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OPTIMAL ALLOCATION OF ACTIVE REDUNDANCIES TO k-OUT-OF-n SYSTEMS WITH HETEROGENEOUS COMPONENTS

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Abstract

In this note we deal with the allocation of independent and identical active redundancies to a k-out-of-n system with the usual stochastic order among its independent components. The optimal policy is proved both to assign more redundancies to the weaker component and to majorize all other policies. This improves the corresponding one in Hu and Wang (2009) and serves as a nice supplement to that in Misra, Dhariyal and Gupta (2009) as well.

Keywords: Majorization; Schur concave; stochastic order

2000 Mathematics Subject Classification: Primary 90B25
Secondary 60E15; 60K10

1. Introduction

In industrial engineering, system security, and reliability, it is of great interest to allocate some redundancies to components of a system so as to optimize the lifetime or increase the reliability of the system. Generally, the following two types of allocation are commonly used: (i) active redundancy (hot standby), in which the redundancies are put in parallel to components of the system and start functioning at the same time as the components are initiated; (ii) standby redundancy (cold standby), in which redundancies are put in standby and start functioning once components fail. Recently, Cha *et al.* (2008) considered the so-called *general standby*, in which the redundancy works in a milder environment in the standby state and, hence, the failure rate is nonzero and smaller than that in the usual environment; therefore, it is just an intermediate stage between the cold and the hot stages. This paper will focus only on the active redundancy. For more on general standby, we refer the reader to Cha *et al.* (2008) and Li *et al.* (2009).

Shaked and Shanthikumar (1992) were among the first to study the problem of allocating m active redundancies to a series system with n components in the situation that lifetimes of components and redundancies are independent and identically distributed. Let $\boldsymbol{r} = (r_1, \dots, r_n)$ be an allocation policy, i.e. r_i redundancies are put in parallel with the ith component in the system ($i = 1, \dots, n$) and $r_1 + \cdots + r_n = m$. They proved that $T_s(\boldsymbol{r})$, the lifetime of the resulting series system with allocation policy $\boldsymbol{r}$, has a Schur-concave survival function with respect to $\boldsymbol{r}$. Afterward, in view of the importance of the hazard rate which describes a system's wear out, Singh and Singh (1997) showed that the failure rate function of $T_s(\boldsymbol{r})$ is Schur convex with respect to the allocation policy $\boldsymbol{r}$. Recently, Hu and Wang (2009) further investigated the allocation of m active redundancies to a k-out-of-n system where lifetimes of all working

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components and active redundancies are independent and identical; they proved that

$$T_{k|n}(\boldsymbol{r}') \leq_{\mathrm{st}} T_{k|n}(\boldsymbol{r}) \quad \text{whenever} \quad \boldsymbol{r} \leq_{\mathrm{m}} \boldsymbol{r}'.$$

Here $\boldsymbol{r}'$ is another allocation policy and $T_{k|n}(\boldsymbol{r})$ is the lifetime of the resulting k-out-of-n system with allocation policy $\boldsymbol{r}$, '$\leq_{\mathrm{st}}$' and '$\leq_{\mathrm{m}}$' denote the usual stochastic order and the majorization order, respectively. Readers are referred to Section 3 for their definitions. It is clear that the conclusion on the series system in Shaked and Shanthikumar (1992) is extended to the general k-out-of-n system.

On the other hand, Misra *et al.* (2009) also studied the allocation of m independent and identical redundancies with distribution G to a series system with lifetimes of n independent components that are stochastically ordered, i.e. $F_1(t) \leq \cdots \leq F_n(t)$ for all t. They proved that the survival function of lifetime $T_s(\boldsymbol{r})$ of the resulting series system is Schur concave with respect to $\boldsymbol{r}$ in $\tilde{\mathcal{R}}_m = \{(r_1, \ldots, r_n) : r_1 \geq \cdots \geq r_n,\ r_1 + \cdots + r_n = m,\ r_i \in N,\ i = 1, \ldots, n\}$. Namely, for $\boldsymbol{r} \in \tilde{\mathcal{R}}_m$ and $\boldsymbol{r}' \in \tilde{\mathcal{R}}_m$,

$$T_s(\boldsymbol{r}') \leq_{\mathrm{st}} T_s(\boldsymbol{r}) \quad \text{whenever} \quad \boldsymbol{r} \leq_{\mathrm{m}} \boldsymbol{r}'. \tag{1}$$

In addition, when $F_i = F$ for $i = 1, \ldots, n$, they also showed that the failure rate function of $T_s(\boldsymbol{r})$ is Schur convex with respect to $\boldsymbol{r}$ if $\ln G(t)/\ln F(t)$ increases in $t \geq 0$; this strengthens the main result of Singh and Singh (1997).

The interest of this paper is twofold. We further investigate the allocation of m active redundancies to a k-out-of-n system in the situation that lifetimes of independent components are stochastically ordered and all redundancies are independent and identically distributed. For the sake of optimizing the survival function of the redundant system, in Section 3 we study relations among those admissible allocation policies which are 'better' than the ones discussed in Misra *et al.* (2009). We show that the survival function of the lifetime of the resulting k-out-of-n system is Schur concave with respect to the admissible allocation policy $\boldsymbol{r}$, and we also present the optimal policy among all allocation policies. On the other hand, when all components are identically distributed, we prove that the survival function of the lifetime of the resulting system is also Schur concave with respect to the allocation policy without any other ancillary conditions, and we present the optimal allocation policy as well. Finally, in Section 4 we present some conclusions on the allocation of redundancies for components without stochastic order and the allocation of heterogeneous redundancies.

Throughout this paper, all random variables are assumed to be nonnegative and have 0 as the common left endpoint of their supports.

2. Assumptions and notation

In this study we deal with a redundant system based on the following assumptions.

(A1) *k-out-of-n system with active redundancies*. The system under study fails once k of its n components fail to operate properly, and active redundancies are respectively in parallel with those working components to which they are allocated.

(A2) *Stochastic order among components*. All components of the system are independent and their lifetimes are stochastically ordered.

(A3) *Independent and identical redundancies*. There are m redundant components to be allocated so as to optimize the lifetime of the system. All the redundancies are independent and identically distributed.

(A4) *Independence between components and redundancies.* The lifetimes of the working components in the system and the redundancies are independent.

For k-out-of-n systems, we refer the reader to Kuo and Zuo (2002) for a comprehensive discussion. We refer readers to Shaked and Shanthikumar (2007) for more on the stochastic order. For ease of reference, we list all the notation which will be employed from now on below.

$\boldsymbol{X} = (X_1, \dots, X_n)$	The vector of random lifetimes of system's components.
$\boldsymbol{Y} = (Y_1, \dots, Y_m)$	The vector of random lifetimes of redundancies.
$F_i, \bar{F}_i,\ i = 1, \dots, n$	The distribution function and survival function of the ith component.
$G, \bar{G}$	The distribution function and survival function of redundancies.
$\boldsymbol{r} = (r_1, \dots, r_n)$	The allocation policy with r_i redundancies allocated to $X_i,\ i = 1, \dots, n$.
$\boldsymbol{r}' = (r_1', \dots, r_n')$	The allocation policy defined in a similar manner to $\boldsymbol{r}$.
$(X_1(\boldsymbol{r}), \dots, X_n(\boldsymbol{r}))$	The vector of lifetimes of components under allocation policy $\boldsymbol{r}$.
$X_{k:n}(\boldsymbol{r})$	The kth order statistic based on $X_1(\boldsymbol{r}), \dots, X_n(\boldsymbol{r})$.
$T_{k\mid n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r})$	The lifetime of a k-out-of-n system with allocation policy $\boldsymbol{r}$.
$\bar{H}_k(t; \boldsymbol{r})$	The survival function of the lifetime $T_{k\mid n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r})$.
$\mathcal{R}_m\ (\bar{\mathcal{R}}_m)$	The set of all (admissible) allocation policies.
$\le_{\mathrm{st}}, \le_{\mathrm{m}}$	The usual stochastic and majorization orders.

Recall that a random variable X with distribution F is said to be smaller than the random variable Y with distribution G in the usual stochastic order (denoted by $X \le_{\mathrm{st}} Y$ or $F \le_{\mathrm{st}} G$) if $\mathrm{P}(X > t) \le \mathrm{P}(Y > t)$ for all t. This useful concept will be employed to compare survival functions of the redundant system with respect to various allocation policies. We refer the reader to Shaked and Shanthikumar (2007) for a comprehensive discussion. According to (A2), components of the system are stochastically ordered. For convenience, say $X_1 \ge_{\mathrm{st}} \cdots \ge_{\mathrm{st}} X_n$. Equivalently,

$$\bar{F}_1(t) \ge \cdots \ge \bar{F}_n(t) \quad \text{for all } t \ge 0. \tag{2}$$

Under the allocation policy $\boldsymbol{r} = (r_1, \dots, r_n)$, r_i active redundancies are allocated to the component $X_i,\ i = 1, \dots, n$. As a result, the survival function of the system under policy $\boldsymbol{r}$ is

$$\bar{H}_k(t; \boldsymbol{r}) = \mathrm{P}(T_{k\mid n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) > t) = \mathrm{P}(X_{k:n}(\boldsymbol{r}) > t) \quad \text{for all } t \ge 0.$$

For any pair of nodes $i < j$, define

$$Z_l = \begin{cases} X_l(\boldsymbol{r}), & 1 \le l \le i-1, \\ X_{l-1}(\boldsymbol{r}), & i+1 \le l \le j-1, \\ X_{l-2}(\boldsymbol{r}), & j+1 \le l \le n. \end{cases}$$

Let $Z_{k:n-2}$ be the kth order statistic based on $Z_1, \dots, Z_{n-2}$, which are independent due to (A3) and (A4). For convenience, set

$$\xi_{s\mid i,j}(t, k) = \mathrm{P}(Z_{k-s+1:n-2} > t) \quad \text{for } 1 \le k \le n \text{ and } s = 1, 2, 3$$

whenever $k \ge s$, and $\xi_{s\mid i,j}(t, k) = 0$ otherwise. The following decomposition of the system's survival function plays an important role in deducing the main results in the sequel.

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Proposition 1. *For a fixed pair of nodes $i < j$,*

$$\begin{aligned}\bar{H}_k(t;\boldsymbol{r}) = {} & \xi_{1|i,j}(t,k)[1-F_i(t)G^{r_i}(t)][1-F_j(t)G^{r_j}(t)] \\ & + \xi_{2|i,j}(t,k)[F_i(t)G^{r_i}(t)+F_j(t)G^{r_j}(t)-2F_i(t)F_j(t)G^{r_i+r_j}(t)] \\ & + \xi_{3|i,j}(t,k)F_i(t)F_j(t)G^{r_i+r_j}(t). \end{aligned} \tag{3}$$

Proof. By the total probability we have, for all $t \geq 0$,

$$\begin{aligned}\bar{H}_k(t;\boldsymbol{r}) &= \mathrm{P}(X_{k:n}(\boldsymbol{r}) > t) \\ &= \mathrm{P}(X_{k:n}(\boldsymbol{r}) > t \mid X_i(\boldsymbol{r}) > t,\ X_j(\boldsymbol{r}) > t)\,\mathrm{P}(X_i(\boldsymbol{r}) > t,\ X_j(\boldsymbol{r}) > t) \\ &\quad + \mathrm{P}(X_{k:n}(\boldsymbol{r}) > t \mid X_i(\boldsymbol{r}) > t,\ X_j(\boldsymbol{r}) \leq t)\,\mathrm{P}(X_i(\boldsymbol{r}) > t,\ X_j(\boldsymbol{r}) \leq t) \\ &\quad + \mathrm{P}(X_{k:n}(\boldsymbol{r}) > t \mid X_i(\boldsymbol{r}) \leq t,\ X_j(\boldsymbol{r}) > t)\,\mathrm{P}(X_i(\boldsymbol{r}) \leq t,\ X_j(\boldsymbol{r}) > t) \\ &\quad + \mathrm{P}(X_{k:n}(\boldsymbol{r}) > t \mid X_i(\boldsymbol{r}) \leq t,\ X_j(\boldsymbol{r}) \leq t)\,\mathrm{P}(X_i(\boldsymbol{r}) \leq t,\ X_j(\boldsymbol{r}) \leq t) \\ &= \mathrm{P}(Z_{k:n-2} > t)[1-F_i(t)G^{r_i}(t)][1-F_j(t)G^{r_j}(t)] \\ &\quad + \mathrm{P}(Z_{k-1:n-2} > t)[1-F_i(t)G^{r_i}(t)]F_j(t)G^{r_j}(t) \\ &\quad + \mathrm{P}(Z_{k-1:n-2} > t)[1-F_j(t)G^{r_j}(t)]F_i(t)G^{r_i}(t) \\ &\quad + \mathrm{P}(Z_{k-2:n-2} > t)F_i(t)F_j(t)G^{r_i+r_j}(t) \\ &= \xi_{1|i,j}(t,k)[1-F_i(t)G^{r_i}(t)][1-F_j(t)G^{r_j}(t)] \\ &\quad + \xi_{2|i,j}(t,k)[F_i(t)G^{r_i}(t)+F_j(t)G^{r_j}(t)-2F_i(t)F_j(t)G^{r_i+r_j}(t)] \\ &\quad + \xi_{3|i,j}(t,k)F_i(t)F_j(t)G^{r_i+r_j}(t). \end{aligned}$$

This is the desired result.

3. Allocation of active redundancies

3.1. The optimal allocation policy

It is evident that we should focus on

$$\mathcal{R}_m = \{(r_1,\dots,r_n) : r_1+\cdots+r_n = m,\ r_i \in N,\ i = 1,\dots,n\},$$

the set of all *allocation policies*.

Theorem 1. *Consider two allocation policies $\boldsymbol{r} \in \mathcal{R}_m$ and $\boldsymbol{r}' \in \mathcal{R}_m$ such that $r_i = r'_j$ and $r_j = r'_i$ for some $1 \leq i < j \leq n$, and $r_l = r'_l$ for $l \notin \{i,j\}$. Then,*

$$T_{k|n}(\boldsymbol{X},\boldsymbol{Y};\boldsymbol{r}) \geq_{\mathrm{st}} T_{k|n}(\boldsymbol{X},\boldsymbol{Y};\boldsymbol{r}') \quad \textit{if and only if} \quad r_i < r_j.$$

Proof. Since

$$r_l = \begin{cases} r'_j, & l = i, \\ r'_i, & l = j, \\ r'_l, & l \notin \{i,j\}, \end{cases}$$

by (3) we have, for all $t \geq 0$,

$$\begin{aligned}\bar{H}_k(t;\boldsymbol{r}) - \bar{H}_k(t;\boldsymbol{r}') = {} & \xi_{1|i,j}(t,k)[1-F_i(t)G^{r_i}(t)][1-F_j(t)G^{r_j}(t)] \\ & - \xi_{1|i,j}(t,k)[1-F_i(t)G^{r'_i}(t)][1-F_j(t))G^{r'_j}(t)] \\ & + \xi_{2|i,j}(t,k)[F_i(t)G^{r_i}(t)+F_j(t)G^{r_j}(t)-2F_i(t)F_j(t)G^{r_i+r_j}(t)] \end{aligned}$$

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$$
\begin{aligned}
&- \xi_{2|i,j}(t,k)[F_i(t)G^{r_i'}(t) + F_j(t)G^{r_j'}(t) - 2F_i(t)F_j(t)G^{r_i'+r_j'}(t)] \\
&+ \xi_{3|i,j}(t,k)F_i(t)F_j(t)[G^{r_i+r_j}(t) - G^{r_i'+r_j'}(t)] \\
= {}& \xi_{1|i,j}(t,k)[G^{r_i}(t) - G^{r_j}(t)][F_j(t) - F_i(t)] \\
&+ \xi_{2|i,j}(t,k)[G^{r_j}(t) - G^{r_i}(t)][F_j(t) - F_i(t)] \\
= {}& [\xi_{1|i,j}(t,k) - \xi_{2|i,j}(t,k)][G^{r_i}(t) - G^{r_j}(t)][F_j(t) - F_i(t)].
\end{aligned}
$$

Since $Z_{k:n-2} \geq Z_{k-1:n-2}$, it holds that

$$\xi_{1|i,j}(t,k) \geq \xi_{2|i,j}(t,k) \quad \text{for all } t \geq 0.$$

On the other hand, by (2) we have

$$F_j(t) \geq F_i(t) \quad \text{for all } t \geq 0.$$

As a result, $\bar{H}_k(t;\boldsymbol{r}) - \bar{H}_k(t;\boldsymbol{r}') \geq 0$ for all $t \geq 0$ if and only if $r_i < r_j$.

Under the same assumption in our model, Misra *et al.* (2009) derived (1) for the series system. However, according to Theorem 1, more redundancies should be allocated to the component which is stochastically smaller. That is, for $X_1 \geq_{\text{st}} X_2 \geq_{\text{st}} \cdots \geq_{\text{st}} X_n$, the allocation policy $\boldsymbol{r} = (r_1, \ldots, r_n)$ with $r_1 \leq \cdots \leq r_n$ performs better than the policy with $r_i > r_j$ for some $i < j$. As a result, it suffices to pay attention to

$$\bar{\mathcal{R}}_m = \{(r_1, \ldots, r_n) \colon r_1 \leq \cdots \leq r_n, r_1 + \cdots + r_n = m,\ r_i \in N,\ i = 1, \ldots, n\},$$

the set of all *admissible allocation policies*.

Theorem 2. *Consider allocation policies* $\boldsymbol{r}' \in \mathcal{R}_m$ *with* $r_j' - r_i' \geq 2$ *for some pair* $1 \leq i < j \leq n$ *and* $\boldsymbol{r} \in \mathcal{R}_m$ *such that* $r_i = r_i' + 1$, $r_j = r_j' - 1$, *and* $r_l = r_l'$ *for* $l \notin \{i, j\}$. *If* $G \geq_{\text{st}} F_1$ *then*

$$T_{k|n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \geq_{\text{st}} T_{k|n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}').$$

Proof. Since

$$r_l = \begin{cases} r_l' + 1, & l = i, \\ r_l' - 1, & l = j, \\ r_l', & l \notin \{i, j\}, \end{cases}$$

by (3) again we have, for all $t \geq 0$,

$$
\begin{aligned}
&\bar{H}_k(t;\boldsymbol{r}) - \bar{H}_k(t;\boldsymbol{r}') \\
&\quad = \xi_{1|i,j}(t,k)[1 - F_i(t)G^{r_i'+1}(t)][1 - F_j(t)G^{r_j'-1}(t)] \\
&\qquad - \xi_{1|i,j}(t,k)[1 - F_i(t)G^{r_i'}(t)][1 - F_j(t)G^{r_j'}(t)] \\
&\qquad + \xi_{2|i,j}(t,k)[F_i(t)G^{r_i'+1}(t) + F_j(t)G^{r_j'-1}(t)] \\
&\qquad - \xi_{2|i,j}(t,k)[F_i(t)G^{r_i'}(t) + F_j(t)G^{r_j'}(t)] \\
&\qquad + \xi_{3|i,j}(t,k)F_i(t)F_j(t)[G^{r_i'+r_j'}(t) - G^{r_i'+1+r_j'-1}(t)]
\end{aligned}
$$

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$$
\begin{aligned}
&= \xi_{1|i,j}(t,k)[(G^{r'_j}(t) - G^{r'_j-1}(t))F_j(t) + (G^{r'_i}(t) - G^{r'_i+1}(t))F_i(t)] \\
&\quad + \xi_{2|i,j}(t,k)[(G^{r'_j-1}(t) - G^{r'_j}(t))F_j(t) + (G^{r'_i+1}(t) - G^{r'_i}(t))F_i(t)] \\
&= (1 - G(t))(\xi_{1|i,j}(t,k) - \xi_{2|i,j}(t,k))[F_i(t)G^{r'_i}(t) - F_j(t)G^{r'_j-1}(t)] \\
&= (1 - G(t))G^{r'_i}(t)F_i(t)(\xi_{1|i,j}(t,k) - \xi_{2|i,j}(t,k))\left[1 - \frac{F_j(t)}{F_i(t)}G^{r'_j-r'_i-1}(t)\right].
\end{aligned}
$$

In view of $r'_j - r'_i \geq 2$ and $G \geq_{\mathrm{st}} F_1 \geq_{\mathrm{st}} F_i$, we have, for all $t \geq 0$,

$$G^{r'_j-r'_i-1}(t) \leq G(t) \leq F_i(t).$$

Then, it holds that, for all $t \geq 0$,

$$\frac{F_j(t)}{F_i(t)}G^{r'_j-r'_i-1}(t) \leq \frac{F_j(t)}{F_i(t)}F_i(t) = F_j(t) \leq 1.$$

Consequently, due to $\xi_{1|i,j}(t,k) \geq \xi_{2|i,j}(t,k)$ for any $t \geq 0$, it immediately follows that

$$\bar{H}_k(t;\boldsymbol{r}) - \bar{H}_k(t;\boldsymbol{r}') \geq 0 \quad \text{for all } t \geq 0.$$

This completes the proof.

According to Theorem 2, it is better for the difference between numbers of redundancies allocated to any two different components in an allocation policy not to exceed 2.

Let $x_{(1)} \leq \cdots \leq x_{(n)}$ be the increasing arrangement of the components of the vector $\boldsymbol{x} = (x_1, \ldots, x_n)$. Recall that $\boldsymbol{x}$ is said to *majorize* $\boldsymbol{y} = (y_1, \ldots, y_n)$ (denoted by $\boldsymbol{x} \geq_{\mathrm{m}} \boldsymbol{y}$) if $\sum_{i=1}^n x_{(i)} = \sum_{i=1}^n y_{(i)}$ and $\sum_{i=1}^n x_{(i)} \leq \sum_{i=1}^n y_{(i)}$ for $j = 1, \ldots, n-1$. It is well known that majorization is quite useful in establishing various inequalities. Readers may refer to Marshall and Olkin (1979) for more on the majorization order. Here, the majorization order enables us to compare the diversity of two allocation policies. The next result provides some insight into comparisons between two admissible policies in $\bar{\mathcal{R}}_m$.

Theorem 3. *Consider two admissible allocation policies $\boldsymbol{r} \in \bar{\mathcal{R}}_m$ and $\boldsymbol{r}' \in \bar{\mathcal{R}}_m$. If $G \geq_{\mathrm{st}} F_1$ then*

$$T_{k|n}(\boldsymbol{X},\boldsymbol{Y};\boldsymbol{r}) \geq_{\mathrm{st}} T_{k|n}(\boldsymbol{X},\boldsymbol{Y};\boldsymbol{r}') \quad \textit{whenever} \quad \boldsymbol{r} \leq_{\mathrm{m}} \boldsymbol{r}'.$$

Proof. Since $\boldsymbol{r} \leq_{\mathrm{m}} \boldsymbol{r}'$, according to Lemma D.1 of Marshall and Olkin (1979, p. 135), there exist $\ell - 2$ admissible allocation policies such that

$$\boldsymbol{r} = \boldsymbol{r}^{(1)} \leq_{\mathrm{m}} \boldsymbol{r}^{(2)} \leq_{\mathrm{m}} \cdots \leq_{\mathrm{m}} \boldsymbol{r}^{(\ell)} = \boldsymbol{r}',$$

and, for $s = 1, \ldots, \ell-1$, $\boldsymbol{r}^{(s)} = (r_1^{(s)}, \ldots, r_n^{(s)})$ and $\boldsymbol{r}^{(s+1)} = (r_1^{(s+1)}, \ldots, r_n^{(s+1)})$ satisfy, for some $1 \leq i, j \leq n$,

$$r_i^{(s)} = r_i^{(s+1)} + 1, \qquad r_j^{(s)} = r_j^{(s+1)} - 1, \quad \text{and} \quad r_l^{(s)} = r_l^{(s+1)} \quad \text{for } l \notin \{i, j\}.$$

Thus, the desired result follows immediately from Theorem 2.

Let $\boldsymbol{r}^* = (r_1^*, \ldots, r_n^*) \in \mathcal{R}_m$ such that $|r_j^* - r_i^*| \leq 1$ for any pair $i \neq j$, and let $\bar{\boldsymbol{r}}^* = (\bar{r}_1^*, \ldots, \bar{r}_n^*) \in \bar{\mathcal{R}}_m$ such that $|\bar{r}_j^* - \bar{r}_i^*| \leq 1$ for any pair $i \neq j$. It should be remarked here that $\boldsymbol{r}^*$ is not unique, whereas $\bar{\boldsymbol{r}}^*$ is unique. For example, when $n = 5$ and $m = 18$, $\boldsymbol{r}^*$ may be $(3, 3, 4, 4, 4)$, $(3, 4, 3, 4, 4)$, or $(3, 4, 4, 3, 4)$, etc.; however, $\bar{\boldsymbol{r}}^* = (3, 3, 4, 4, 4)$.

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The proof of the following Theorem is straightforward and hence omitted for briefness.

Theorem 4. *For any* $\boldsymbol{r} \in \mathcal{R}_m$, $\boldsymbol{r} \geq_{\mathrm{m}} \boldsymbol{r}^*$. *In particular, for any* $\boldsymbol{r} \in \bar{\mathcal{R}}_m$, $\boldsymbol{r} \geq_{\mathrm{m}} \bar{\boldsymbol{r}}^*$.

Now, in combination with Theorem 1, Theorem 3, and Theorem 4, we reach the optimal allocation policy in the sense of the usual stochastic order.

Theorem 5. *For any allocation policy* $\boldsymbol{r} \in \mathcal{R}_m$, *if* $G \geq_{\mathrm{st}} F_1$ *then*

$$T_{k|n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \geq_{\mathrm{st}} T_{k|n}(\boldsymbol{X}, \boldsymbol{Y}; \bar{\boldsymbol{r}}^*).$$

That is, $\bar{\boldsymbol{r}}^*$ *is the optimal.*

The stochastic order condition $G \geq_{\mathrm{st}} F_1$ actually claims that redundancies are not worse than those active components. This is a bit restrictive and not always the case in practice. However, as can be seen in the following example, this condition may not be dropped.

Example 1. (*Stochastic order between the component and standby.*) For a series system with three components having survival functions

$$\bar{F}_1(t) = \mathrm{e}^{-0.2t}, \qquad \bar{F}_2(t) = \mathrm{e}^{-0.5t}, \qquad \bar{F}_3(t) = \mathrm{e}^{-2t},$$

consider three active redundancies having common survival function $\bar{G}(t) = \mathrm{e}^{-t}$. Then it is easy to verify that

$$\bar{F}_3(t) \leq \bar{G}(t) \leq \bar{F}_2(t) \leq \bar{F}_1(t) \quad \text{for all } t \geq 0.$$

Owing to Theorem 1, we consider only allocation policies with their elements being arranged in ascending order. The survival curves of the redundant system corresponding to the three admissible allocation policies $\boldsymbol{r}_1 = (0, 0, 3)$, $\boldsymbol{r}_2 = (0, 1, 2)$, and $\boldsymbol{r}_3 = (1, 1, 1)$ are plotted in Figure 1. As can be seen, the three corresponding survival curves cross each other, and none of them is superior to the other two in the sense of the usual stochastic order. That is, the optimal allocation does not exist.

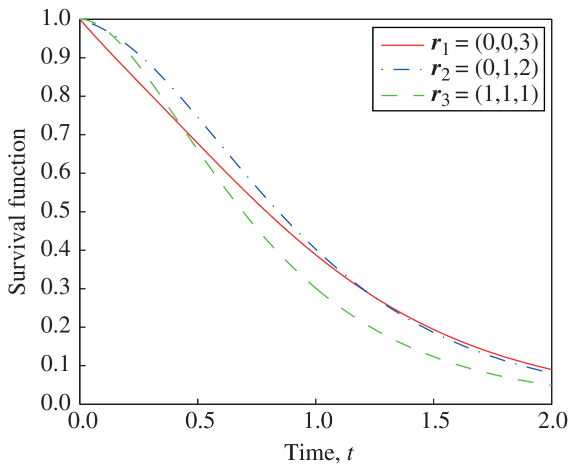

FIGURE 1: Survival curves of a series system with allocation $\boldsymbol{r}_i$, $i = 1, 2, 3$.

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3.2. The case with homogeneous components

Recall that a function $\phi\colon R^n \to R$ is said to be *Schur concave* if, for $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^n$, $\phi(\boldsymbol{x}) \le \phi(\boldsymbol{y})$ whenever $\boldsymbol{x} \ge_{\mathrm{m}} \boldsymbol{y}$. The following necessary and sufficient condition for a permutation symmetric differentiable function to be Schur concave is quite useful. For a comprehensive discussion on the theory and application of the majorization order as well as the Schur-concave function, we refer the reader to Marshall and Olkin (1979).

Lemma 1. (Roberts and Varberg (1973).) *Let $f(\boldsymbol{x})$ be symmetric and have continuous partial derivatives for $\boldsymbol{x} = (x_1, \dots, x_n) \in I^n$, where I is an open interval. Then $f\colon I^n \to R$ is Schur concave if and only if*

$$(x_i - x_j)\left(\frac{\partial f(\boldsymbol{x})}{\partial x_i} - \frac{\partial f(\boldsymbol{x})}{\partial x_j}\right) \le 0$$

for $\boldsymbol{x} \in I^n$ such that $x_i \ne x_j$ with $1 \le i < j \le n$.

The next theorem studies the optimal allocation policy of active redundancies for k-out-of-n systems with identical components. As can be seen, the stochastic order condition between the redundancies and the components may be neglected.

Theorem 6. *Suppose that $F_i = F$ for $i = 1, \dots, n$, and consider two allocation policies $\boldsymbol{r} \in \mathcal{R}_m$ and $\boldsymbol{r}' \in \mathcal{R}_m$. Then,*

$$T_{k|n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \ge_{\mathrm{st}} T_{k|n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}') \quad \textit{whenever} \quad \boldsymbol{r} \le_{\mathrm{m}} \boldsymbol{r}'.$$

Proof. Since $F_i = F$ for $i = 1, \dots, n$, (3) becomes

$$\begin{aligned}\bar{H}_k(t; \boldsymbol{r}) = {} & \xi_{1|i,j}(t,k)[1 - F(t)G^{r_i}(t)][1 - F(t)G^{r_j}(t)] \\ & + \xi_{2|i,j}(t,k)[F(t)G^{r_i}(t) + F(t)G^{r_j}(t) - 2F^2(t)G^{r_i+r_j}(t)] \\ & + \xi_{3|i,j}(t,k)F^2(t)G^{r_i+r_j}(t)\end{aligned}$$

for any pair $1 \le i, j \le n$ and $t \ge 0$. Then,

$$\begin{aligned}\frac{\partial \bar{H}_k(t; \boldsymbol{r})}{\partial r_i} = {} & -\xi_{1|i,j}(t,k)(1 - F(t)G^{r_j}(t))F(t)G^{r_i}(t)\ln G(t) \\ & + \xi_{2|i,j}(t,k)[F(t)G^{r_i}(t) - 2F^2(t)G^{r_i+r_j}(t)]\ln G(t) \\ & + \xi_{3|i,j}(t,k)F^2(t)G^{r_i+r_j}(t)\ln G(t),\end{aligned}$$

and, thus, owing to the fact that $\xi_{1|i,j}(t,k) \ge \xi^k_{2,|i,j}(t)$ for all $t \ge 0$, we have

$$\begin{aligned}& (r_i - r_j)\left(\frac{\partial \bar{H}_k(t; \boldsymbol{r})}{\partial r_i} - \frac{\partial \bar{H}_k(t; \boldsymbol{r})}{\partial r_j}\right) \\ & \quad = (r_i - r_j)(G^{r_j}(t) - G^{r_i}(t))F(t)(\xi_{1|i,j}(t,k) - \xi_{2|i,j}(t,k))\ln G(t) \\ & \quad \le 0.\end{aligned}$$

That is, the survival function $\bar{H}_k(t; \boldsymbol{r})$ of the resulting system is Schur concave.

Note that, since $\bar{H}_k(t; \boldsymbol{r})$ is symmetric with respect to $\boldsymbol{r} \in \mathcal{R}_m$, by Lemma 1, we reach the desired conclusion.

From Theorem 4 and Theorem 6, we immediately have Theorem 7, below.

Theorem 7. *For any allocation policy* $\boldsymbol{r} \in \mathcal{R}_m$,

$$T_{k|n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \geq_{\text{st}} T_{k|n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}^*).$$

That is, $\boldsymbol{r}^*$ *is the optimal allocation policy.*

Hu and Wang (2009) showed that, when $\bar{G}(t) = \bar{F}_i(t) = \bar{F}(t)$ for $i = 1, \dots, n$,

$$T_{k|n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \geq_{\text{st}} T_{k|n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}') \quad \text{whenever} \quad \boldsymbol{r} \leq_{\text{m}} \boldsymbol{r}'.$$

As is seen, this is just a particular case of both Theorem 3 and Theorem 6.

4. Two related conclusions

The model in Section 2 assumes that all components are ordered in the sense of the usual stochastic order. However, in practice, this may not be applicable due to components wearing out differently. In fact, this assumption can be neglected in the case below: $m = rn$ redundancies are allocated to a k-out-of-n system with heterogeneous components. From the proof of Theorem 2, we have Corollary 1.

Corollary 1. *Suppose that there are* $m = rn$ *redundancies. If* $G \geq_{\text{st}} F_i$ *for* $i = 1, \dots, n$ *then* $\boldsymbol{r}^* = (r, \dots, r)$ *is the optimal allocation policy. Formally,*

$$T_{k|n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \geq_{\text{st}} T_{k|n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}^*) \quad \textit{for any } \boldsymbol{r} \in \mathcal{R}_m.$$

On the other hand, for the case with stochastically ordered components and heterogeneous redundancies, let $G_i(t)$ denote the distribution function of the lifetime Y_i of the ith redundancy, $i = 1, \dots, n$. Denote by $T'(\boldsymbol{X}, \boldsymbol{Y}; (i_1, \dots, i_n))$ the entire lifetime of the system with G_{i_l} being allocated to the lth component with survival function $\bar{F}_l$, $l = 1, \dots, n$. Then, we have the following corollary, which may be proved in a similar manner to that of Theorem 1.

Corollary 2. *Consider a k-out-of-n system with n components such that* $X_1 \geq_{\text{st}} \cdots \geq_{\text{st}} X_n$*, and suppose that there are n redundancies arranged as* $Y_1 \leq_{\text{st}} \cdots \leq_{\text{st}} Y_n$*. Then,*

$$T'(\boldsymbol{X}, \boldsymbol{Y}; (1, \dots, n)) \geq_{\text{st}} T'(\boldsymbol{X}, \boldsymbol{Y}; (i_1, \dots, i_n))$$

for any permutation $(i_1, \dots, i_n)$ *of* $(1, \dots, n)$.

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