S_{2m_j+1}$ are not equivalent if $i \neq j$;
(iii) the image $\phi(W_F \times SL_2(\mathbb{C}))$ is not contained in any proper Levi subgroup of $SO_{2n+1}(\mathbb{C})$.

The local Langlands parameters in $\Phi^{(d)}(Sp_{2n})$ are called *discrete*.

Theorem 4.9 *There is a surjective map* ι *(which extends the one in Theorem* 4.5*) from* $\Pi^{(dg)}(Sp_{2n})$ *to the set* $\Phi^{(d)}(Sp_{2n})$*. The map* ι *preserves the local factors:*

$$L(\sigma \times \tau, s) = L(\iota(\sigma) \otimes r^{-1}(\tau), s),$$

$$\epsilon(\sigma \times \tau, s, \psi) = \epsilon(\iota(\sigma) \otimes r^{-1}(\tau), s, \psi)$$

for all $\sigma \in \Pi^{(dg)}(Sp_{2n})$ *and all irreducible generic representations* τ *of* $GL_{k_\tau}(F)$*, with all* $k_\tau \in \mathbb{Z}_{>0}$*. Here* $r^{-1}(\tau)$ *is the irreducible admissible representation of* $W_F \times SL_2(\mathbb{C})$ *of dimension* k_τ*, corresponding to* τ *by the local Langlands reciprocity map for* GL_{k_τ}*.*

4.3 Elliptic Tempered Generic Representations

First we recall the classification of elliptic tempered generic representations of G from [Hb] and [M2, Lemma 3.3].

Take any $\sigma^{(2)} \in \Pi^{(dg)}(Sp_{2n''})$. Then by Theorem 4.8, there exists $\rho^{(2)} \in \Pi^{(dg)}(GL_{2n''+1})$ such that $l(\sigma^{(2)}) = \rho^{(2)}$. By (4.4) and the proof of Theorem 4.8, we can write $\rho^{(2)}$ as follows:

$$\rho^{(2)} = \times_{i=1}^{r} St(\tau_i, 2m_i + 1) = \times_{\tau \in P(\rho^{(2)})} \times_{j=1}^{d_\tau} St(\tau, 2m_j(\tau) + 1).$$

Let $\beta_1, \dots, \beta_c$ (with possible repetitions) be irreducible, self-dual, supercuspidal representations of $GL_{k_{\beta_1}}(F), \dots, GL_{k_{\beta_c}}(F)$, respectively. Take a sequence of pairwise, inequivalent, square-integrable representations

$$\{St(\beta_i, 2e_i + 1)\}_{i=1}^{c}, 2e_i \in \mathbb{Z}_{\geq 0}$$

of $GL_{k_{\beta_i}(2e_i+1)}(F) (i = 1, 2, \dots, c)$, such that

$$St(\beta_i, 2e_i + 1) \notin \{St(\tau_i, 2m_i + 1) | 1 \leq i \leq r\},$$

and one of the following properties holds:

- $St(\beta_i, 2e_i + 1) \in A_2(\rho^{(2)})$, which implies that $e_i = 0$;
- $L(\sigma^{(0)} \times \beta_i, s)$ has a pole as $s = 0$ and $e_i \geq 1$;
- $L(\beta_i, \mathrm{Sym}^2, s)$ has a pole at $s = 0$, $L(\sigma^{(0)} \times \beta_i, s)$ is holomorphic at $s = 0$, and $e_i \in \mathbb{Z}_{\geq 0}$;

- $L(\beta_i, \wedge^2, s)$ has a pole at $s = 0$ and $e_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$.

Then the unique generic constituent σ of

$$St(\beta_1, 2e_1 + 1) \times \cdots \times St(\beta_c, 2e_c + 1) \rtimes \sigma^{(2)} \tag{4.11}$$

is an elliptic tempered representation of $Sp_{2n}(F)$ $(n = n'' + \sum_{i=1}^{c}(2e_i + 1)k_{\beta_i})$.

This is the way that all elliptic, tempered, generic representations of G are obtained and the inducing data $\{St(\beta_i, 2e_i + 1)\}_{i=1}^{c}$ and $\sigma^{(2)}$ are uniquely determined ([M2]).

Recall that $\sigma^{(0)}$ is the irreducible supercuspidal generic representation of $Sp_{2n'}(F)$ such that $l(\sigma^{(0)}) = \times_{\tau \in A(\rho^{(2)})}\tau$ as in (4.5).

Let $\Pi^{(etg)}(Sp_{2n})$ be the set of equivalence classes of irreducible, elliptic, tempered, generic representations of G. Let $\Pi^{(etg)}(GL_{2n+1})$ be the set of equivalence classes of tempered representations of $GL_{2n+1}(F)$ of the form

$$St(\lambda_1, 2h_1 + 1) \times St(\lambda_2, 2h_2 + 1) \times \cdots \times St(\lambda_f, 2h_f + 1), \tag{4.12}$$

where each representation $St(\lambda_i, 2h_i + 1)$ in (4.12) is self-dual and appears either once or twice, and satisfies the following conditions. For each i,

(i) if $L(\lambda_i, \wedge^2, s)$ has a pole at $s = 0$, then $h_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$;
(ii) if $L(\lambda_i, \mathrm{Sym}^2, s)$ has a pole at $s = 0$, then $h_i \in \mathbb{Z}_{\geq 0}$.

Theorem 4.10 *There is a surjective map l (which extends the one in Theorem 4.8) from $\Pi^{(etg)}(Sp_{2n})$ to $\Pi^{(etg)}(GL_{2n+1})$ and satisfying:*

$$L(\sigma \times \pi, s) = L(l(\sigma) \times \pi, s), \quad \epsilon(\sigma \times \pi, s, \psi) = \epsilon(l(\sigma) \times \pi, s, \psi),$$

for any $\sigma \in \Pi^{(etg)}(Sp_{2n})$ and any irreducible generic representation π of $GL_k(F)$ with all $k \in \mathbb{Z}_{>0}$.

Next let us write down the parameters of representations in $\Pi^{(etg)}(Sp_{2n})$. Let $\Phi^{(etg)}(Sp_{2n})$ be the subset of $\Phi(Sp_{2n})$ consisting of elements of the form

$$\phi = \Big[\bigoplus_{i=1}^{c} \phi_i \otimes S_{2e_i+1}\Big] \oplus \phi_{\sigma^{(2)}} \oplus \Big[\bigoplus_{i=1}^{c} \phi_i \otimes S_{2e_i+1}\Big]$$

with the property that the image $\phi(W_F \times SL_2(\mathbb{C}))$ is a proper Levi subgroup of $Sp_{2n}(\mathbb{C})$ if and only if $c \neq 0$, where $\phi_{\sigma^{(2)}}$ is the parameter corresponding to the irreducible, square-integrable, generic representation $\sigma^{(2)}$ occurring in σ. Then we have the following result.

Theorem 4.11 *There is a surjective map ι (which extends the one in Theorem 4.9) from $\Pi^{(etg)}(Sp_{2n})$ to the set $\Phi^{(etg)}(Sp_{2n})$. And the map ι preserves the local factors:*

$$L(\sigma \times \tau, s) = L(\iota(\sigma) \otimes r^{-1}(\tau), s),$$

$$\epsilon(\sigma \times \tau, s, \psi) = \epsilon(\iota(\sigma) \otimes r^{-1}(\tau), s, \psi)$$

for all $\sigma \in \Pi^{(etg)}(Sp_{2n})$ and all irreducible generic representations τ of $GL_{k_\tau}(F)$, with all $k_\tau \in \mathbb{Z}_{>0}$. Here $r^{-1}(\tau)$ is the irreducible admissible representation of $W_F \times SL_2(\mathbb{C})$ of dimension k_τ, corresponding to τ by the local Langlands reciprocity map for GL_{k_τ}.

4.4 Tempered Generic Representations

First we recall Muic's description of tempered generic representations of G. Let $\sigma^{(et)} \in \Pi^{(etg)}(Sp_{2n'''})$, then by Theorem 4.10, there exists $\rho^{(et)} \in \Pi^{(etg)}(GL_{2n'''+1})$ such that $\rho^{(et)} = l(\sigma^{(et)})$. We keep the notation, describe $\sigma^{(et)}$ as the unique generic constituent of (4.11), and express $\rho^{(et)} = l(\sigma^{(et)})$ as (4.12).

Let $\eta_1, \eta_2, \ldots, \eta_d$ (with possible repetitions) be irreducible, unitary, supercuspidal representations of $GL_{k_{\eta_1}}(F), GL_{k_{\eta_2}}(F), \ldots, GL_{k_{\eta_d}}(F)$, respectively. From these $\eta_i's$, take a sequence of irreducible square-integrable representations $\{St(\eta_i, 2p_i + 1)\}_{i=1}^d$ of $GL_{k_{\eta_1}(2p_i+1)}(F)$ with $2p_i \in \mathbb{Z}_{\geq 0}$ and $i = 1, 2, \ldots, d$, satisfying one of the following properties:

- $St(\eta_i, 2p_i + 1) \in \{St(\beta_j, 2e_j + 1) | 1 \leq j \leq c\}$;
- $St(\eta_i, 2p_i + 1) \in \{St(\tau_j, 2m_j + 1) | 1 \leq j \leq r\}$;
- $\eta_i \ncong \widetilde{\eta}_i$;
- $L(\eta_i, \wedge^2, s)$ has a pole at $s = 0$ and $p_i \in \mathbb{Z}_{\geq 0}$;
- $L(\eta_i, \mathrm{Sym}^2, s)$ has a pole at $s = 0$ and $p_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$.

Then the induced representation

$$\sigma = St(\eta_1, 2p_1 + 1) \times St(\eta_2, 2p_2 + 1) \times \cdots \times St(\eta_d, 2p_d + 1) \rtimes \sigma^{(et)}$$

is an irreducible tempered generic representation of

$$Sp_{2n}(F)\left(n = n''' + \sum_{i=1}^{d} k_{\eta_1}(2p_i + 1)\right).$$

This is the way that all irreducible, tempered, generic representations of G are obtained, and the inducing data $\{St(\eta_i, 2p_i+1)\}_{i=1}^d$ and $\sigma^{(et)}$ are uniquely determined up to replacements $St(\eta_i, 2p_i + 1) \leftrightarrow St(\widetilde{\eta}_i, 2p_i + 1)$ in case $\eta_i \ncong \widetilde{\eta}_i$, ([M2, Theorem 4.1]).

Let $\Pi^{(tg)}(Sp_{2n})$ be the equivalence classes of irreducible, tempered, generic representations of G. Let $\Pi^{(tg)}(GL_{2n+1})$ be the set of equivalence classes of tempered representations of $GL_{2n+1}(F)$ of the form

$$St(\lambda_1, 2h_1 + 1) \times St(\lambda_2, 2h_2 + 1) \times \cdots \times St(\lambda_f, 2h_f + 1), \tag{4.13}$$

where $\lambda_1, \lambda_2, \ldots \lambda_f$ are unitary supercuspidal representations, and $2h_i \in \mathbb{Z}_{\geq 0}$ such that for $1 \leq i \leq f$:

(i) if $\lambda_i \ncong \widetilde{\lambda}_i$,, then $St(\lambda_i, 2h_i + 1)$ occurs in (4.13) as many times as $St(\widetilde{\lambda}_i, 2h_i + 1)$ does;

(ii) if $L(\lambda_i, \wedge^2, s)$ has a pole at $s = 0$, and $h_i \in \mathbb{Z}_{\geq 0}$, then $St(\lambda_i, 2h_i + 1)$ occurs an even number of times in (4.13);

(iii) if $L(\lambda_i, \mathrm{Sym}^2, s)$ has a pole at $s = 0$, and $h_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$, then $St(\lambda_i, 2h_i + 1)$ occurs an even number of times in (4.13).

Theorem 4.12 *There is a surjective map l (which extends the one in Theorem 4.10) from $\Pi^{(tg)}(Sp_{2n})$ to $\Pi^{(tg)}(GL_{2n+1})$. And l preserves local factors:*

$$L(\sigma \times \pi, s) = L(l(\sigma) \times \pi, s), \quad \epsilon(\sigma \times \pi, s, \psi) = \epsilon(l(\sigma) \times \pi, s, \psi),$$

for any $\sigma \in \Pi^{(tg)}(Sp_{2n})$ and any irreducible generic representation π of $GL_k(F)$ with all $k \in \mathbb{Z}_{>0}$.

Next, we write down the parameters for representations in $\Pi^{(tg)}(Sp_{2n})$.
By Theorem 4.12, for each $\sigma \in \Pi^{(tg)}(Sp_{2n})$,

$$\begin{aligned} l(\sigma) = St(\eta_1, 2p_1 + 1) \times \cdots \times St(\eta_d, 2p_d + 1) \times l(\sigma^{(et)}) \\ \cdot St(\widetilde{\eta_d, 2p_d} + 1) \times \cdots \times St(\widetilde{\eta_1, 2p_1} + 1). \end{aligned}$$

Then the local Langlands parameter of σ is

$$\phi_{\sigma^{(et)}} \oplus \bigoplus_{i=1}^{d} \left[\phi_{\eta_i} \times S_{2p_i+1} \oplus \widetilde{\phi_{\eta_i}} \times S_{2p_i+1} \right].$$

Let $\Phi^{(t)}(Sp_{2n})$ be the subset of $\Phi(Sp_{2n})$ consisting of the local Langlands parameters ϕ with the property that $\phi(W_F)$ is bounded in $SO_{2n+1}(\mathbb{C})$. The parameters in $\Phi^{(t)}(Sp_{2n})$ are called *tempered*. Then we prove the following result: the local Langlands parameters corresponding to representations in $\Pi^{(tg)}(Sp_{2n})$ are exactly the tempered parameters.

Theorem 4.13 *There is a surjective map ι (which extends the one in Theorem 4.11) from $\Pi^{(tg)}(Sp_{2n})$ to the set $\Phi^{(t)}(Sp_{2n})$. It preserves the local factors:*

$$\begin{aligned} L(\sigma \times \tau, s) &= L(\iota(\sigma) \otimes r^{-1}(\tau), s), \\ \epsilon(\sigma \times \tau, s, \psi) &= \epsilon(\iota(\sigma) \otimes r^{-1}(\tau), s, \psi) \end{aligned}$$

for all $\sigma \in \Pi^{(tg)}(Sp_{2n})$ and all irreducible generic representations τ of $GL_{k_\tau}(F)$, with all $k_\tau \in \mathbb{Z}_{>0}$. Here $r^{-1}(\tau)$ is the irreducible admissible representation of $W_F \times SL_2(\mathbb{C})$ of dimension k_τ, corresponding to τ by the local Langlands reciprocity map for GL_{k_τ}.

The basic idea of the proof is that, given a $\phi \in \Phi^{(t)}(Sp_{2n})$, compose it with the embedding $SO_{2n+1}(\mathbb{C}) \hookrightarrow GL_{2n+1}(\mathbb{C})$, then it is a $2n+1$-dimensional representation of $W_F \times SL_2(\mathbb{C})$. Then, we can decompose ϕ, since it preserves a non-degenerate symmetric bilinear form.

4.5 Generic Representations

First, we continue with Muic's description of generic representations of G.
We consider self-dual representations of $GL_{2n+1}(F)$ of the form

$$\delta(\Sigma_1) \times \cdots \times \delta(\Sigma_f) \times \rho^{(t)} \times \delta(\widetilde{\Sigma_f}) \times \cdots \times \delta(\widetilde{\Sigma_1}),$$

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where $\rho^{(t)}$ is an irreducible self-dual tempered representation of $GL_{2n^*+1}(F)$ and

$$\begin{aligned}\Sigma_1 &= [\nu^{-q_1}\xi_1, \nu^{-q_1+w_1}\xi_1],\\ \Sigma_2 &= [\nu^{-q_2}\xi_2, \nu^{-q_2+w_2}\xi_2],\\ &\dots,\\ \Sigma_f &= [\nu^{-q_f}\xi_f, \nu^{-q_f+w_f}\xi_f],\end{aligned} \tag{4.14}$$

where $\xi_1, \xi_2, \dots, \xi_f$ are irreducible, unitary, and supercuspidal, with possible repetitions, $q_i \in \mathbb{R}$, $w_i \in \mathbb{Z}_{\geq 0}$, and $q_i \neq \frac{w_i}{2}$.

Assume that $\rho^{(t)} \in \Pi^{(tg)}(GL_{2n^*+1})$. Then by Theorem 4.12 there exists a $\sigma^{(t)} \in \Pi^{(tg)}(Sp_{2n^*})$ such that $l(\sigma^{(t)}) = \rho^{(t)}$. Let $\rho^{(2)}$ be the lift of the irreducible square-integrable generic representation $\sigma^{(2)}$, which is related to $\sigma^{(t)}$. Let $\sigma^{(0)}$ be the irreducible, generic, supercuspidal representation occurring in $\sigma^{(2)}$, whose lift is denoted by $l(\sigma^{(0)}) = \rho^{(0)}$. Then, by Theorems 4.2, 4.8, 4.10, and 4.12, the representation $\rho^{(t)}$ is completely determined up to isomorphism by the following three families of irreducible square-integral representations of $GL_*(F)$:

$$\{St(\tau_j, 2m_j+1)\}_{j=1}^{r}, \quad \{St(\beta_j, 2e_j+1)\}_{j=1}^{c}, \quad \{St(\eta_j, 2p_j+1)\}_{j=1}^{d}. \tag{4.15}$$

Definition 4.14 Let $\{\Sigma_j\}_{j=1}^{f}$ and $\rho^{(t)}$ be given as above. Then the sequence $\{\Sigma_j\}_{j=1}^{f}$ is called an Sp_{2n}-generic sequence of segments with respect to $\rho^{(t)}$ if it satisfies the following conditions:

(i) the segment Σ_i is not linked to either Σ_j or $\widetilde{\Sigma_j}$ for $1 \leq i \neq j \leq f$;

(ii) for $1 \leq i \leq f$, Σ_i is not linked to any segment, which corresponds to a representation in any of the families

$$\{St(\tau_j, 2m_j+1)\}_{j=1}^{r}, \{St(\beta_j, 2e_j+1)\}_{j=1}^{c}, \{St(\eta_j, 2p_j+1)\}_{j=1}^{d},$$
$$\{St(\widetilde{\eta}_j, 2p_j+1) | \eta_j \ncong \widetilde{\eta}_j, 1 \leq j \leq d\};$$

(iii) one of the following three conditions holds

(a) $\xi_i \ncong \widetilde{\xi}_i$,

(b) Σ_i is linked to an element of $A_2(\rho^{(2)})$,

(c) $(\xi_i, \sigma^{(0)})$ is $(C\alpha)$ $(\alpha = 0, \frac{1}{2}, 1)$, but $\pm\alpha \notin \{-q_i, -q_i+1, \dots, -q_i+w_i\}$, that is

(3C1) if $L(\sigma^{(0)} \times \xi_i, s)$ has a pole at $s = 0$, then

$$\pm 1 \notin \{-q_i, -q_i+1, \dots, -q_i+w_i\};$$

(3C0) if $L(\xi_i, \mathrm{Sym}^2, s)$ has a pole at $s = 0$, but $L(\sigma^{(0)} \times \xi_i, s)$ has no pole at $s = 0$, then $0 \notin \{-q_i, -q_i+1, \dots, -q_i+w_i\}$;

(3C$\frac{1}{2}$) if $L(\xi_i, \wedge^2, s)$ has a pole at $s = 0$, then

$$\pm\tfrac{1}{2} \notin \{-q_i, -q_i+1, \dots, -q_i+w_i\}.$$

Given a Sp_{2n}-generic sequence of segments, put $\pi_i = \delta(\Sigma_i), i = 1, 2, \dots, f$, then the representation σ of G defined by $\sigma := \pi_1 \times \pi_2 \times \cdots \times \pi_f \rtimes \sigma^{(t)}$ is irreducible and generic. Moreover, all irreducible generic representations of G can be obtained in this way. And the set $\{\pi_1, \pi_2, \dots, \pi_f, \sigma^{(t)}\}$ is uniquely determined ([M2]). After rearranging the data, if the exponent of $\delta(\Sigma_i)$ is negative, then replace Σ_i by $\widetilde{\Sigma_i}$ to get a positive exponent. We may assume that the exponents of $\delta(\Sigma_1), \delta(\Sigma_2), \dots, \delta(\Sigma_f)$ are positive and in non-increasing order (the Langlands inducing data), *i.e.,*

$$\frac{w_1}{2} - q_1 \geq \frac{w_2}{2} - q_2 \geq \cdots \geq \frac{w_f}{2} - q_f > 0.$$

Let $\Pi^{(g)}(Sp_{2n})$ be the set of equivalence classes of irreducible generic representations of G. Let $\Pi^{(g)}(GL_{2n+1})$ be the set of equivalence classes of irreducible self-dual representations of $GL_{2n+1}(F)$, which are Langlands quotients of representations

$$\delta(\Sigma_1) \times \cdots \times \delta(\Sigma_f) \times \rho^{(t)} \times \delta(\widetilde{\Sigma_f}) \times \cdots \times \delta(\widetilde{\Sigma_1}), \tag{4.16}$$

where $\{\Sigma_j\}_{j=1}^{f}$ are of the form (4.14); $\xi_1, \xi_2, \dots, \xi_f$ are irreducible unitary and supercuspidal with possible repetitions, $q_i \in \mathbb{R}$, $w_i \in \mathbb{Z}_{\geq 0}$, $q_i \neq \frac{w_i}{2}$, and $\rho^{(t)}$ is determined by three families of irreducible square-integrable representations of the form (4.15) satisfying the following:

(i) $\frac{w_1}{2} - q_1 \geq \frac{w_2}{2} - q_2 \geq \cdots \geq \frac{w_f}{2} - q_f > 0.$

(ii) The only possible linkages among the segments

$$\Sigma_1, \Sigma_2, \dots, \Sigma_f, \widetilde{\Sigma_f}, \dots, \widetilde{\Sigma_2}, \widetilde{\Sigma_1}$$

may occur between Σ_i and $\widetilde{\Sigma_i}$ for some index i.

(iii) The representations $\delta(\Sigma_i) \times \rho^{(t)}$ and $\delta(\widetilde{\Sigma_i}) \times \rho^{(t)}$ are irreducible for all $1 \leq i \leq f$.

(iv) Assume ξ_i is self-dual and $2q_i \in \mathbb{Z}$, such that if $L(\xi_i, \wedge^2, s)$ has a pole at $s = 0$, then $q_i \in \frac{1}{2} + \mathbb{Z}$, and if $L(\xi_i, \mathrm{Sym}^2, s)$ has a pole at $s = 0$, then $q_i \in \mathbb{Z}$. Then Σ_i is not linked to $\widetilde{\Sigma_i}$. Moreover, if $L(\rho^{(0)} \times \xi_i, s)$ has a pole at $s = 0$, and $q_i \in \mathbb{Z}$, then $-q_i \geq 2$, or $q_i = -1$ and $\xi_i \in A_2(\rho^{(2)})$.

Theorem 4.15 *There is a surjective map l (which extends the one in Theorem* 4.12*) from $\Pi^{(g)}(Sp_{2n})$ to $\Pi^{(g)}(GL_{2n+1})$. And it preserves the local factors*

$$L(\sigma \times \pi, s) = L(l(\sigma) \times \pi, s), \quad \epsilon(\sigma \times \pi, s, \psi) = \epsilon(l(\sigma) \times \pi, s, \psi),$$

for any $\sigma \in \Pi^{(g)}(Sp_{2n})$ and any irreducible generic representation π of $GL_k(F)$ with all $k \in \mathbb{Z}_{>0}$.

Remark 4.16 The irreducibility of (4.16) is equivalent to the genericity of $\rho = l(\sigma)$.

At last, we write down the corresponding parameters.

Let $\Phi^{(g)}(Sp_{2n})$ be the subset of $\Phi(Sp_{2n})$ that consists of elements of the form

$$\phi_\sigma = \iota(\sigma^{(t)}) \oplus \bigoplus_{i=1}^{f} \Big[|\cdot|^{-q_i + \frac{w_i}{2}} r^{-1}(\xi_i) \otimes S_{w_i+1} \oplus |\cdot|^{q_i - \frac{w_i}{2}} r^{-1}(\widetilde{\xi_i}) \otimes S_{w_i+1} \Big],$$

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where the sequence $\{\Sigma_j = [\nu^{-q_j}\xi_j, \nu^{-q_j+w_j}\xi_j]\}_{j=1}^{f}$ is an Sp_{2n}-generic sequence of segments with respect to $\rho^{(t)} = l(\sigma^{(t)})$, ι is the reciprocity map given in Theorem 4.13 for irreducible, tempered, generic representations in $\Pi^{(tg)}(Sp_{2n})$; r is the reciprocity map for $GL_*(F)$, and $|\cdot|^s$ is the character of W_F normalized as in [T] via local class field theory.

Theorem 4.17 *There is a surjective map ι (which extends the one in Theorem 4.13) from $\Pi^{(g)}(Sp_{2n})$ to $\Phi^{(g)}(Sp_{2n})$. The map ι preserves the local factors:*

$$L(\sigma \times \tau, s) = L(\iota(\sigma) \otimes r^{-1}(\tau), s),$$
$$\epsilon(\sigma \times \tau, s, \psi) = \epsilon(\iota(\sigma) \otimes r^{-1}(\tau), s, \psi),$$

for all $\sigma \in \Pi^{(g)}(Sp_{2n})$ and all irreducible generic representations τ of $GL_{k_\tau}(F)$, with all $k_\tau \in \mathbb{Z}_{>0}$. Here $r^{-1}(\tau)$ is the irreducible admissible representation of $W_F \times SL_2(\mathbb{C})$ of dimension k_τ corresponding to τ by the local Langlands reciprocity map for GL_{k_τ}.

5 Representations Attached to Parameters

In this section, as in [JngS2], we associate one irreducible representation of G with each local Langlands parameter $\phi \in \Phi(Sp_{2n})$. The key idea is to analyze the structure of each local Langlands parameter.

Proposition 5.1 *Given a $\phi \in \Phi(Sp_{2n})$. Then either $\phi \in \Phi^{(t)}(Sp_{2n})$, or*

$$\phi = \phi^{(t)} \oplus \phi^{(n)}, \tag{5.1}$$

where $\phi^{(t)} \in \Phi^{(t)}(Sp_{2n^*})$ ($n^* < n$) and $\phi^{(n)} \in \Phi(SO_{2(n-n^*)})$, which is of the form

$$\phi^{(n)} = \bigoplus_{i=1}^{f} \Big[|\cdot|^{-q_i+\frac{w_i}{2}} \phi_i \otimes S_{w_i+1} \oplus |\cdot|^{q_i-\frac{w_i}{2}} \widetilde{\phi_i} \otimes S_{w_i+1} \Big], \tag{5.2}$$

where $f \in \mathbb{Z}_{>0}$, $w_1, w_2, \dots, w_f \in \mathbb{Z}_{\geq 0}$, $q_1, q_2, \dots, q_f \in \mathbb{R}$ such that for $1 \leq i \leq f$, $q_i \neq \frac{w_i}{2}$, ϕ_i is an irreducible bounded representation of W_F, and for $1 \leq i \leq f-1$,

$$\tfrac{w_i}{2} - q_i \geq \tfrac{w_{i+1}}{2} - q_{i+1} > 0,$$

$|\cdot|^s$ is the character of W_F normalized as in [T] *via local class field theory.*

Proof Given a parameter $\phi \in \Phi(Sp_{2n})$, assume $V = \mathbb{C}^{2n+1}$ is the corresponding non-degenerate orthogonal space of dimension $2n+1$, with an orthogonal form $\langle \cdot, \cdot \rangle$.

Let V_1 be the direct sum of all irreducible subspaces, which are stable under the action of $W_F \times SL_2(\mathbb{C})$ and in which $\phi(W_F)$ is bounded. Let V_2 be the direct sum of all irreducible subspaces, which are stable under the action of $W_F \times SL_2(\mathbb{C})$ and in which $\phi(W_F)$ is unbounded. Then $V = V_1 \oplus V_2$.

As in [JngS2], one can see that both subspaces V_1 and V_2 are non-degenerate with respect to the restriction of the non-degenerate orthogonal form $\langle \cdot, \cdot \rangle$.

Denote by $\phi^{(t)}$ the sub-representation of $W_F \times SL_2(\mathbb{C})$ on V_1, and by $\phi^{(n)}$ the sub-representation of $W_F \times SL_2(\mathbb{C})$ on V_2. Then there are two cases:

(i) $\phi^{(t)} \in \Phi^{(t)}(Sp_{2n^*})$ and $\phi^{(n)} \in \Phi(SO_{2(n-n^*)})$;
(ii) $\phi^{(t)} \in \Phi^{(t)}(SO_{2n^*})$ and $\phi^{(n)} \in \Phi(Sp_{2(n-n^*)})$.

We want to prove that the case (ii) cannot occur. Otherwise, $\phi^{(n)}$ is an odd dimensional orthogonal representation, decompose it into irreducible representations

$$\phi^{(n)} = \bigoplus_{i=1}^{m} |\cdot|^{-q_i+\frac{w_i}{2}}\phi_i \otimes S_{w_i+1}.$$

Since $\phi^{(n)}$ is an orthogonal representation, it is stable under the involution $\theta(g) = J^{-1}g^{t,-1}J$, where J is the orthogonal form of order $2(n-n^*)+1$, that is, a square matrix of order $2(n-n^*)+1$ whose second diagonal are 1 and 0 elsewhere. So, either θ send $|\cdot|^{-q_i+\frac{w_i}{2}}\phi_i \otimes S_{w_i+1}$ to itself or to $|\cdot|^{q_i-\frac{w_i}{2}}\widetilde{\phi_i} \otimes S_{w_i+1}$. Since $\phi^{(n)}$ is odd dimensional, there exists i such that

$$\left(|\cdot|^{-q_i+\frac{w_i}{2}}\phi_i \otimes S_{w_i+1}\right)^{\theta} = |\cdot|^{-q_i+\frac{w_i}{2}}\phi_i \otimes S_{w_i+1}.$$

But, since S_{w_i+1} is self-dual,

$$\left(|\cdot|^{-q_i+\frac{w_i}{2}}\phi_i \otimes S_{w_i+1}\right)^{\theta} = \left(|\cdot|^{-q_i+\frac{w_i}{2}}\phi_i\right)^{\theta} \otimes S_{w_i+1}.$$

On the other hand,

$$\left(|\cdot|^{-q_i+\frac{w_i}{2}}\phi_i\right)^{\theta} = |\cdot|^{q_i-\frac{w_i}{2}}\widetilde{\phi_i}.$$

So, $-q_i + \frac{w_i}{2} = q_i - \frac{w_i}{2}$, and $\phi_i = \widetilde{\phi_i}$. Hence $-q_i + \frac{w_i}{2} = 0$, which means $|\cdot|^{-q_i+\frac{w_i}{2}}\phi_i \otimes S_{w_i+1} = \phi_i \otimes S_{w_i+1}$ is tempered, a contradiction!

Therefore, we can see that $\phi^{(t)} \in \Phi^{(t)}(Sp_{2n^*})$, and $\phi^{(n)}$ is of the form (5.2). This completes the proof. ∎

Let $\Pi'(Sp_{2n})$ is the set of equivalence classes of irreducible admissible representations of G, which are Langlands quotients of induced representations

$$\delta(\Sigma_1) \times \delta(\Sigma_2) \times \cdots \times \delta(\Sigma_f) \rtimes \sigma^{(t)},$$

where $\sigma^{(t)}$ is an irreducible generic tempered representation of $Sp_{2n^*}(F)$, and $\Sigma_1, \Sigma_2, \ldots, \Sigma_f$ are imbalanced segments, whose exponents are positive and in non-increasing order.

Then we have the following result, which is Theorem 1.1.

Theorem 5.2 *There is a surjective map ι (which extends the one in Theorem 4.13) from $\Pi'(Sp_{2n})$ to the set $\Phi(Sp_{2n})$. And it preserves the local factors*

$$L(\sigma \times \tau, s) = L(\iota(\sigma) \otimes r^{-1}(\tau), s),$$

$$\epsilon(\sigma \times \tau, s, \psi) = \epsilon(\iota(\sigma) \otimes r^{-1}(\tau), s, \psi),$$

for all $\sigma \in \Pi'(Sp_{2n})$ and all irreducible admissible representations τ of $GL_{k_\tau}(F)$, with all $k_\tau \in \mathbb{Z}_{>0}$. Here $r^{-1}(\tau)$ is the irreducible admissible representation of $W_F \times SL_2(\mathbb{C})$ of dimension k_τ, corresponding to τ by the local Langlands reciprocity map for GL_{k_τ}.

Proof Given a $\phi \in \Phi(Sp_{2n})$, by Proposition 5.1, it can be written as $\phi = \phi^{(t)} \oplus \phi^{(n)}$. By Theorem 4.13, there exists $\sigma^{(t)} \in \Pi^{(tg)}(Sp_{2n^*})$ such that

$$\iota(\sigma^{(t)}) = \phi^{(t)}. \tag{5.3}$$

Using the local Langlands reciprocity map r for $GL_k(F)$, define

$$\Sigma_i = [\nu^{-q_i} r(\phi_i), \nu^{-q_i+w_i} r(\phi_i)], 1 \le i \le f. \tag{5.4}$$

Let σ be the Langlands quotient of the induced representation $\delta(\Sigma_1) \times \delta(\Sigma_2) \times \ldots \delta(\Sigma_f) \rtimes \sigma^{(t)}$, and define $\iota(\sigma) = \phi$. ■

6 A Conjecture of Gross–Prasad and Rallis

In this section, we give an application of the above results to a conjecture of Gross–Prasad [GP] and Rallis [Ku]. For general formulation and discussion of this conjecture, see [JngS2]. We will prove the Sp_{2n}-case of this conjecture; the method is the same as in [JngS2]. Note that for $\mathbf{G} = Sp_{2n}$, ϕ is generic if the representation σ attached to ϕ in Theorem 5.2 is generic. By the classification of irreducible generic representations of $G = Sp_{2n}(F)$ in [M2], we have the following characterization of the genericity of the local Langlands parameters of Sp_{2n}.

Proposition 6.1 *For any local Langlands parameter $\phi\colon W_F \times SL_2(\mathbb{C}) \to SO_{2n+1}(\mathbb{C})$, the representation σ attached to ϕ in Theorem* 5.2 *is generic if and only if $\sigma^{(t)}$ and $\Sigma_i (i = 1, 2, \ldots, f)$ defined in* (5.3) *and* (5.4) *satisfy the conditions of Definition* 4.14 *(with $\rho^{(t)} = l(\sigma^{(t)})$).*

The following theorem is the G-case of the conjecture, which is Theorem 1.2 stated in the introduction. It gives a criterion for determining the genericity of the representation attached to each ϕ in Section 5.

Theorem 6.2 *For any local Langlands parameter $\phi\colon W_F \times SL_2(\mathbb{C}) \to SO_{2n+1}(\mathbb{C})$, the representation σ attached to ϕ in Theorem* 5.2 *is generic if and only if the associated adjoint L-function $L(\mathrm{Ad}_{SO_{2n+1}} \circ \phi, s)$ is regular at $s = 1$.*

Proof *Step (1).* Assume that σ is generic. Write $\phi = \phi^{(t)} \oplus \phi^{(n)}$ as in (5.1) and (5.2). Put

$$\theta = \bigoplus_{i=1}^{f} |\cdot|^{\frac{w_i}{2} - q_i} \phi_i \otimes S_{w_i+1}.$$

Then, $\phi^{(n)} = \theta \oplus \widetilde{\theta}$, and we have the following decomposition of $L(\mathrm{Ad}_{SO_{2n+1}} \circ \phi, s)$

$$\begin{aligned} L(\mathrm{Ad}_{SO_{2n+1}} \circ \phi, s) = L(\theta \otimes \widetilde{\theta}, s) L(\theta \otimes \phi^{(t)}, s) L(\widetilde{\theta} \otimes \phi^{(t)}, s) \\ \cdot\ L(\mathrm{Ad}_{SO_{2n^*+1}} \circ \widetilde{\phi}, s) L(\wedge^2 \circ \theta, s) L(\wedge^2 \circ \widetilde{\theta}, s). \end{aligned} \tag{6.1}$$

We will show that each factor in the above product is holomorphic at $s = 1$.

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By Theorems 4.12 and 4.13 and by [HT] and [H1], we have

$$L(\theta \otimes \widetilde{\theta}, s) = L(r(\theta) \times \widetilde{r(\theta)}, s),$$
$$L(\theta \otimes \phi^{(t)}, s) = L(r(\theta) \times \rho^{(t)}, s),$$
$$L(\widetilde{\theta} \otimes \phi^{(t)}, s) = L(\widetilde{r(\theta)} \times \rho^{(t)}, s),$$

where $\rho^{(t)} = l(\sigma^{(t)})$, $\phi^{(t)} = y(\sigma^{(t)})$, and $r(\theta) = \delta(\Sigma_1) \times \cdots \times \delta(\Sigma_f)$. By Definition 4.14(i) and (ii), the representations $r(\theta)$ and $\pi = r(\theta) \times \rho^{(t)}$ are irreducible and generic; assume that π is a representation of $GL_{n_1}(F)$. Then from the known GL_n-case of the conjecture, we know that $L(\pi \times \widetilde{\pi}, s) = L(\mathrm{Ad}_{GL_{n_1}} \circ r^{-1}(\pi), s)$ is holomorphic at $s = 1$. On the other hand, $L(\pi \times \widetilde{\pi}, s)$ has the following decomposition

$$L(\pi \times \widetilde{\pi}, s) = L(r(\theta) \times \widetilde{r(\theta)}, s) L(r(\theta) \times \rho^{(t)}, s) L(\widetilde{r(\theta)} \times \rho^{(t)}, s) \cdot L(\rho^{(t)} \times \rho^{(t)}, s). \tag{6.2}$$

Since the last L-factor in (6.2) does not vanish at $s = 1$,

$$L(r(\theta) \times \widetilde{r(\theta)}, s) L(r(\theta) \times \rho^{(t)}, s) L(\widetilde{r(\theta)} \times \rho^{(t)}, s)$$

is holomorphic at $s = 1$. Note that this product occurs in (6.1).

From Theorem 5.2, we know that $\rho^{(t)} = l(\sigma^{(t)})$ is an irreducible, tempered, generic representation of $GL_{2n^*+1}(F)$. Then from the known GL_n-case of the conjecture we have

$$L(\rho^{(t)} \times \widetilde{\rho^{(t)}}, s) = L(\mathrm{Ad}_{GL_{2n^*+1}} \circ \phi^{(t)}, s)$$

is regular at $s = 1$. Since as polynomials in q^{-s}, $L(\mathrm{Ad}_{SO_{2n^*+1}} \circ \phi^{(t)}, s)^{-1}$ divides $L(\mathrm{Ad}_{GL_{2n^*+1}} \circ \phi^{(t)}, s)^{-1}$, $L(\mathrm{Ad}_{SO_{2n^*+1}} \circ \phi^{(t)}, s)$ is holomorphic at $s = 1$.

From Proposition 5.1, we know that θ has positive exponents, so the L-function $L(\theta \otimes \theta, s)$ is holomorphic at $s = 1$. Since $L(\theta \otimes \theta, s) = L(\mathrm{Sym}^2 \circ \theta, s) L(\wedge^2 \circ \theta, s)$, and $L(\mathrm{Sym}^2 \circ \theta, s)$ does not vanish at $s = 1$, $L(\wedge^2 \circ \theta, s)$ must be holomorphic at $s = 1$.

At last, we have to show that $L(\wedge^2 \circ \widetilde{\theta}, s)$ is regular at $s = 1$. Let $\theta_i = \phi_i \otimes S_{w_i+1}$, then we have the decomposition

$$L(\wedge^2 \circ \widetilde{\theta}, s) = \prod_{i=1}^{f} L(\wedge^2 \circ \widetilde{\theta}_i, s - w_i + 2q_i) \cdot \prod_{1 \leq i < j \leq f} L\Big(\widetilde{\theta}_i \otimes \widetilde{\theta}_j, s - \frac{w_i + w_j}{2} + q_i + q_j\Big).$$

For $1 \leq i < j \leq f$, by [JngS2, (0.17)], we have

$$L\Big(\widetilde{\theta}_i \otimes \widetilde{\theta}_j, s - \frac{w_i + w_j}{2} + q_i + q_j\Big) = L\Big(\widetilde{St}_i \otimes \widetilde{St}_j, s - \frac{w_i + w_j}{2} + q_i + q_j\Big) = L\big(\delta(\widetilde{\Sigma}_i) \times \delta(\widetilde{\Sigma}_j), s\big),$$

where $\delta(\widetilde{\Sigma}_i) = \nu^{q_i - \frac{w_i}{2}} \widetilde{St}_i$.

By [JngS2, Proposition 7.1], the following statement is true: for $i < j$, Σ_i and Σ_j are linked if and only if

$$L(\delta(\Sigma_i) \times \delta(\widetilde{\Sigma_j}), s) L(\delta(\Sigma_j) \times \delta(\widetilde{\Sigma}_i), s)$$

has a pole at $s = 1$.

Since there is no linkage between $\widetilde{\Sigma}_i$ and Σ_j (Definition 4.14(i)), so by the statement above, we have that $L(\delta(\widetilde{\Sigma}_i) \times \delta(\widetilde{\Sigma_j}), s)$ is holomorphic at $s = 1$; that is,

$$L\Big(\widetilde{\theta}_i \otimes \widetilde{\theta}_j, s - \frac{w_i + w_j}{2} + q_i + q_j\Big)$$

is holomorphic at $s = 1$.

Next, we want to calculate the L-factor $L(\wedge^2 \circ \widetilde{\theta}_i, z)$ for $i = 1, 2, \ldots, f$ and $z = s - w_i + 2q_i$. First, we know that

$$\wedge^2 \circ (\widetilde{\phi}_i \otimes S_{w_i+1}) = (\wedge^2 \circ \widetilde{\phi}_i) \otimes (\mathrm{Sym}^2 \circ S_{w_i+1}) \oplus (\mathrm{Sym}^2 \circ \widetilde{\phi}_i) \otimes (\wedge^2 \circ S_{w_i+1}).$$

The following formula can be found in [FH]:

$$\mathrm{Sym}^2(\mathrm{Sym}^m \mathbb{C}^2) = \bigoplus_{k=0}^{[\frac{m}{2}]} \mathrm{Sym}^{2m-4k} \mathbb{C}^2,$$

$$\wedge^2(\mathrm{Sym}^m \mathbb{C}^2) = \bigoplus_{k=0}^{[\frac{m-1}{2}]} \mathrm{Sym}^{2(m-1)-4k} \mathbb{C}^2.$$

And since S_{w_i+1} is the irreducible representation Sym^{w_i} of $SL_2(\mathbb{C})$,

$$\wedge^2 \circ (\widetilde{\phi}_i \otimes S_{w_i+1}) = \Big[\bigoplus_{k=0}^{[\frac{w_i}{2}]} (\wedge^2 \circ \widetilde{\phi}_i) \otimes S_{2w_i-4k+1}\Big] \otimes \Big[\bigoplus_{k=0}^{[\frac{w_i-1}{2}]} (\mathrm{Sym}^2 \circ \widetilde{\phi}_i) \otimes S_{2(w_i-1)-4k+1}\Big].$$

Therefore,

$$L(\wedge^2 \circ \widetilde{\theta}_i, z) = \prod_{k=0}^{[\frac{w_i}{2}]} L(\wedge^2 \circ \widetilde{\phi}_i, z + w_i - 2k) \cdot \prod_{k=0}^{[\frac{w_i-1}{2}]} L(\mathrm{Sym}^2 \circ \widetilde{\phi}_i, z + w_i - 2k - 1). \tag{6.3}$$

That is,

$$L(\wedge^2 \circ \widetilde{\theta}_i, s - w_i + 2q_i) = \prod_{k=0}^{[\frac{w_i}{2}]} L(\wedge^2 \circ \widetilde{\phi}_i, s + 2q_i - 2k) \cdot \prod_{k=0}^{[\frac{w_i-1}{2}]} L(\mathrm{Sym}^2 \circ \widetilde{\phi}_i, s + 2q_i - 1 - 2k). \tag{6.4}$$

We assume that ϕ_i is self-dual since if ϕ_i is not self-dual, then all these L-factors in (6.4) are holomorphic on the real line, and in particular at $s = 1$.

From Theorem 4.3, we know that $L(\mathrm{Sym}^2 \circ\phi_i, z)$ has a pole at $z = 0$ if and only if $L(r(\phi_i), \mathrm{Sym}^2, z)$ has a pole at $z = 0$ and $L(\wedge^2 \circ \phi_i, z)$ has a pole at $z = 0$ if and only if $L(r(\phi_i), \wedge^2, z)$ has a pole at $z = 0$.

If $L(r(\phi_i), \wedge^2, s + 2q_i - 2k)$ has a pole at $s = 1$ for some $0 \le k \le [\frac{w_i}{2}]$, then $1 + 2q_i - 2k = 0$, since $r(\phi_i)$ is an irreducible self-dual supercuspidal representation of $GL_*(F)$. So that

$$-q_i = \tfrac{1}{2} - k \in \tfrac{1}{2} + \mathbb{Z}_-. \tag{6.5}$$

Since $-q_i + w_i \ge -q_i + \frac{w_i}{2} > 0$, we know from (6.5) that $-q_i + w_i \in \frac{1}{2} + \mathbb{Z}_{\ge 0}$. Since by (6.5), $-q_i \le \frac{1}{2}$, we can see that

$$\tfrac{1}{2} \in \{-q_i, -q_i + 1, \dots, -q_i + w_i\}.$$

This contradicts Definition 4.14(iii)(c). On the other hand, Definition 4.14(iii)(a) and (b) are not valid, so we can see that $L(r(\phi_i), \wedge^2, s + 2q_i - 2k)$ is holomorphic at $s = 1$, for all $0 \le k \le [\frac{w_i}{2}]$.

If $L(r(\phi_i), \mathrm{Sym}^2, s + 2q_i - 2k - 1)$ has a pole at $s = 1$, for some $0 \le k \le [\frac{w_i - 1}{2}]$, then $-q_i = -k \in \mathbb{Z}_-$. So $-q_i + w_i \ge 1$. Then we can see that $0, 1 \in \{-q_i, -q_i + 1, \dots, -q_i + w_i\}$. This contradicts Definition 4.14(iii)(b) and (c). On the other hand, Definition 4.14(iii)(a) is not valid, so, we can see that $L(r(\phi_i), \mathrm{Sym}^2, s + 2q_i - 2k - 1)$ is holomorphic at $s = 1$ for all $0 \le k \le [\frac{w_i - 1}{2}]$.

Therefore the adjoint L-function $L(\mathrm{Ad}_{SO_{2n+1}} \circ \phi, s)$ is holomorphic at $s = 1$ when $\phi \in \Phi(Sp_{2n})$ is a generic parameter. Hence, Step (1) is proved.

Step (2). Assume that the adjoint L-function $L(\mathrm{Ad}_{SO_{2n+1}} \circ \phi, s)$ is regular at $s = 1$, we will prove that ϕ is generic.

Assume that ϕ is not generic, then $\rho^{(t)} = l(\sigma^{(t)})$ and $\Sigma_i (i = 1, 2, \dots, f)$ do not satisfy the conditions in Definition 4.14. We will consider all the cases one by one.

If Definition 4.14(i) is not satisfied, then there exist $1 \le i \ne j \le f$, such that Σ_i is linked to Σ_j or $\widetilde{\Sigma_j}$. Then, from the proof of Step (1), we know that the product

$$L\big(\delta(\Sigma_i) \times \delta(\widetilde{\Sigma_j}), s\big)\, L\big(\delta(\Sigma_j) \times \delta(\widetilde{\Sigma}_i), s\big)\, L\big(\delta(\Sigma_i) \times \delta(\Sigma_j), s\big)\, L\big(\delta(\widetilde{\Sigma}_i) \times \delta(\widetilde{\Sigma_j}), s\big) \tag{6.6}$$

has a pole at $s = 1$. This means that $L(\mathrm{Ad}_{SO_{2n+1}} \circ \phi, s)$ has a pole at $s = 1$, but the product in (6.6) is a factor in $L(\mathrm{Ad}_{SO_{2n+1}} \circ \phi, s)$, a contradiction!

If Definition 4.14(ii) is not satisfied, then the representation $r(\theta) \times \rho^{(t)}$ is reducible and its Langlands quotient π is non-generic. Then following from the GL_n-case of the conjecture, we know that the product in (6.2) has a pole at $s = 1$. And, since the last factor is holomorphic at $s = 1$, the pole at $s = 1$ must occur in the product of the first three factors. On the other hand, the product of the first three factors occurs in $L(\mathrm{Ad}_{SO_{2n+1}} \circ \phi, s)$, so we can see that $L(\mathrm{Ad}_{SO_{2n+1}} \circ \phi, s)$ has a pole at $s = 1$, a contradiction!

If there is an integer $1 \le i \le f$ such that Definition 8.1(iii) is not satisfied, then ϕ_i is self-dual; Σ_i is not linked to an element of $A_2(\rho^{(2)})$, and $r(\phi_i)$ does not satisfy condition (iii)(c), where $\rho^{(2)} = l(\sigma^{(2)})$ and $\sigma^{(2)}$ is the irreducible discrete series generic representation occurring in $\sigma^{(t)}$. Let $\sigma^{(0)}$ be the irreducible, supercuspidal, generic representation occurring in $\sigma^{(2)}$. Put $\xi_i = r(\phi_i)$, then ξ_i is self-dual.

Assume that $(\xi_i, \sigma^{(0)})$ is $(C1)$, but one of $\pm 1 \in \{-q_i, -q_i + 1, \dots, -q_i + w_i\}$. Then, $q_i \in \mathbb{Z}$. Since $L(\xi_i, \mathrm{Sym}^2, s)$ has a pole at $s = 0$, (6.3) will have a pole at $s = 1$ if there exists one $0 \le k' \le [\frac{w_i - 1}{2}]$ such that

$$1 + 2q_i - 2k' - 1 = 0,$$

that is $k' = q_i$. If $q_i < 0$, then $q_i \le -1$, and so $-q_i \ge 1$; this means $-q_i = 1$. Since $L(\rho^{(0)} \times \xi_i, s)$ has a pole at $s = 1$, this means Σ_i is linked to an element of $A(\rho^{(2)})$. Then Σ_i is linked to an element of $A_0(\rho^{(2)}) \cup A_1(\rho^{(2)})$. So, Σ_i is linked to a segment of $\rho^{(t)}$, which means Definition 4.14(ii) is not satisfied. But, in this case, we have already shown that $L(\mathrm{Ad}_{SO_{2n+1}} \circ \phi, s)$ has a pole at $s = 1$, a contradiction! So, we have $q_i \ge 0$. Since $q_i < \frac{w_i}{2}$, we have

$$0 \le q_i \le \left[\frac{w_i - 1}{2}\right], q_i \in \mathbb{Z}.$$

So, the second product of (6.3) has a pole at $s = 1$, for its factor corresponding to $k' = q_i$ has a pole at $s = 1$, a contradiction!

If $(\xi_i, \sigma^{(0)})$ is $(C0)$, but $0 \in \{-q_i, -q_i + 1, \dots, -q_i + w_i\}$, then we can see that $q_i \in \mathbb{Z}$. So, $-q_i \le 0$; that is, $q_i \ge 0$. On the other hand, $q_i < \frac{w_i}{2}$, so, we have $0 \le q_i \le [\frac{w_i - 1}{2}], q_i \in \mathbb{Z}$. Then as in the last case, we can see that the product (6.3) has a pole at $s = 1$, a contradiction!

If $(\xi_i, \sigma^{(0)})$ is $(C\frac{1}{2})$, but one of $\pm\frac{1}{2} \in \{-q_i, -q_i + 1, \dots, -q_i + w_i\}$. Then $q_i \in \frac{1}{2} + \mathbb{Z}$ and $-q_i \le \frac{1}{2}$. That is $q_i + \frac{1}{2} \ge 0$. Since $L(\xi_i, \wedge^2, s)$ has a pole at $s = 0$, (6.3) will have a pole at $s = 1$ if there exists one $0 \le k' \le [\frac{w_i}{2}]$ such that $1 + 2q_i - 2k' = 0$; that is, $k' = q_i + \frac{1}{2}$. On the other hand, $q_i < \frac{w_i}{2}$, so we have

$$0 \le q_i + \tfrac{1}{2} \le [\tfrac{w_i}{2}], q_i \in \tfrac{1}{2} + \mathbb{Z}.$$

So, for the same reason as above, the product (6.3) has a pole at $s = 1$, a contradiction!

Therefore, ϕ is generic. This completes the proof. ■

7 Genericity and Arthur Parameters

The local Arthur parameter (A-parameter) for G is of the following form (direct sum of irreducible representations):

$$\psi : W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \to SO_{2n+1}(\mathbb{C})$$

$$\psi = \bigoplus_{i=1}^{k} \phi_i \otimes S_{m_i} \otimes S_{n_i},$$

satisfying the following conditions:

(i) $\phi_i(W_F)$ is bounded and consists of semi-simple elements;
(ii) the restrictions of ψ to the two copies of $SL_2(\mathbb{C})$ are analytic.

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By Arthur's conjecture (see [A]), for each A-parameter ψ, there is a conjectual A-packet corresponding to ψ, which is also a finite set of $\Pi(G)$ satisfying certain conditions.

For each A-parameter ψ, Arthur associated a local Langlands parameter (L-parameter) ϕ_ψ as follows

$$\phi_\psi(w,x) = \psi\left(w, x, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right),$$

and

$$\phi(w) \otimes S_m(x) \otimes S_n\left(\begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right) = \bigoplus_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} |w|^j \phi(w) \otimes S_m(x).$$

Arthur also showed that $\psi \mapsto \phi_\psi$ is injective.

Let $R(G)$ be the Grothendieck group of the category of all smooth finite length representations of G, then the Aubert duality operator D_G on $R(G)$ is defined as follows (see [Aub])

$$D_G = \Sigma_{\theta \subseteq \Delta} (-1)^{|\theta|} i^G_{M_\theta} \circ r^G_{M_\theta},$$

where Δ is the set of simple roots, M_θ is the standard Levi corresponding to the subset θ, $i^G_{M_\theta}$ and $r^G_{M_\theta}$ are normalized induction and Jacquet functors, respectively. For any $\sigma \in \Pi(G)$, let $\widehat{\sigma} = \pm D_G(\sigma)$ take the sign such that $\widehat{\sigma}$ is a positive element in $R(G)$. This is called the Aubert involution of σ.

In this section, first, given any $\sigma \in \Pi^{(g)}(Sp_{2n})$ with L-parameter ϕ_σ, we will compute its Aubert involution $\widehat{\sigma}$ and the corresponding L-parameter $\phi_{\widehat{\sigma}}$. Then we will show that for each A-parameter ψ and the corresponding L-parameter ϕ_ψ, the representation attached to ϕ_ψ in Section 5 is generic if and only if ϕ_ψ is tempered, *i.e.*, Theorem 1.3. And if ϕ_ψ is tempered, then there is also an A-parameter $\widehat{\psi}$ such that $\phi_{\widehat{\psi}} = \phi_{\widehat{\sigma}}$, and ψ and $\widehat{\psi}$ are symmetric. Ban proved the SO_{2n+1}-case of these results in [Ban2]; we use the same method.

Denote

$$\begin{aligned}
\times_{\tau \in P'} \times_{i=1}^{e_\tau} \delta(D_i(\tau)) &= \times_{i=1}^{k} \delta(\Sigma_i),\\
\{St(\beta_j, 2e_j+1)\}_{j=1}^{c} &= \{\delta(\Sigma_i)\}_{i=k+1}^{l},\\
\{St(\eta_j, 2p_j+1)\}_{j=1}^{d} &= \{\delta(\Sigma_i)\}_{i=l+1}^{m},\\
\{\delta(\Sigma_i)\}_{i=1}^{f} &= \{\delta(\Sigma_i)\}_{i=m+1}^{p},\\
\times_{\tau \in A(\rho^{(2)})} \tau &= \{\delta(\Sigma_i)\}_{i=p+1}^{q},
\end{aligned}$$

where $\Sigma_i = [\nu^{-a_i}\tau_i, \nu^{b_i}\tau_i]$.

Let $I_1 = \{1, \dots, k\}$, $I_2 = \{k+1, \dots, l\}$, $I_3 = \{l+1, \dots, m\}$, $I_4 = \{m+1, \dots, p\}$, $I_5 = \{p+1, \dots, q\}$, and $I_1' = \{i \in I_1 | a_i \geq 0\}$, $I_1'' = \{i \in I_1 | a_i = -1\}$, $I_5' = I_5 / \{j \in I_5 | \tau_j \cong \tau_i,$ for some $i \in I_1''\}$, $I = I_1 \cup I_2 \cup I_3 \cup I_4$, $I_0 = \{i \in I | 0 \in [-a_i, b_i]\}$.

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For $j_i \in [-a_i, b_i]$, $j_i \neq 0$, let $\epsilon_{j_i} = 1$ if $j_i > 0$, and let $\epsilon_{j_i} = -1$ if $j_i < 0$. Rewrite the multiset $\{(|j_i|, \epsilon_{j_i}) | i \in I, j_i \in [-a_i, b_i], j_i \neq 0\}$ in a non-increasing order with respect to the first j_i, denote the ultimate set by $\{(\alpha_1, \epsilon_1), \dots, (\alpha_t, \epsilon_t)\}$. For $\alpha_s = |j_i|$, let $\tau_{\alpha_s} = \tau_i$, and

$$\tau_{\alpha_s}^{\alpha_s} = \begin{cases} \tau_{\alpha_s}, & \text{if } \alpha_s = 1, \\ \widetilde{\tau_{\alpha_s}}, & \text{if } \alpha_s = -1. \end{cases}$$

Let σ' be the unique generic constituent of $\times_{i \in I_0} \tau_i \rtimes \sigma^{(0)}$.

Lemma 7.1 *We have that $\widehat{\sigma}$ is the Langlands quotient of the following induced representation*

$$\nu^{\alpha_1} \tau_{\alpha_1}^{\epsilon_1} \times \cdots \times \nu^{\alpha_t} \tau_{\alpha_t}^{\epsilon_t} \rtimes \widehat{\sigma}'.$$

Proof Let

$$\sigma_1 = (\nu^{b_1} \tau_1 \times \cdots \times \nu^{-a_1} \tau_1) \times \cdots \times (\nu^{b_p} \tau_p \times \cdots \times \nu^{-a_p} \tau_p) \rtimes \sigma^{(0)}.$$

By the classification theory of generic representations of G in Section 3, σ is a subrepresentation of σ_1. Then by [Ban1, Corollary 4.2], $\widehat{\sigma}$ is a quotient of σ_1.

By the Langlands classification theory of representations of G, we can write $\widehat{\sigma}$ as the unique Langlands quotient of the induced representation

$$\nu^{\beta_1} St_1 \times \cdots \times \nu^{\beta_z} St_z \rtimes \sigma_2,$$

where the St_i's are irreducible square-integrable and σ_2 is irreducible tempered, and $\beta_1 \geq \cdots \geq \beta_z > 0$.

Since the Aubert involution commutes with parabolic induction in the Grothendieck group and $\sigma = \widehat{\widehat{\sigma}}$ [Aub], σ is the unique irreducible generic constituent of

$$\nu^{-\beta_1} \widehat{St_1} \times \cdots \times \nu^{-\beta_z} \widehat{St_z} \rtimes \widehat{\sigma}_2,$$

where $\widehat{St_i}$ is the Aubert involution (*i.e.,* the Zelevinsky involution) of St_i. Then from [Rod, Theorems 2 and 3] or [M2, Lemma 1.2], both $\widehat{St_i}$ and $\widehat{\sigma}_2$ are generic. On the other hand, for each $1 \leq i \leq z$, St_i is an irreducible, square-integrable, generic representation, by [Z, Theorem 9.7], $\widehat{St_i}$ is generic if and only if it is supercuspidal, *i.e.,* St_i is supercuspidal.

Since σ_2 is irreducible tempered, it can be written as a subrepresentation of the induced representation $St_{z+1} \times \cdots \times St_w \rtimes \sigma_3$, where St_j's and σ_3 are all square-integrable.

By [Jan, Theorem 1.1], σ_3 can be written as a subrepresentation of

$$\delta(\Sigma_{w+1}) \times \cdots \times \delta(\Sigma_u) \rtimes \sigma_4,$$

where σ_4 is supercuspidal and Σ_k's, σ_4 satisfy the conditions of [Jan, Theorem 1.1].

Then $\widehat{\sigma}_2$ is the unique generic constituent of

$$\widehat{St_{z+1}} \times \dots \widehat{St_w} \times \zeta(\Sigma_{w+1}) \times \cdots \times \zeta(\Sigma_u) \rtimes \widehat{\sigma}_4,$$

where $\zeta(\Sigma_k) = \widehat{\delta(\Sigma_k)}$, $w+1 \le k \le u$. Note that $\widehat{\sigma_4} = \sigma_4$. By a similar argument as above, we can see that St_j, $z+1 \le j \le w$ and $\delta(\Sigma_k)$, $w+1 \le k \le u$ are all supercuspidal. By the proof of [BZh, Lemma 4.2], we can see that $\{w+1, \dots, u\} = \varnothing$. Therefore, $\widehat{\sigma}$ is a subquotient of

$$\sigma_5 = \nu^{\beta_1} St_1 \times \cdots \times \nu^{\beta_z} St_z \times St_{z+1} \times \cdots \times St_w \rtimes \sigma_4,$$

where St_j, $1 \le k \le w$ are supercuspidal unitary, and σ_4 is also supercuspidal. Note that, by the classification of irreducible generic representations, $\sigma_4 = \sigma^{(0)}$.

By [C, Corollary 6.3.7],

$$\{\beta_1, \dots, \beta_z\} = \{\alpha_1, \dots, \alpha_t\}, \quad \{St_1, \dots, St_z\} = \{\tau_{\alpha_1}^{\epsilon_1}, \dots, \tau_{\alpha_t}^{\epsilon_t}\}, \quad \text{and}$$
$$\{St_j, z+1 \le j \le w\} = \{\tau_i^{\eta_i} | i \in I_0, \eta_i = \pm 1\}.$$

So, $\widehat{\sigma}$ is the Langlands quotient of

$$\nu^{\alpha_1} \tau_{\alpha_1}^{\epsilon_1} \times \cdots \times \nu^{\alpha_t} \tau_{\alpha_t}^{\epsilon_t} \rtimes \sigma_6,$$

where σ_6 is a subrepresentation of $\times_{i \in I_0} \tau_i^{\eta_i} \rtimes \sigma^{(0)}$. By a similar argument in the proof of [Ban2, Lemma 5.1], we can see that

$$\times_{i \in I_0} \tau_i^{\eta_i} \rtimes \sigma^{(0)} = \times_{j=z+1}^{w} St_j \rtimes \sigma^{(0)},$$

and $\sigma_6 = \widehat{\sigma'}$. This completes the proof. ■

Based on the above lemma and results in previous sections, it is easy to give the L-parameter of $\widehat{\sigma}$.

Theorem 7.2 *The L-parameter of $\widehat{\sigma}$ is*

$$\phi_{\widehat{\sigma}} = \Big(\bigoplus_{i \in I} \bigoplus_{j=-a_i}^{b_j} \big(|\cdot|^j \phi_i \oplus |\cdot|^{-j} \widetilde{\phi_i} \big) \Big) \oplus \big(\bigoplus_{i \in I_5} \phi_i \big),$$

where $\phi_i = r^{-1}(\tau_i)$, r is the local Langlands reciprocity map for GL as in [HT, H1].

The following result is Theorem 1.3.

Theorem 7.3 *For each A-parameter ψ and the corresponding L-parameter ϕ_ψ, the representations attached to ϕ_ψ in Section 5 are generic if and only if ϕ_ψ is tempered.*

Proof Assume $\psi = \bigoplus_{i=1}^{\nu} \phi_i' \otimes S_{m_i} \otimes S_{n_i}$, then by definition

$$\begin{aligned}
\phi_\psi &= \bigoplus_{i=1}^{\nu} \bigoplus_{j_i=-\frac{n_i-1}{2}}^{\frac{n_i-1}{2}} |\cdot|^{j_i} \phi_i' \otimes S_{m_i} \\
&= \bigoplus_{n_i \text{ even}} \bigoplus_{j_i=\frac{1}{2}}^{\frac{n_i-1}{2}} |\cdot|^{j_i} \phi_i' \otimes S_{m_i} \oplus |\cdot|^{-j_i} \widetilde{\phi_i'} \otimes S_{m_i} \\
&\quad \oplus \bigoplus_{n_i \text{ odd}} \Big(\bigoplus_{j_i=1}^{\frac{n_i-1}{2}} |\cdot|^{j_i} \phi_i' \otimes S_{m_i} \oplus |\cdot|^{-j_i} \widetilde{\phi_i'} \otimes S_{m_i} \Big) \oplus \phi_i' \otimes S_{m_i}.
\end{aligned}$$

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Assume that σ is the representation attached to ϕ_ψ in Section 5. Since ϕ_ψ is tempered, then σ is obviously generic, so it suffices to assume that σ is generic and to show $n_i = 1$, $1 \le i \le v$ and $I_4 = \varnothing$.

Note that

$$\phi_\sigma = \bigoplus_{i\in I_1} \phi_i \otimes S_{2b_i+1} \oplus \bigoplus_{i\in I_1'} \phi_i \otimes S_{2a_i+1} \oplus \bigoplus_{i\in I_5'} \phi_i \oplus \bigoplus_{i\in I_2\cup I_3} \phi_i \otimes S_{2b_i+1} \oplus \phi_i \otimes S_{2b_i+1}$$

$$\oplus \bigoplus_{i\in I_4} |\cdot|^{\frac{b_i-a_i}{2}} \phi_i \otimes S_{a_i+b_i+1} \oplus |\cdot|^{-\frac{b_i-a_i}{2}} \phi_i \otimes S_{a_i+b_i+1}.$$

If there exists i such that $n_i > 4$, first consider it to be even, then the following linked segments

$$\Sigma_1' = [\nu^{-\frac{m_i}{2}+1}\tau_i', \nu^{\frac{m_i}{2}}\tau_i'], \quad \Sigma_2' = [\nu^{-\frac{m_i}{2}+2}\tau_i', \nu^{\frac{m_i}{2}+1}\tau_i'],$$

where $\tau_i' = r(\phi_i')$, are in the segments corresponding to the index I_4, but this contradicts Definition 4.14(i). Similarly, there is also a contradiction when n_i is odd.

If $1 < n_i \le 3$ and $(m_i, n_i) \ne (1,3)$, then by the similar argument in the proof of [Ban2, Theorem 5.4], we can see that $\phi_i' \otimes S_{m_i} \otimes S_{n_i}$ is orthogonal. In particular, ϕ_i' is self-dual. When $n_i = 2$, the segment corresponding to $j_i = \frac{1}{2}$ is $\Sigma = [\nu^{-\frac{m_i}{2}+1}\tau_i', \nu^{\frac{m_i}{2}}\tau_i']$, where τ_i' is self-dual. By Definition 4.14, Σ has to satisfy condition (iii)(b) or (c). One can easily see that (iii)(b) is not true for Σ. Since, $\phi_i' \otimes S_{m_i} \otimes S_{n_i}$ is orthogonal, and $n_i = 2$, $\phi_i' \otimes S_{m_i}$ is symplectic. If m_i is even, then ϕ_i' is orthogonal; that is, $L(\tau_i', \mathrm{Sym}^2, s)$ has a pole at $s = 0$, but $0, 1 \in \{-\frac{m_i}{2}+1, \dots, \frac{m_i}{2}\}$, so $(3C1)$ and $(3C0)$ fail. If m_i is odd, then ϕ_i' is symplectic; that is, $L(\tau_i', \wedge^2, s)$ has a pole at $s = 0$, but $\frac{1}{2} \in \{-\frac{m_i}{2} + 1, \dots, \frac{m_i}{2}\}$, so $(3C\frac{1}{2})$ also fails, a contradiction! Similarly, there is also a contradiction when $n_i = 3$.

If $(m_i, n_i) = (1,3)$, then the segment corresponding to $j_i = 1$ is $\Sigma = [\nu\tau_i']$ and there is also a term ϕ_i' in ψ whose the corresponding representation is τ_i'. Since $\Sigma = [\nu\tau_i']$ and τ_i' are linked, this contradicts either Definition 4.14(i) or (ii). This completes the proof. ∎

Remark 7.4 Note that by the structure of irreducible generic representations of $GL_n(F)$ (see [BZ, Z]), Theorem 7.3 also holds for the GL_n-case.

Next, for any $\sigma \in \Pi^{(tg)}(Sp_{2n})$, we will consider the symmetry of A-parameters.

Theorem 7.5 *Given any $\sigma \in \Pi^{(tg)}(Sp_{2n})$ with L-parameter ϕ_σ and an A-parameter ψ such that $\phi_\sigma = \phi_\psi$, then there is also an A-parameter $\widehat{\psi}$ such that $\phi_{\widehat{\sigma}} = \phi_{\widehat{\psi}}$, and ψ and $\widehat{\psi}$ are symmetric.*

Proof First, it is easy to see that

$$\psi = \bigoplus_{i\in I_1} \phi_i \otimes S_{2b_i+1} \otimes S_1 \oplus \bigoplus_{i\in I_1'} \phi_i \otimes S_{2a_i+1} \otimes S_1 \oplus \bigoplus_{i\in I_5'} \phi_i \otimes S_1 \otimes S_1$$

$$\oplus \bigoplus_{i\in I_2\cup I_3} \phi_i \otimes S_{2b_i+1} \otimes S_1 \oplus \phi_i \otimes S_{2b_i+1} \otimes S_1.$$

Let

$$\widehat{\psi} = \bigoplus_{i\in I_1} \phi_i \otimes S_1 \otimes S_{2b_i+1} \oplus \bigoplus_{i\in I_1'} \phi_i \otimes S_1 \otimes S_{2a_i+1} \oplus \bigoplus_{i\in I_5'} \phi_i \otimes S_1 \otimes S_1$$
$$\oplus \bigoplus_{i\in I_2\cup I_3} \phi_i \otimes S_1 \otimes S_{2b_i+1} \oplus \phi_i \otimes S_1 \otimes S_{2b_i+1}.$$

Then ψ and $\widehat{\psi}$ are symmetric. It suffices to show that $\phi_{\widehat{\sigma}} = \phi_{\widehat{\psi}}$, which can be easily seen from the definition and Theorem 7.2. ■

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