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ON A PROBLEM OF PRAEGER AND SCHNEIDER

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Abstract

This note provides an affirmative answer to Problem 2.6 of Praeger and Schneider ['Group factorisations, uniform automorphisms, and permutation groups of simple diagonal type', *Israel J. Math.* **228**(2) (2018), 1001–1023]. We will build groups G (abelian, nonabelian and simple) for which there are two automorphisms α, β of G such that the map

$$T = T_\alpha \times T_\beta : G \longrightarrow G \times G, \quad g \mapsto (g^{-1}g^\alpha, g^{-1}g^\beta)$$

is surjective.

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1. Introduction

The purpose of this paper is to provide an affirmative answer to [3, Problem 2.6], by proving the following theorems.

THEOREM 1.1. *Let G be a group and let $\alpha, \beta \in \mathsf{Aut}(G)$. Then it is possible to embed G in a group $\widetilde{G}$ with $\widetilde{\alpha}, \widetilde{\beta} \in \mathsf{Aut}(\widetilde{G})$ such that the map*

$$T = T_{\widetilde{\alpha}} \times T_{\widetilde{\beta}} : \widetilde{G} \longrightarrow \widetilde{G} \times \widetilde{G}, \quad \widetilde{g} \mapsto (\widetilde{g}^{-1}\widetilde{g}^{\widetilde{\alpha}}, \widetilde{g}^{-1}\widetilde{g}^{\widetilde{\beta}})$$

is surjective, $\widetilde{\alpha}|_G = \alpha$ and $\widetilde{\beta}|_G = \beta$. Moreover, $|\widetilde{G}| = \max\{\aleph_0, |G|\}$.

THEOREM 1.2. *Under the hypotheses of Theorem 1.1, if G is countable, then $\widetilde{G}$ can be made simple.*

For completeness we will also prove the following result.

THEOREM 1.3. *Let G be an abelian group and let $\alpha, \beta \in \mathsf{Aut}(G)$. Then it is possible to embed G in an abelian group $\widetilde{G}$ with $\widetilde{\alpha}, \widetilde{\beta} \in \mathsf{Aut}(\widetilde{G})$ such that the map $T_{\widetilde{\alpha}} \times T_{\widetilde{\beta}}$ is surjective, $\widetilde{\alpha}|_G = \alpha$ and $\widetilde{\beta}|_G = \beta$.*

We will now try to motivate our results in the light of the paper [3].

The O'Nan–Scott theorem is an invaluable tool in the theory of finite permutation groups. Its proof is closely linked to the classification of finite simple groups (CFSG) and therefore any extension to the case of infinite groups is generally extremely difficult.

In a recent paper [3] devoted to quasiprimitive permutation groups of simple diagonal type, Praeger and Schneider applied new ideas which allowed them to extend some results to infinite groups and at the same time to make the proofs independent of CFSG. A fundamental tool in their arguments is the concept of uniform automorphism. An automorphism α of a group G is called uniform if the associated map

$$T_\alpha : G \to G, \quad g \mapsto g^{-1}g^\alpha$$

is surjective.

Let $G = G_1 \times G_2 \times \cdots \times G_k$ be a direct product of k groups and, for $i \in \{1, 2, \ldots, k\}$, let $\pi_i : G \to G_i$ be the coordinate projection. A subgroup H of G is a *strip* if, for all i, the restriction $\pi_i|_H$ is injective or $H^{\pi_i} = 1$. If H is a strip, we define $J = \{j \mid H^{\pi_j} \neq 1\}$. The strip H is called *nontrivial* if $|J| \geq 2$ and *full* if $H^{\pi_j} = G_j$ whenever $j \in J$.

Let G be a group and let X, Y be direct products of pairwise disjoint nontrivial full strips in G^k with $k \geq 2$. If $G^k = XY$, then G^k is said to be factorised by (direct products of) strips. Lemma 2.4 of [3] asserts that if $d \geq 1$, then

$$G^{2d} \text{ has a factorisation by strips} \iff G \text{ has a uniform automorphism.}$$

Example 1.4 [3, Lemmas 2.3 and 2.4]. Let G be a group and α a uniform automorphism of G and define $X = \{(g, g) \mid g \in G\}$ and $Y = \{(h, h^\alpha) \mid h \in G\}$. Let $(x, y) \in G \times G$. Choose h such that $h^{-1}h^\alpha = x^{-1}y$ and let $g = xh^{-1}$. Then $gh^\alpha = xh^{-1}h^\alpha = y$ and $gh = x$. So $(g, g)(h, h^\alpha) = (x, y)$ and therefore $G \times G = XY$.

The strips involved in Example 1.4 (and in [3, Lemma 2.4]) have length two. In order to factorise G^6, say, as $G^6 = XY$ where X is a direct product of two strips of length three and Y is a direct product of three strips of length two, it is necessary to assume that G admits uniform automorphisms with very particular properties.

Example 1.5 [3, Example 2.5]. Consider a group G and suppose that there are two (uniform) automorphisms α and β of G such that the map

$$T_\alpha \times T_\beta : G \to G \times G, \quad g \mapsto (g^{-1}g^\alpha, g^{-1}g^\beta)$$

is surjective. Set

$$X = \{(t, t, t, s, s, s) \mid t, s \in G\}, \quad Y = \{(u, v, w, u, v^\alpha, w^\beta) \mid u, v, w \in G\}.$$

Let $(x_1, x_2, x_3, x_4, x_5, x_6) \in G^6$ and choose $t \in G$ such that

$$tt^{-\alpha} = x_1x_4^{-1}x_5(x_2^{-1})^\alpha \quad \text{and} \quad tt^{-\beta} = x_1x_4^{-1}x_6(x_3^{-1})^\alpha.$$

Let $s = x_4x_1^{-1}t$, $u = t^{-1}x_1$, $v = t^{-1}x_2$ and $w = t^{-1}x_3$. Then it follows by the assumptions above that $(t, t, t, u, u, u) \cdot (u, v, w, u, v^\alpha, w^\beta) = (x_1, x_2, x_3, x_4, x_5, x_6)$. Therefore $G^6 = XY$.

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We remark that Example 1.5 is only hypothetical, as it depends on the following problem posed in [3].

Problem 1.6 [3, Problem 2.6]. Exhibit a group G that admits a pair (α, β) of automorphisms such that the map

$$T = T_\alpha \times T_\beta : G \longrightarrow G \times G, \quad g \mapsto (g^{-1}g^\alpha, g^{-1}g^\beta) \tag{$\circledast$}$$

is surjective (or prove that no such group exists).

If G is a nontrivial finite group, then $|G \times G| > |G|$, and hence such automorphisms cannot exist for finite G.

In this paper we will give an affirmative answer to Problem 1.6 and therefore we show that Example 1.5 is effective. We will build groups G of every infinite cardinality endowed with two automorphisms α and β that satisfy property ($\circledast$). Moreover, with a refinement of our construction (Theorem 1.2), we will show that G can be simple.

2. Uniform automorphisms

In a paper of the late 1950s, Zappa [8] introduced the definition of *uniform* automorphism with the aim of extending to infinite groups some results hitherto known only for finite groups. An automorphism α of a group G is fixed point free if $C_G(\alpha) = 1$, that is, if the map $T_\alpha : G \to G, g \mapsto g^{-1}g^\alpha$ is injective. If the map T_α is onto, then α is called uniform. For finite groups these two properties are evidently equivalent; but not for infinite groups, as some examples will show.

If $\iota : \mathbb{Q} \to \mathbb{Q}, x \mapsto -x$, then $\mathbb{Z}$ is an ι-invariant subgroup of $\mathbb{Q}$. It is easy to verify the following assertions.

Example 2.1. If $\alpha \in \mathsf{Aut}(G)$ is fixed point free and $H \le G$ is α-invariant, then α induces a fixed point free automorphism on H. So ι is fixed point free on $\mathbb{Z}$ but it is not fixed point free on $\mathbb{Q}/\mathbb{Z}$.

Example 2.2. If $\alpha \in \mathsf{Aut}(G)$ is uniform and $N \trianglelefteq G$ is α-invariant, then α induces a uniform automorphism on G/N. So ι is uniform on $\mathbb{Q}/\mathbb{Z}$ but it is not uniform on $\mathbb{Z}$.

In general, for an infinite group, to possess a uniform automorphism is a stronger condition than to possess a fixed point free automorphism. As an example, consider $G = \langle x, y \rangle$, the free group with two generators x and y. The automorphism α defined by $x^\alpha = y$ and $y^\alpha = x$ is fixed point free but not uniform. This happens because the normal subgroup $N = \langle x^2, xy^{-1}, y^2 \rangle$ is α-invariant but α induces the identity on $G/N \simeq C_2$. It can be proved that a free group of finite rank cannot admit uniform automorphisms.

It is known that if a finite group G admits a fixed point free automorphism, then G is solvable. A brief proof of this fact is provided in [6]. (The proof relies on the fact that each simple group has a cyclic Sylow group and therefore CFSG is required.)

It has been conjectured that a finitely generated group with a uniform automorphism cannot be simple. Zappa in [8] proved that if a polycyclic group G has a uniform automorphism of prime order, then G is a finite p'-group and this result has been

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extended in [1] (see also [4]) in which it is proved that any finitely generated hyperabelian group having a uniform automorphism of prime-power order must be finite. It is interesting to observe that the automorphism α of $G = \mathbb{Z} \times \mathbb{Z}$ (in additive notation) defined by $(x, y)^\alpha = (y, -x + y)$ is uniform and has order six.

To the best of our knowledge there is only one class of examples of simple groups that admits a uniform automorphism (Example 2.3 below) and none of these groups are finitely generated.

Example 2.3. Let G be a connected algebraic group over the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ and let α be the automorphism naturally induced on G by the Frobenius map $\overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p, k \mapsto k^p$. Then, by a deep result of Lang (see [7, 4.4.17]), α is uniform and we can easily verify that α has infinite order.

Let $G = \mathrm{SL}_n(\overline{\mathbb{F}}_p)$ with $n \geq 2$. Since $Z(G)$ is characteristic in G it follows that $G/Z(G) = \mathrm{PSL}_n(\overline{\mathbb{F}}_p)$ is a simple group admitting a uniform automorphism.

We remark that in (⋇) the automorphisms α and β of G must necessarily be uniform and therefore we do not believe that it is possible to find an example in which G is finitely generated and simple (see Remark 3.5).

3. The proof of Theorems 1.1, 1.2 and 1.3

We begin this section with the following simple lemma.

Lemma 3.1. *Let $(G_\rho)_{\rho<\tau}$ be a set of groups for some limit ordinal τ and suppose that $\alpha_\rho, \beta_\rho \in \mathsf{Aut}(G_\rho)$. Suppose that the following three conditions are satisfied:*

(a) *if $\rho < \sigma < \tau$, then $G_\rho \leq G_\sigma$;*
(b) *if $\rho < \sigma < \tau$, then $\alpha_\sigma|_{G_\rho} = \alpha_\rho$ and $\beta_\sigma|_{G_\rho} = \beta_\rho$;*
(c) *if $\rho < \sigma < \tau$, then $G_\rho \times G_\rho \subseteq T_{\alpha_\sigma}(G_\sigma) \times T_{\beta_\sigma}(G_\sigma)$.*

Let $G = \bigcup_{\rho<\tau} G_\rho$ be the direct limit of $(G_\rho)_{\rho<\tau}$ and let $\alpha, \beta \in \mathsf{Aut}(G)$ be the automorphisms naturally induced by $(\alpha_\rho)_{\rho<\tau}$ and $(\beta_\rho)_{\rho<\tau}$ on G. Then the map $T_\alpha \times T_\beta : G \to G \times G$ is surjective.

Proof. For $g, h \in G$, there is $\rho < \tau$ such that $g, h \in G_\rho$. From the hypothesis there is $x \in G_{\rho+1}$ such that $(g, h) = (x^{-1}x^\alpha, x^{-1}x^\beta)$. □

In our proofs we will use free products and their universal property. We recall the definition.

Definition 3.2. Let $\{X_i \mid i \in I\}$ be a set of groups. A free product of the X_i is a group P and a family of homomorphisms $j_i : X_i \to P$ such that, for every group Y and every family of homomorphisms $f_i : X_i \to Y$, there is a unique homomorphism $\phi : P \to Y$ with $\phi \circ j_i = f_i$ for all $i \in I$, that is, the diagram

$$\begin{tikzcd} & P = \ast_{i \in I} X_i \arrow[d, dashed, "\phi"] \\ X_i \arrow[ur, hook, "j_i"] \arrow[r, "f_i"'] & Y \end{tikzcd}$$

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is commutative (see [5, Theorem 11.51]). Since a free product exists and is unique up to isomorphism, we can speak of the free product of $\{X_i \mid i \in I\}$ and we write $P = *_{i\in I} X_i$.

Lemma 3.3. *Let G be a group and let $\alpha, \beta \in \mathsf{Aut}(G)$. Then we can construct an overgroup $\widehat{G}$ of G and $\widehat{\alpha}, \widehat{\beta} \in \mathsf{Aut}(\widehat{G})$ such that $\widehat{\alpha}|_G = \alpha$, $\widehat{\beta}|_G = \beta$ and*

$$G \times G \subseteq T_{\widehat{\alpha}}(\widehat{G}) \times T_{\widehat{\beta}}(\widehat{G}).$$

Moreover, $|\widehat{G}| = \max\{\aleph_0, |G|\}$.

Proof. If G is infinite, let $G_0 = G$, $\alpha_0 = \alpha$ and $\beta_0 = \beta$. If G is finite define $G_0 = G * \langle x\rangle$ (or $G_0 = G \times \langle x\rangle$) where $\langle x\rangle$ is an infinite cyclic group, $\alpha_0|_G = \alpha$, $\beta_0|_G = \beta$ and $x^{\alpha_0} = x = x^{\beta_0}$.

We can well-order the elements of $G_0 \times G_0$,

$$G_0 \times G_0 = \{(g_\rho, h_\rho) \mid 0 \leq \rho < \tau\},$$

and since G_0 is infinite we can do this with τ a limit ordinal. Moreover, we can impose the condition $(g_0, h_0) = (1, 1)$.

For every $\rho < \tau$ we define a chain of groups $(G_\rho)_{i\in\rho}$ with $\alpha_\rho, \beta_\rho \in \mathsf{Aut}(G_\rho)$ such that if $\rho < \sigma < \tau$, then

(i) $G_\rho \leq G_\sigma$;
(ii) $\alpha_\sigma|_{G_\rho} = \alpha_\rho$ and $\beta_\sigma|_{G_\rho} = \beta_\rho$;
(iii) there is an element $y_\rho \in G_\sigma$ such that

$$(y_\rho^{-1} y_\rho^{\alpha_\sigma}, y_\rho^{-1} y^{\beta_\sigma}) = (g_\rho, h_\rho).$$

If $\rho = 0$, then G_0, α_0 and β_0 have already been defined, and since $(g_0, h_0) = (1, 1)$ we can set $y_0 = 1$. Suppose that G_ρ, α_ρ and β_ρ have been constructed for all $\rho < \sigma$. If σ is a limit ordinal, simply define $G_\sigma = \bigcup_{\rho<\sigma} G_\rho$ and let $\alpha_\sigma, \beta_\sigma \in \mathsf{Aut}(G_\sigma)$ be the automorphisms naturally induced by $(\alpha_\rho)_{\rho<\sigma}$ and $(\beta_\rho)_{\rho<\sigma}$ in G_σ. If σ is not a limit ordinal we can choose ρ such that $\sigma = \rho + 1$. Let $\langle x_{\rho+1}\rangle$ be an infinite cyclic group such that $G_\rho \cap \langle x_{\rho+1}\rangle = 1$ and define $G_{\rho+1} = G_\rho * \langle x_{\rho+1}\rangle$. It is clear that the maps

$$\alpha_\rho^{\bigstar} : G_\rho \to G_{\rho+1}, \quad g \mapsto g^{\alpha_\rho}; \quad \gamma : \langle x_{\rho+1}\rangle \to G_{\rho+1}, \quad x_{\rho+1} \mapsto x_{\rho+1} g_{\rho+1}$$

and

$$\beta_\rho^{\bigstar} : G_\rho \to G_{\rho+1}, \quad g \mapsto g^{\beta_\rho}; \quad \delta : \langle x_{\rho+1}\rangle \to G_{\rho+1}, \quad x_{\rho+1} \mapsto x_{\rho+1} h_{\rho+1}$$

define homomorphisms. Let $j_1 : G_\rho \to G_{\rho+1}$ and $j_2 : \langle x_{\rho+1}\rangle \to G_{\rho+1}$ be the natural embeddings. From the universal property of free products we can draw the diagrams shown in Figure 1. We can easily verify that conditions (i) and (ii) are fulfilled and, moreover,

$$x_{\rho+1}^{-1} x_{\rho+1}^{\alpha_{\rho+1}} = x_{\rho+1}^{-1} x_{\rho+1} g_\rho = g_\rho \quad \text{and} \quad x_{\rho+1}^{\beta_{\rho+1}} = x_{\rho+1}^{-1} x_{\rho+1} h_\rho = h_\rho,$$

so condition (iii) is also satisfied with $y_\rho = x_{\rho+1}$.

$$\begin{tikzcd}
 & G_\rho * \langle x_{\rho+1} \rangle \arrow[dd, dashed, "\alpha_{\rho+1}"] \\
\langle x_{\rho+1} \rangle \arrow[ur, hook, "j_2"] \arrow[dr, "\gamma"'] & \\
G_\rho \arrow[uur, bend left=60, "j_1"] \arrow[r, "\alpha^\star_\rho"'] & G_\rho * \langle x_{\rho+1} \rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
 & G_\rho * \langle x_{\rho+1} \rangle \arrow[dd, dashed, "\beta_{\rho+1}"] \\
\langle x_{\rho+1} \rangle \arrow[ur, hook, "j_2"] \arrow[dr, "\delta"'] & \\
G_\rho \arrow[uur, bend left=60, "j_1"] \arrow[r, "\beta^\star_\rho"'] & G_\rho * \langle x_{\rho+1} \rangle
\end{tikzcd}$$

FIGURE 1. Diagrams for the proof of Lemma 3.3.

Define

$$\widehat{G} = \bigcup_{\rho<\tau} G_\rho.$$

Let $g \in \widehat{G}$ and let $\rho < \tau$ be such that $g \in G_\rho$. Since condition (ii) is fulfilled we can define $\widehat{\alpha}, \widehat{\beta} \in \mathrm{Aut}(\widehat{G})$ by putting $g^{\widehat{\alpha}} = g^{\alpha_\rho}$ and $g^{\widehat{\beta}} = g^{\beta_\rho}$. Then Lemma 3.1 gives us the conclusion. Moreover, $|G_\sigma| = \max\{\aleph_0, |G_\rho|\}$ for every $\rho < \sigma < \tau$ and hence $|\widehat{G}| = \max\{\aleph_0, |G|\}$. □

Proof of Theorem 1.1. Let G be a group and let $\alpha, \beta \in \mathrm{Aut}(G)$. We define a chain of groups $(G_i)_{i\in\mathbb{N}}$ with $\alpha_n, \beta_n \in \mathrm{Aut}(G_n)$ satisfying the conditions of Lemma 3.1 (with $\tau = \omega$).

If $n = 0$, then we define $G_0 = G$, $\alpha_0 = \alpha$ and $\beta_0 = \beta$. If $n > 0$ then, with the notation of Lemma 3.3, we define $G_n = \widehat{G_{n-1}}$, $\alpha_n = \widehat{\alpha_{n-1}}$ and $\beta_n = \widehat{\beta_{n-1}}$.

Let $\widetilde{G} = \bigcup_{n\in\mathbb{N}} G_n$ and let $\widetilde{\alpha}, \widetilde{\beta} \in \mathrm{Aut}(\widetilde{G})$ be the automorphisms naturally induced by $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ on $\widetilde{G}$. From Lemma 3.1 we can conclude that the map $T_{\widetilde{\alpha}} \times T_{\widetilde{\beta}} : \widetilde{G} \to \widetilde{G} \times \widetilde{G}$ is surjective and $|\widetilde{G}| = \max\{\aleph_0, |G|\}$. □

Proof of Theorem 1.3. To prove Theorem 1.3 we can use the same technique adopted for Theorem 1.1. The only difference is that a direct sum is used instead of a free product. With the same hypotheses as in Definition 3.2, we know that if X_i $(i \in I)$ and Y are abelian groups, then the diagram

$$\begin{tikzcd}
 & P = \bigoplus_{i\in I} X_i \arrow[d, dashed, "\phi"] \\
X_i \arrow[ur, hook, "j_i"] \arrow[r, "f_i"'] & Y
\end{tikzcd}$$

is commutative [5, Theorem 10.9]. □

In order to prove Theorem 1.2, the previous construction must be slightly modified. We recall the following result proved using the technique of HNN extensions.

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THEOREM 3.4 [2, Theorem IV.3.4]. *Every countable group G can be embedded in a countable, simple (and divisible) group $G^\bullet$.*

REMARK 3.5. A remarkable theorem of P. Hall asserts that every countable group can be embedded in a finitely generated simple group [2, Theorem IV.3.5]. Unfortunately, we cannot use this result to produce examples of finitely generated simple groups that satisfy condition (⋇). This is because all our proofs employ direct limits of groups.

PROOF OF THEOREM 1.2. Let G be a countable group and $\alpha, \beta \in \mathsf{Aut}(G)$. Let $G_0 = G$, $\alpha_0 = \alpha$ and $\beta_0 = \beta$. For every $n > 0$, let

$$K_n = (G_{n-1} \rtimes \langle \alpha_{n-1}, \beta_{n-1} \rangle)^\bullet$$

be the group defined in Theorem 3.4 and let $\alpha_n^\bullet, \beta_n^\bullet \in \mathsf{Aut}(K_n)$ be the inner automorphisms induced by α_{n-1} and β_{n-1} on K_n. Applying Theorem 1.1, we can define $G_n = \widetilde{K_n}$ and $\alpha_n = \widetilde{\alpha^\bullet}_n, \beta_n = \widetilde{\beta^\bullet}_n$ such that $T_{\alpha_n} \times T_{\beta_n} : G_n \to G_n \times G_n$ is surjective. We remark that $|G_n| = \aleph_0$.

Let $\widehat{G} = \bigcup_{n \in \mathbb{N}} G_n$ and let $\widehat{\alpha}, \widehat{\beta}$ be the automorphisms naturally induced by α_n and β_n in $\widehat{G}$. From Lemma 3.1, the map $T_{\widehat{\alpha}} \times T_{\widehat{\beta}} : \widehat{G} \to \widehat{G} \times \widehat{G}$ is surjective. Since a countable union of countable sets is again countable, it follows that $|\widehat{G}| = \aleph_0$.

If $N \neq 1$ is a normal subgroup of $\widehat{G}$, let $m \in \mathbb{N}$ be such that $N \cap G_m \neq 1$. Then for every $n \geq m$, we have $1 \neq N \cap K_{n+1} \trianglelefteq K_{n+1}$ and, since K_{n+1} is simple, $K_{n+1} \leq N$. Hence $G_n \leq K_{n+1} \leq N$ for every $n \in \mathbb{N}$ and so $N = \widehat{G}$. This proves that $\widehat{G}$ is simple. □

REMARK 3.6. Our constructions can be easily generalised to exhibit groups G with r automorphisms $\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathsf{Aut}(G)$ such that the map

$$T_{\alpha_1} \times T_{\alpha_2} \times \cdots \times T_{\alpha_r} : G \longrightarrow G^r$$

is surjective.

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