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AN APPROXIMATION METHOD FOR MONOTONE LIPSCHITZIAN OPERATORS IN HILBERT SPACES

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Abstract

Suppose H is a complex Hilbert space and K is a nonempty closed convex subset of H. Suppose T: $K \to H$ is a monotonic Lipschitzian mapping with constant $L \geqslant 1$ such that, for x in K and h in H, the equation $x + Tx = h$ has a solution q in K. Given x_0 in K, let $\{C_n\}_{n=0}^{\infty}$ be a real sequence satisfying: (i) $C_0 = 1$, (ii) $0 \leqslant C_n < L^{-2}$ for all $n \geqslant 1$, (iii) $\Sigma_n C_n(1 - C_n)$ diverges. Then the sequence $\{p_n\}_{n=0}^{\infty}$ in H defined by $p_n = (1 - C_n)x_n + C_n Sx_n$, $n \geqslant 0$, where $\{x_n\}_{n=0}^{\infty}$ in K is such that, for each $n \geqslant 1$, $\|x_n - P_{n-1}\| = \inf_{x \in K} \|p_{n-1} - x\|$, converges strongly to a solution q of $x + Tx = h$. Explicit error estimates are given. A similar result is also proved for the case when the operator T is *locally* Lipschitzian and monotone.

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Let X be an arbitrary Banach space. An operator T with domain $D(T)$ and range $R(T)$ in X is said to be *monotone* [7] if

(1) $\|x - y\| \leqslant \|x - y + t(Tx - Ty)\|$ for every $x, y \in D(A)$ and $t > 0$.

If $X = H$, a complex Hilbert space, condition (1) reduces to $\mathrm{Re}\langle x - y, Tx - Ty\rangle \geqslant 0$ for all x, y in H. Operators satisfying (1) are sometimes referred to as *accretive* (see e.g. [2]). The accretive operators were introduced by T. Kato [7] and F. E. Browder [2] in 1967. In 1968, Browder proved that if T: $X \to X$ is locally Lipschitzian and accretive, then $(I + T)(X) = X$; this result was subsequently generalized by R. H. Martin [12] in 1970 to the continuous accretive operators. In 1974 K. Deimling [5] generalized Martin's result by showing that if V is an open

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subset of X and T a continuous mapping of V into X, and if T is locally closed, locally one-to-one and locally accretive, then $T(V)$ is open. For some interesting applications of this result the reader may consult [9] or [10].

An early fundamental result in the theory of monotone operators on Hilbert space due to Zarantonello [14] states that the operator equation $x + Tx = h$ has a unique solution x in H for each h in H, provided that T is monotonic and Lipschitzian. Recently Dotson [6] has shown that if $T\colon H \to H$ is monotonic and has Lipschitz constant 1 (in this case the operator T is called *nonexpansive* in the terminology of [8]), then an iterative process of the type introducd by W. R. Mann [11], under certain conditions, converges strongly to the unique solution of the equation.

Our object in this paper is to construct an iterative process which converges strongly to a solution of the operator equation $x + Tx = f$ for f in H and x in K where $T\colon K \to H$ is a monotonic Lipschitzian operator with Lipschitz constant $L \geqslant 1$, and where K is a nonempty closed convex subset of H. Thus, our result generalizes Dotson's theorem both in the domain of definition of the operator and in the range of its Lipschitz constant. Furthermore, we prove a convergence result for the equation $x + Tx = f$ when T is *locally* Lipschitzian and monotone.

THEOREM 1. *Suppose H is a complex Hilbert space and K a nonempty closed convex subset of H. Suppose $T\colon K \to H$ is a monotonic Lipschitzian mapping with constant $L \geqslant 1$ such that, for x in K, and h in H, the equation $x + Tx = h$ has a solution q in K. Define $S\colon K \to H$ by $Sx = -Tx + h$ for all x in K. Given x_0 in K, let $\{C_n\}_{n=0}^{\infty}$ be a real sequence satisfying*: (i) $C_0 = 1$, (ii) $0 \leqslant C_n < L^{-2}$ *for all* $n \geqslant 1$, (iii) $\Sigma_n C_n(1 - C_n)$ *diverges. Then the sequence $\{p_n\}_{n=0}^{\infty}$ in H defined by $p_n = (1 - C_n)x_n + C_n Sx_n$, $n \geqslant 0$, where $\{x_n\}_{n=0}^{\infty}$ in K is such that, for each $n \geqslant 1$, $\|x_n - p_{n-1}\| = \inf_{x \in K} \|p_{n-1} - x\|$, converges strongly to a solution q of $x + Tx = h$.*

PROOF. We observe that q is a fixed point of S and that $\|Sx - Sy\| \leqslant L\|x - y\|$ for all x, y in K. Moreover, monotonicity of T implies that $\operatorname{Re}\langle Sx - Sy, x - y\rangle \leqslant 0$ for all x, y in K. Let $R\colon H \to K$ be the map which assigns to each point x of H the unique point of K which is nearest to x. Then R is nonexpansive [4]. Starting with $x_0 \in K$ we obtain Sx_0 in H and so compute p_0 from $p_0 = (1 - C_0)x_0 + C_0 Sx_0$ in H. Then $x_1 = R(p_0)$ lies in K, so that $p_1 = (1 - C_1)x_1 + C_1 Sx_1$. By continuing this process we generate the sequence $\{p_n\}_{n=0}^{\infty}$ in H. Observe that

$$(2) \qquad \|x_n - q\| = \|R(p_{n-1}) - R(q)\| \leqslant \|p_{n-1} - q\| \quad \text{for each } n \geqslant 1.$$

Moreover,

$$\begin{aligned}\|p_n - q\|^2 &= \|(1-C_n)(x_n - q) + C_n(Sx_n - Sq)\|^2 \\ &= (1-C_n)^2\|x_n - q\|^2 + C_n^2\|Sx_n - Sq\|^2 \\ &\quad + 2C_n(1-C_n)\mathrm{Re}\langle Sx_n - Sq, x_n - q\rangle \\ &\leqslant \left\{(1-C_n)^2 + L^2C_n^2\right\}\|x_n - q\|^2,\end{aligned}$$

since $\mathrm{Re}\langle Sx_n - Sq, x_n - q\rangle \leqslant 0$, $C_n \in [0,1)$ and $\|Sx_n - Sq\| \leqslant L\|x_n - q\|$. Thus, using (2) we obtain,

$$\begin{aligned}(3)\qquad \|p_n - q\|^2 &\leqslant \left\{(1-C_n)^2 + L^2C_n^2\right\}\|p_{n-1} - q\|^2 \\ &= \left(1 - \left[C_n(1-C_n) + C_n(1 - L^2C_n)\right]\right)\|p_{n-1} - q\|^2 \\ &\leqslant \prod_{k=1}^{n}\left[1 - \left\{C_k(1-C_k) + C_k(1 - L^2C_k)\right\}\right]\|p_0 - q\|^2,\end{aligned}$$

and for all k, $C_k(1-C_k) + C_k(1-L^2C_k) \leqslant \frac{1}{4} + \frac{1}{4L^2} < 1$ (since $L \geqslant 1$). Moreover, the divergence of $\Sigma_k C_k(1 - C_k)$ implies that

$$\prod_{k=1}^{n}\left[1 - \left\{C_k(1-C_k) + C_k(1-L^2C_k)\right\}\right] \to 0 \text{ as } n \to \infty.$$

Hence $\{p_n\}_{n=0}^{\infty}$ converges strongly to q, completing the proof of the theorem.

REMARKS.

(i) Our theorem generalizes the theorem of [6] to mappings with Lipschitz constant $L \geqslant 1$ and to mappings which may only be defined on nonempty closed convex subsets K of H and which take values in H.

(ii) With the notation of the theorem, if $K = H$, the iteration scheme of the theorem can be simplified to $x_{n+1} = (1 - C_n)x_n + C_nSx_n$, $x_0 \in H$, $n \geqslant 0$. In this case, a theorem of Zarantonello [14] guarantees the existence of a unique fixed point, say q, of S in H. Then, it follows that

$$\|x_{n+1} - q\|^2 \leqslant \left[1 - \left\{C_n(1-C_n) + C_n(1 - C_nL^2)\right\}\right]\|x_n - q\|^2,$$

and, as in the proof of the above theorem, $\{x_n\}_{n=1}^{\infty}$ converges strongly to q.

There are some particular choices of C_n and an alternate method which give the additional information of an error estimate. Choose $C_n = 1/(n + L^2)$, $n \geqslant 1$. Then, clearly, $C_n < L^{-2}$ for all $n \geqslant 1$. It is easy to see that $\Sigma C_n(1 - C_n)$ diverges. Let q denote a solution of $x + Tx = h$. Then, as in the proof of the theorem, using the same notation, from (3) we obtain,

$$(4)\qquad \|p_n - q\|^2 \leqslant \left[\frac{(n + L^2 - 1)^2}{(n+L^2)^2} + \frac{L^2}{(n+L^2)^2}\right]\|p_{n-1} - q\|^2.$$

Observe that inequality (3) also yields $\|p_n - q\| \leqslant \|p_{n-1} - q\|$ for all $n \geqslant 1$ (since for all k, $C_k(1 - C_k) + C_k(1 - L^2 C_k) < 1$), so that (4) yields

$$(5) \qquad (n + L^2)^2 \|p_n - q\|^2 - (n + L^2 - 1)^2 \|p_{n-1} - q\|^2 \leqslant L^2 \|p_0 - q\|^2.$$

Summing inequality (5) for $n = 1$ to N and observing that the left hand side telescopes, we obtain

$$(N + L^2)^2 \|p_N - q\|^2 - L^2 \|p_0 - q\|^2 \leqslant NL^2 \|p_0 - q\|^2,$$

so that for each $N = 1, 2, 3, \ldots,$ we have

$$\|p_N - q\|^2 \leqslant \frac{L^2}{(N + L^2)} \|p_0 - q\|^2.$$

Thus, $\{p_n\}_{n=1}^{\infty}$ converges to q, and for each n we have

$$\|p_n - q\| \leqslant \left(\frac{L^2}{n + L^2}\right)^{1/2} \|p_0 - q\|.$$

DEFINITION. Let $D(T)$ denote the domain of a map T. Then T: $D(T) \to H$ is called *locally Lipschitzian* with constant $L \geqslant 1$ if, for each q in $D(T)$, there is an $\varepsilon > 0$ such that

$$(6) \qquad \|Tx - Ty\| \leqslant L\|x - y\| \text{ whenever } \|x - q\| \leqslant \varepsilon \text{ and } \|y - q\| \leqslant \varepsilon.$$

THEOREM 2. *Supose T: $D(T) \to H$ is a locally Lipschitzian (with Lipschitz constant $L \geqslant 1$) monotone operator with $D(T) \subseteq H$ open, and let $f \in H$. Suppose the equation $x + Tx = f$ has a solution q in $D(T)$, and define S by $Sx = -Tx + f$. Let $\{C_n\}_{n=0}^{\infty}$ be a real sequence satisfying* (i) $C_0 = 1$, (ii) $0 \leqslant C_n < L^{-2}$ *for all* $n \geqslant 1$, *and* (iii) $\Sigma_n C_n(1 - C_n)$ *diverges. For $q \in B \subseteq H$, where B is closed and convex, define the sequences $\{p_n\}_{n=1}^{\infty}$ in H and $\{X_n\}_{n=0}^{\infty}$ in B by* (a) $X_0 \in B$ *arbitrary*, (b) $p_{n+1} = (1 - C_n)X_n + C_n SX_n$, *and* (c) X_n *is the point in B such that* $\|X_n - p_{n-1}\| = \inf_{x \in B} \|p_{n-1} - x\|$. *Then, for any initial guess X_0 in B, the sequence $\{p_n\}_{n=1}^{\infty}$ converges strongly to a solution q in B of $x + Tx = f$.*

PROOF. Let q be a solution of $x + Tx = f$. Since T is locally Lipschitzian, given any $\varepsilon > 0$, choose $\hat{\varepsilon} \in (0, \varepsilon)$ so that (6) is satisfied. Let $B = \{X \in H$: $\|q - X\| \leqslant \hat{\varepsilon}\}$. Then B is closed and convex. Since $\{X_n\}_{n=0}^{\infty}$ is contained in B, we have

$$\|SX_n - Sq\| = \|TX_n - Tq\| \leqslant L\|X_n - q\| \quad \text{for all } n.$$

The rest of the argument is now exactly as in the proof of Theorem 1 and is, therefore, omitted.

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