CORRESPONDENCE.

AN ASSURANCE FALLACY.

To the Editor of the Assurance Magazine.

SIR,—The following problem presents several points of interest.

An assurance of A pounds is to be effected on (x), at an annual premium (ϖ), subject to the condition that interest on the premiums paid up to and including the year of death is to be allowed by the Office, at the rate involved in the tables employed, which rate it is assumed is that realized by the Office. Required ϖ.

Attempt a solution thus:—Since all the interest realized is to be handed over to (x) or his representatives, the Office has obviously nothing but the bare premiums out of which to pay the sum assured. It is, therefore, as regards the Office, the same thing as if no interest were made; and we consequently need take account only of the average number of premiums that will be received from each policyholder. This number being $1+e'_x$ (where e'_x is the *curtate* mean duration of lives aged x), we have

$$\varpi(1+e'_x)=\text{A};$$

whence

$$\varpi=\frac{\text{A}}{1+e'_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

This is a very singular result. It is independent of the rate of interest; and yet it is obvious that the higher the rate realized by the Office the

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greater will be the annual return to (x), and consequently the less the cost of the assurance to him. The foregoing equation therefore cannot be true, and the process by which it is attained must be fallacious.*

But where, then, is the fallacy? It is in the assumption, tacitly made in the so-called solution, that the interest realized by the Office and that payable to the policyholders are identical. They are so, however, only as to *rate*, but not as to *amount*, except during the first year. At the end of that period the premium fund is so reduced by payment of death claims, that the interest yielded by it is no longer sufficient to meet that due to the policyholders. The deficiency, therefore, must be made good from the premiums themselves, and these therefore require to be increased to meet this charge.

The reasons why I have commenced with an erroneous solution, are—first, that an impression prevails, as I am informed, that this solution is a correct one; and secondly, that the problem belongs to a class which appear to invite the application of what are called common sense notions, while such applications usually lead, as in the present case, unless skilfully managed, to erroneous conclusions.

I now give a legitimate solution of the problem. The benefit consists of, first, a uniform assurance of A, the term corresponding to which is AM_x; and secondly, of an increasing annuity of ϖi, $2\varpi i$, $3\varpi i$, &c., which makes its last payment at the end of the year of death. The term given by this annuity, *minus* its last payment, is $\varpi i S_x$, and that given by the last payment is $\varpi i R_x$. Hence, the payment term being ϖN_{x-1}, we have

$$\varpi N_{x-1} = AM_x + \varpi i(S_x + R_x).$$

From this we obtain

$$\varpi = \frac{AM_x}{N_{x-1} - i(S_x + R_x)}.$$

Now,
$$N_{x-1} - i(S_x + R_x) = N_{x-1} - i(S_x + vS_{x-1} - S_x)$$
$$= N_{x-1} - (1-v)S_{x-1} = R_x.$$
$$\therefore \ \varpi = \frac{AM_x}{R_x} \quad . \ . \ . \ . \ . \ . \ . \quad (2).$$

Of the value of ϖ thus determined it would be easy to show that for any value of x, except the oldest age in the table (for which ϖ is always equal to A), it increases *with* (not *as*) i, the rate of interest.

Since, when i diminishes without limit, $\frac{M_x}{R_x}$ approaches without limit to $\frac{D_x}{N_{x-1}}$; therefore, when $i=0$, *i.e.*, when money bears no interest, we have

$$\varpi = \frac{AD_x}{N_{x-1}} = \frac{A}{1+\mathring{e}'_x},$$

which agrees with (1). From this it appears that, although not true generally, (1) is true in the case of money bearing no interest. In this

* The reasoning here does not seem quite conclusive.—ED. *J. I. A.*

case, however, no interest being realized there is none payable to the policyholders.

The following table shows the premium per cent., by the Carlisle rate of mortality, at several rates of interest. The commutation table for $i=0$ will be found at p. 145, vol. xiii. of the *Journal of the Institute of Actuaries.*

Age.	$i=0.$	$i=\cdot 03.$	$i=\cdot 04.$	$i=\cdot 05.$
30	2·8604	3 7290	4·1146	4 5707
50	4·6281	5 4515	5 7719	6·1156
70	10·3371	11·6040	12 0426	12·4872
90	26·4432	28·5903	29 3248	29·9864

For further elucidation of this somewhat curious problem I have worked out the following example at length, by the Carlisle table, at 5 per cent. The age is 90, and the sum assured £100. By (2) we get for the annual premium

$$\varpi=\frac{147\cdot 9288}{4\cdot 933192}=29\cdot 98643;\text{ whence }\varpi i=1\cdot 4993215.$$

142ϖ		4258·0731			1732·4009
5 per cent.		*212 9037			*86·6200
		4470·9768			1819·0209
$\varpi i\times 142$	*212·9037		$5\varpi i\times 40$	*299 8643	
100×37	3700·	3912·9037	100×10	1000·	1299·8643
		558·0731			519·1566
105ϖ		3148 5752	30ϖ		899·5929
		3706·6483			1418 7495
		*185·3324			*70 9375
		3891 9807			1489 6870
$2\varpi i\times 105$	*314·8575		$6\varpi i\times 30$	*269·8779	
100×30	3000·	3314·8575	100×7	700·	969·8779
		577·1232			519·8091
75ϖ		2248·9822	23ϖ		689·6879
		2826·1054			1209·4970
		*141·3053			*60·4794
		2967·4107			1269·9719
$3\varpi i\times 75$	*337·3473		$7\varpi i\times 23$	*241·3908	
100×21	2100·	2437·3473	100×5	500·	741·3908
		530 0634			528·5811
54ϖ		1619 2672	18ϖ		539·7557
		2149 3306			1068·3368
		*107·4665			*53·4168
		2256·7971			1121·7536
$4\varpi i\times 54$	*323·8534		$8\varpi i\times 18$	*215·9023	
100×14	1400·	1723·8534	100×4	400·	615·9023
		532 9437			505·8513
40ϖ		1199·4572	14ϖ		419·8100
		1732·4009			925·6613

		925·6613
		*46 2830
		971 9443
$9\varpi i\times 14$	*188 9143	
100×3	300·	488 9145
		483 0308
11ϖ		329 8507
		812 8815
		*40 6440
		853 5255
$10\varpi i\times 11$	*164 9254	
100×2	200·	364 9254
		488 6001
9ϖ		269 8779
		758·4780
		*37 9239
		796 4019
$11\varpi i\times 9$	*148·4328	
100×2	200	348 4328
		417 9691
7ϖ		209 9050
		657 8741
		*32 8937
		690 7678
$12\varpi i\times 7$	*125 9430	
100×2	200·	325 9430
		364 8248
5ϖ		149·9322
		514 7570

		514·7570
		*25 7378
		540 4948
$13\varpi i\times 5$	*97·4559	
100×2	200·	297·4559
		243·0389
3ϖ		89 9593
		332 9982
		*16 6499
		349 6481
$14\varpi i\times 3$	*62 9715	
100×2	200·	262 9715
		86 6766
ϖ		29 9864
		116 6630
		*5 8331
		122·4961
$15\varpi i\times 1$	*22·4898	
100×1	100·	122·4898

Little explanation of the above is needed. At the outset the premium is received from the tabular number alive at 90, viz., 142, and a year's interest is added, giving a total in hand at the end of the first year of £4,470. This is immediately reduced by the payment of, first, £212 9037, interest on the premiums, and secondly, £3,700, the claims arising on 37 deaths, to £558·0731. The premium is again received from the 105 survivors, a year's interest is added, and the outgoings of the second year, amounting to £3314 8575, are deducted, leaving £577·1232 in hand at the commencement of the third year. In this way the scheme works itself out at the end of the fifteenth year.

It is visible now that after the first year the interest which the office realizes is altogether insufficient to meet that which it has to pay. And it is singular to note that, after the first few years, the ratio of the interest receivable (by the Office) to the interest payable, closely approximates to that of 1 : 4.* Whether this is accidental, or whether the like would be observed in other circumstances, I am at present unable to say.

Returning to equation (2), and writing it thus,

$$\varpi R_x = AM_x,$$

* To facilitate this comparison I have marked the interest on both sides with asterisks.

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we see that the transaction resolves itself into an exchange or commutation of one assurance on (x) for another, viz., a uniform assurance of A payable *by* the Office, and an increasing assurance of ϖ, 2ϖ, &c. ($n\varpi$ in the nth year), payable *to* the Office. And this is correct, as it is obviously the same thing, theoretically, whether the premiums be paid annually, interest being allowed upon them, or in the aggregate at the end of the year of death. In practice, however, there is a great distinction between the two modes of payment. No Office would consent to defer the receipt of premium till the emergence of the claim, as they would in a great many cases have then more to receive than to pay.

It is interesting, however, to watch the operation of this mode of payment in a particular case; and I have therefore worked it out for the same age as before, 90, and at the same rate, 5 per cent. The premium also is of course the same, 29·98643.

100×37	3700·				
$\varpi \times 37$	1109·4979	2590·5021			2852·6285
5 per cent.		129 5250			142·6314
		2720·0271			2995·2599
100×30	3000·		100×3	300·	
$2\varpi \times 30$	1799·1858	1200 8142	$9\varpi \times 3$	809·6336	−509·6336
		3920 8413			2485·6263
		196 0420			124 2813
		4116 8833			2609·9076
100×21	2100·		100×2	200·	
$3\varpi \times 21$	1889·1451	210·8549	$10\varpi \times 2$	599·7286	−399·7286
		4327·7382			2210·1790
		216 3869			110·5090
		4544·1251			2320·6880
100×14	1400·		100×2	200·	
$4\varpi \times 14$	1679 2401	−279·2401	$11\varpi \times 2$	659·7015	−459·7015
		4264·8850			1860·9865
		213·2442			93 0493
		4478 1292			1954 0358
100×10	1000·		100×2	200·	
$5\varpi \times 10$	1499·3215	−499·3215	$12\varpi \times 2$	719·6743	−519·6743
		3978 8077			1434·3615
		198 9404			71 7181
		4177·7481			1506·0796
100×7	700·		100×2	200·	
$6\varpi \times 7$	1259·4301	−559·4301	$13\varpi \times 2$	779·6472	−579·6472
		3618·3180			926·4324
		180 9159			46·3216
		3799 2339			972·7540
100×5	500·		100×2	200·	
$7\varpi \times 5$	1049 5250	−549·5250	$14\varpi \times 2$	839·6200	−639·6200
		3249 7089			333·1340
		162·4854			16·6567
		3412·1943			349·7907
100×4	400·		100×1	100·	
$8\varpi \times 4$	959·5658	−559·5658	$15\varpi \times 1$	449·7965	−349·7965
		2852 6285			

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The great distinction between this mode of arranging the transaction and the other is that there the Office was put in funds at the outset, enabling it to meet all claims as they arose, while here it is in advance from first to last.

If it is required to *load* the premium of this problem, we must proceed as in all cases in which the Office makes a return to the assured. It is not sufficient to apply the required loading to the value of ϖ, determined as above, since this would leave the additional interest which has to be returned unprovided for. The loading must, as in all such cases, be applied to the benefit side of the fundamental equation.

Let the required loading be k per pound. Then,

$$\varpi N_{x-1} = (1+k)\{AM_x + \varpi i(S_x+R_x)\};$$

whence,

$$\varpi = \frac{(1+k)AM_x}{N_{x-1}-(1+k)i(S_x+R_x)}.$$

But, $N_{x-1}-(1+k)i(S_x+R_x) = N_{x-1}-(1+k)i(S_x+vS_{x-1}-S_x)$

$$= N_{x-1}-(1+k)(1-v)S_{x-1} = (1+k)\{N_{x-1}-(1-v)S_{x-1}\}-kN_{x-1}$$
$$= (1+k)R_x - kN_{x-1}.$$

$$\therefore\ \varpi = \frac{(1+k)AM_x}{(1+k)R_x-kN_{x-1}} = \frac{AM_x}{R_x-\dfrac{k}{1+k}N_{x-1}} \quad . \quad . \quad . \quad . \quad . \quad (3).$$

This is obviously greater than $\dfrac{AM_x}{R_x}$; but it can be shown to be also greater than $\dfrac{(1+k)AM_x}{R_x}$, which is what the net premium becomes when the loading is *directly* applied to it. Thus,

$$\frac{AM_x}{R_x-\dfrac{k}{1+k}N_{x-1}} > (1+k)\frac{AM_x}{R_x},$$

if $\qquad R_x > (1+k)R_x - kN_{x-1},$

if $\qquad kN_{x-1} > kR_x,$

if $\qquad N_{x-1} > R_x;$

and this last we know to be true.

If no interest is earned, M_x and R_x, as before, assume their limiting values, and (3) becomes

$$\varpi = \frac{AD_x}{N_{x-1}-\dfrac{k}{1+k}N_{x-1}} = (1+k)\frac{AD_x}{N_{x-1}}.$$

In this case, therefore, it suffices to apply the loading *directly* to the net premium; which is in accordance with the remark already made, the interest returnable by the Office being here *nil*.

I append a table of loaded premiums, corresponding to that already given of net premiums. The loading is 10 per cent., that is $k = \cdot 1$,

Age.	$i=0$.	$i=\cdot03$.	$i=\cdot04$.	$i=\cdot05$.
30	3·1464	4 5002	5·1881	6·1767
50	5·0909	6 3154	6 8336	7·4329
70	11·3708	13 0714	13 6844	14·3199
90	29·0875	31 8114	32 7722	33 6255

I am, Sir,
Your most obedient servant,

P. GRAY.

London, 2nd Sept., 1867.

*** A short note on the problem which forms the subject of this letter will be found in vol. v., p. 348.

VALUE OF A POLICY—FORMULÆ—MILNE.

To the Editor of the Assurance Magazine.

DEAR SIR,—There is a theorem which I suppose must be in the heads of many actuaries, but I cannot find it in any of the books. It is that the values of a policy, as it runs on, are proportional to the falls in the value of the annuity. That is, if a_x be the value of an annuity of £1 at the age x, the age of creation of the policy, the values of the policy at the ages y and z are as a_x-a_y to a_x-a_z. That this theorem is not commonly expressed seems due to the value at the age y being usually written $1-\dfrac{1+a_y}{1+a_x}$ instead of $\dfrac{a_x-a_y}{1+a_x}$.

I shall be curious to see whether any one will produce a statement of this simple form. I find it occasionally very useful to take out from the table, without any writing, that the policy-value of $1+a_x$ at death is a_x-a_y at the age y, the age x being that of commencement. When a formula represents two different results, it is a useful exercise of ingenuity to deduce one result directly from the other. Now a_x-a_y is the value to (x) of a counter-survivorship—as we may call it—of the following kind. The executors of the first who dies pay an annuity of £1 to the survivor; and $(a_x-a_y)\div(1+a_x)$ is the whole-life premium which (x) should pay to be put in this position. How, from the nature of this contract, does it follow that one payment of this premium, over and above the annual premium which (x) should pay, admits (y) to a policy of £1 at the premium for the age (x)?

Easy forms, corollaries from common forms, are things for *second editions.* A person who is engaged in a great effort, and has a heavy system of tables to look after, does not watch offshoots. Now none of the best known works—except only those of Price and Morgan, which lay no stress on formulæ—have arrived at second editions: this may be said of Baily, G. Davies, Milne, and David Jones.

It is much to be regretted that Milne did not, in his later years, occupy himself with a reconstruction of the algebraical part of his work. But it is hardly known how completely he abandoned the subject. In May,