Can. J. Math. Vol. **49** (4), 1997 pp. 696–707.

GEODESIC FLOW ON IDEAL POLYHEDRA

CHARALAMBOS CHARITOS AND GEORGIOS TSAPOGAS

ABSTRACT. In this work we study the geodesic flow on n-dimensional ideal polyhedra and establish classical (for manifolds of negative curvature) results concerning the distribution of closed orbits of the flow.

1**. Introduction and statements of results.** The geodesic flow on Riemannian manifolds has been studied extensively through the development of geometry. Early this century Hedlund, Hopf, Morse and others studied geodesics on surfaces of constant negative curvature partially answering a basic dynamical question, namely, the distribution of periodic vectors of geodesic flow. This work was extended and generalized in various ways until P. Eberlein in [7] proved Theorems 1 and 2 below for complete manifolds with finite volume and sectional curvature ≤ 0 which satisfy the property that for any two points $x \neq y$ in the boundary $\partial \tilde{M}$ of the universal cover of M there exists a unique geodesic joining x with y. The line of approach in [7] was to analyze the action of $\pi_1(M)$ on the boundary $\partial \tilde{M}$. Then the limit set of the action of $\pi_1(M)$ on $\partial \tilde{M}$, which is actually equal to the whole $\partial \tilde{M}$, was linked to the geodesic flow.

In the present work, following the same approach as in [7] we study the geodesic flow on *finite ideal polyhedra of any dimension n*. These complete spaces consist of finitely many ideal hyperbolic polytopes glued together by isometries along their $(n-1)$-faces. Important examples of ideal polyhedra have appeared in Thurston's work, see [12], [11, Section 10.3], where 3-manifolds, which are complements of links and knots in $\mathbb{S}^3$, are constructed by gluing together finitely many ideal tetrahedra. In consequence, these finite volume 3-manifolds are equipped by a complete hyperbolic structure. Moreover, the 2-skeleton of these 3-manifolds are examples of 2-dimensional ideal polyhedra.

If X is a n-dimensional ideal polyhedron (or, more generally, a hyperbolic metric space) the geodesic flow is defined by the map

$$\Phi: \mathbb{R} \times GX \longrightarrow GX$$

where the action of $\mathbb{R}$ is given by right translation, *i.e.* for all $t \in \mathbb{R}$ and $\gamma \in GX$, $\Phi(t, \gamma) \equiv \Phi_t(\gamma) := \gamma_t$, where $\gamma_t: \mathbb{R} \longrightarrow X$ is the geodesic defined by $\gamma_t(s) = \gamma(s+t)$, $s \in \mathbb{R}$. Recall that GX consists of all (local) isometries $\gamma: \mathbb{R} \longrightarrow X$, when X is (not) simply connected. The topology on GX is the topology of uniform convergence on compact sets. We will be calling $g \in GX$ a closed geodesic, if it is a periodic map. Each closed geodesic g induces a local isometry $S^1(r) \longrightarrow X$ where $S^1(r)$ is a circle with radius

Received by the editors April 1997; revised February 10, 1997.
AMS subject classification: Primary: 57M20; secondary: 53C23.

https://doi.org/10.4153/CJM-1997-033-8 Published online by Cambridge University Press

$r = \frac{\text{period}(g)}{2\pi}$. For the geodesic flow on GX, we prove the following results, when X is a finite complete n-dimensional ideal polyhedron (see Section 2 below for a precise definition).

THEOREM 1. *There exists a geodesic γ in GX whose orbit $\mathbb{R}\gamma$ under the geodesic flow is dense in GX.*

THEOREM 2. *The closed geodesics are dense in GX.*

Let X be a proper geodesic CAT(-1) space, ∂X its boundary (see for instance [1] for definitions and first properties) and Γ be a discrete group of isometries of X acting properly discontinuously on X. The *limit set* Λ of such an action is studied by M. Coornaert in [4], using the classification of the isometries of X into three types, namely, elliptic, parabolic and hyperbolic, as in the manifold case (*cf.* [8]). Following the geometrical approach of [7] and using results from [4], collected in the next paragraph, we carry out the proofs of Theorems 1 and 2. A key element in the proof is the fact that the non-wandering set Ω_X is equal to the whole GX (see Proposition 10 below). This fact is equivalent to $\Lambda\big(\pi_1(X)\big) = \partial\tilde{X}$, where $\tilde{X}$ is the universal cover of X. We show the latter equality by constructing a continuous surjective map

$$f\colon \partial\tilde{\Delta} \longrightarrow \partial\tilde{X}$$

where $\tilde{\Delta}$ (resp. Δ) is the infinite graph (resp. finite graph) obtained by joining the centers of the polytopes of $\tilde{X}$ (resp. X). We then use the fact that the limit set of the action of $\pi_1(X)$ on $\tilde{\Delta}$ is the whole $\partial\tilde{\Delta}$, and the map f to show that $\Lambda\big(\pi_1(X)\big) = \partial\tilde{X}$.

2. Preliminaries.

An n-dimensional ideal polyhedron is a complete locally finite union of ideal hyperbolic polytopes glued together isometrically along their $(n-1)$-faces with at least two germs of polytopes along each $(n-1)$-face. This is naturally a metric space (see [2]).

If X is an ideal polyhedron of dimension n then X has curvature less or equal to -1 (see [3, Proposition 1]). Its universal covering $\tilde{X}$ satisfies the CAT(-1) inequality (see [10, Corollary 2.11]). Moreover, $\tilde{X}$ has the property (see [3, Proposition 2])

$$\forall x, y \in \tilde{X} \cup \partial\tilde{X}, \quad \exists \text{ a unique geodesic } \gamma_{xy} \text{ joining } x \text{ with } y. \tag{1}$$

We now gather results from [4] which we will use in the sequel. Let Y denote a proper hyperbolic metric space and Γ a discrete group of isometries of Y. Let $y \in Y$ be arbitrary point. The *limit set* $\Lambda(\Gamma)$ of the group Γ is defined to be $\Lambda(\Gamma) = \overline{\Gamma y} \cap \partial Y$. The limit set has been studied extensively (see [4, Chapter II], [6, Chapter 2.1]). Each isometry of Y is either elliptic or, parabolic or, hyperbolic. If ϕ is hyperbolic then $\phi^n(y)$ converges to a point $\phi(+\infty) \in \partial Y$ (resp. $\phi(-\infty) \in \partial Y$) as $n \longrightarrow +\infty$ (resp. $n \longrightarrow -\infty$) with $\phi(+\infty) \neq \phi(-\infty)$. If ϕ is parabolic then $\phi^n(y)$ converges to a single point in ∂Y for $|n| \longrightarrow \infty$.

Denote by Fix_h the set of points in ∂Y which are fixed by hyperbolic elements of Γ.

We will use the following three results from [4]:

https://doi.org/10.4153/CJM-1997-033-8 Published online by Cambridge University Press

PROPOSITION 3. Fix_h *is countable and* Γ*-invariant. Moreover, if* $|\Lambda(\Gamma)| = \infty$*, then* Fix_h *is dense in* $\Lambda(\Gamma)$.

PROPOSITION 4. *There exists an orbit of* Γ *dense in* $\Lambda(\Gamma)\times\Lambda(\Gamma)$*, provided* $|\Lambda(\Gamma)| = \infty$.

PROPOSITION 5. *The set* $\{(\phi(+\infty), \phi(-\infty)) : \phi \in \Gamma$ *is hyperbolic*$\}$ *is dense in* $\Lambda(\Gamma) \times \Lambda(\Gamma)$*, provided* $|\Lambda(\Gamma)| = \infty$.

Note that the action of Γ on $\partial Y \times \partial Y$ is given by the product action, *i.e.* for $\phi \in \Gamma$ and $(\xi_1, \xi_2) \in \partial Y \times \partial Y$, $\phi(\xi_1, \xi_2) = (\phi\xi_1, \phi\xi_2)$.

Denote by Φ the geodesic flow on GX, as described in the introduction, by Ω the *non-wandering set* of Φ and by p the covering projection $\tilde{X} \longrightarrow X$. Recall that a point x in GX belongs to the non-wandering set Ω of the geodesic flow $\Phi: \mathbb{R} \times GX \longrightarrow GX$, if there exist sequences $\{x_n\} \subset GX$ and $\{t_n\} \subset \mathbb{R}$, such that $t_n \longrightarrow \infty$, $x_n \longrightarrow x$ and $\Phi_{t_n}(x_n) \longrightarrow x$.

PROPOSITION 6. *Let X be an n-dimensional ideal polyhedron and Γ a discrete group of isometries acting on $\tilde{X}$ such that $X \approx \tilde{X}/\Gamma$ and $|\Lambda(\Gamma)| = \infty$. Let $\gamma \in GX, \tilde{\gamma} \in G\tilde{X}$ be given such that $p \circ \tilde{\gamma} = \gamma$. Then*

$$\gamma \in \Omega \Leftrightarrow \tilde{\gamma}(\infty) \in \Lambda(\Gamma) \quad \textit{and} \quad \tilde{\gamma}(-\infty) \in \Lambda(\Gamma)$$

PROOF. Assuming $\gamma \in \Omega$, there exists sequences $\{\gamma_n\} \subset GX$ and $\{t_n\} \subset \mathbb{R}$, such that $t_n \longrightarrow \infty$, $\gamma_n \longrightarrow \gamma$ and $\Phi_{t_n}(\gamma_n) \longrightarrow \gamma$. Set $\delta_n \equiv \Phi_{t_n}(\gamma_n)$ and let $\{\widetilde{\gamma_n}\}, \{\widetilde{\delta_n}\}$ be lifts of $\{\gamma_n\}, \{\delta_n\}$, respectively, such that $\widetilde{\gamma_n} \longrightarrow \tilde{\gamma}$ and $\widetilde{\delta_n} \longrightarrow \tilde{\gamma}$. For each $n \in \mathbb{N}$,

$$p\big(\widetilde{\gamma_n}(t_n)\big) = \delta_n(0) = p\big(\widetilde{\delta_n}(0)\big)$$

and, hence, there exists $\phi_n \in \Gamma$ such that $\phi_n\big(\widetilde{\gamma_n}(t_n)\big) = \widetilde{\delta_n}(0)$. Therefore, for some sequence $\{\phi_n\} \subset \Gamma, (\phi_n \circ \widetilde{\gamma_n})(t_n) \longrightarrow \tilde{\gamma}(0)$. Thus, $d_{\tilde{X}}\Big(\phi_n^{-1}\big(\tilde{\gamma}(0)\big), \widetilde{\gamma_n}(t_n)\Big) \longrightarrow 0$. Since $\widetilde{\gamma_n}(t_n) \longrightarrow \tilde{\gamma}(\infty)$, it follows from [4, Chapter I, Proposition 3.1] that $\phi_n^{-1}\big(\tilde{\gamma}(0)\big) \longrightarrow \tilde{\gamma}(\infty)$. Similarly we show $\phi_n\big(\tilde{\gamma}(0)\big) \longrightarrow \tilde{\gamma}(-\infty)$.

Assume now $\big(\tilde{\gamma}(\infty), \tilde{\gamma}(-\infty)\big) \in \Lambda(\Gamma) \times \Lambda(\Gamma)$. By Proposition 5, there exists a sequence $\{\phi_n\} \subset \Gamma: \phi_n(+\infty) \longrightarrow \tilde{\gamma}(+\infty)$ and $\phi_n(-\infty) \longrightarrow \tilde{\gamma}(-\infty)$. Let $\widetilde{\gamma_n}$ be the geodesic joining $\phi_n(+\infty)$ with $\phi_n(-\infty)$. Parametrize each $\widetilde{\gamma_n}$ such that $\widetilde{\gamma_n} \longrightarrow \tilde{\gamma}$. Set $t_n = d\Big(\widetilde{\gamma_n}(0), (\phi_n)^{-n}\big(\widetilde{\gamma_n}(0)\big)\Big)$ and $\gamma_n = p \circ \widetilde{\gamma_n}$. Clearly, $\gamma_n \longrightarrow \gamma$ and $t_n \longrightarrow \infty$. Moreover, since $p \circ \phi_n = p$,

$$\Phi_{t_n}(\gamma_n) = p\big(\phi_n(\widetilde{\gamma_n})\big) = p(\widetilde{\gamma_n}) \longrightarrow p(\tilde{\gamma}) = \gamma,$$

which completes the proof of the proposition. ∎

3**. The limit set.** This section is devoted into establishing Corollary 9 below *i.e.*, that the limit set of the action of Γ on $\tilde{X}$ is the whole $\partial\tilde{X}$. Let X be a finite complete n-dimensional ideal polyhedron and $\tilde{X}$ its universal cover. Denote by $\tilde{\Delta}$ the (infinite) graph obtained by joining the centers of the polytopes of $\tilde{X}$ and by Δ the (finite) graph obtained by joining the centers of the polytopes of X. Set $\Gamma = \pi_1(X)$. The action of

https://doi.org/10.4153/CJM-1997-033-8 Published online by Cambridge University Press

Γ on $\tilde{X}$ restricts to an action of Γ on $\tilde{\Delta}$ such that $\tilde{\Delta}/\Gamma \approx \Delta$. Observe that $\tilde{\Delta}$ is not necessarily a tree *i.e.*, $\tilde{\Delta}$ is not necessarily the universal covering of Δ. However, the action of Γ on $\tilde{\Delta}$ admits a fundamental domain of finite diameter. Consider a maximal tree T in $\tilde{\Delta}$. The natural inclusion $T \hookrightarrow \tilde{\Delta}$ is a cobounded (*i.e.* the distance function $d(\cdot, T)$ is bounded in $\tilde{\Delta}$) quasi-isometry. Hence, $\tilde{\Delta}$ is a (-1)-hyperbolic space in the sense of Gromov (see [6, Chapter 3, Theorem 2.2]). Hence, Γ acts cocompactly and properly discontinuously on the hyperbolic space $\tilde{\Delta}$. It follows (see [6, Chapter 4 Theorem 4.1]) that $\partial\tilde{\Delta}$ is homeomorphic to $\partial\Gamma$, where $\partial\Gamma$ denotes the boundary of Γ. On the other hand, Γ acts on itself and the limit set $\Lambda_\Gamma(\Gamma)$ of this action is equal to $\partial\Gamma$. Consider the map $\Gamma \longrightarrow \tilde{\Delta}$ given by $\gamma \longrightarrow \gamma(p)$ for some $p \in \tilde{\Delta}$ fixed. This map is a quasi-isometry, hence induces a homeomorphism $\partial\Gamma \longrightarrow \partial\tilde{\Delta}$ which takes $\Lambda_\Gamma(\Gamma)$ into $\Lambda_{\tilde{\Delta}}(\Gamma)$. It follows that

$$\Lambda_{\tilde{\Delta}}(\Gamma) = \partial\tilde{\Delta} \tag{2}$$

We next construct a continuous surjective map $\partial\tilde{\Delta} \longrightarrow \partial\tilde{X}$ (*cf.* Proposition 8 below) in order to obtain property 2 for the action of Γ on $\tilde{X}$.

Let $\delta\colon [0,\infty) \longrightarrow \tilde{\Delta}$ be a geodesic ray. δ can be viewed as a piece-wise geodesic ray in $\tilde{X}$. Let $\{T_i\}$ be the (infinite) sequence of polytopes in $\tilde{X}$ intersected by the image of δ. Write $[0,\infty)$ as a union of subintervals, $[0,\infty) = \bigcup_i I_i$, and enumerate $\{T_i\}$ such that $\forall i$

$\delta(I_i) \subset T_i$ and $I_i \cap I_{i+1}$ consists of a single point, say t_i, and

T_i, T_{i+1} have a common $(n-1)$-face containing $\delta(t_i)$.

This procedure can always be performed because δ does not have a back and forth in $\tilde{X}$ and the intersection of its image with the skeleta of $\tilde{X}$ is transverse. Recall that a curve $\gamma\colon I \longrightarrow X$ has a back and forth if $\exists t_1, t_2 \in I : \gamma\big((t_1,t_2)\big)$ lies in the interior of a single polytope T of X and $\gamma(t_1), \gamma(t_2)$ belong to the same k-face of T, for some $k \leq n-1$.

We will be calling $\{T_i\}$ the ordered sequence of polytopes intersected by δ. Glue together these polytopes along their faces as follows: identify isometrically the face of T_i which contains $\delta(t_i)$ with the face of T_{i+1} which contains $\delta(t_i)$ such that these two points are identified after gluing. We call the resulting space the *developing hypersurface associated to the curve* δ and denote it by S_δ. As S_δ is isometric to a subset of the hyperbolic ball $\mathbb{H}^n$, namely, a (infinite) ideal hyperbolic polytope, we may view S_δ as a subset of $\mathbb{H}^n$. As $\partial\mathbb{H}^n$ is homeomorphic with S^{n-1}, we will identify $\partial\mathbb{H}^n$ with S^{n-1}, by means of this homeomorphism.

LEMMA 7. *Given a geodesic ray $\delta\colon [0,\infty) \longrightarrow \tilde{\Delta}$, the set of ideal vertices of $S_\delta \subset \mathbb{H}^n$,*

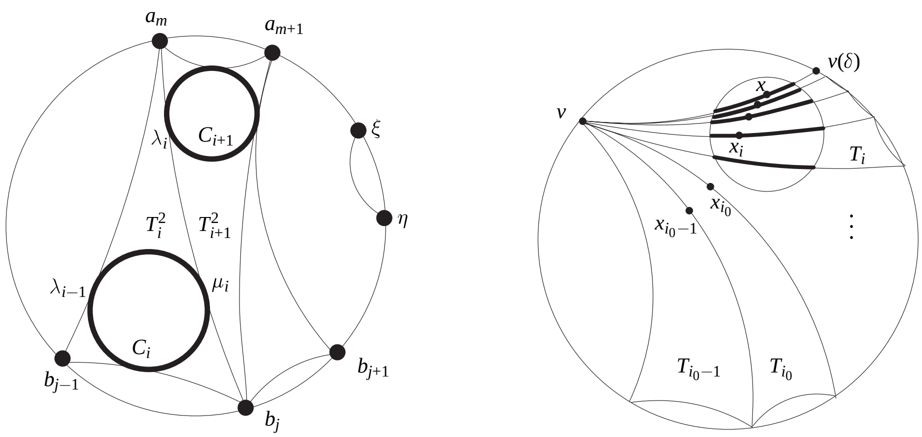
$$A = \{a_k \in \partial\mathbb{H}^n / a_k \text{ is an ideal vertex of some polytope } T_i\}$$

has exactly one accumulation point in $\partial\mathbb{H}^n$, denoted by $v(\delta)$. Moreover, if

(i) $v \in \partial\mathbb{H}^n$ such that v is the ideal vertex of infinitely many polytopes of S_δ then $v = v(\delta)$.

(ii) if $\{x_n\}$ is a sequence in $S_\delta = \bigcup_{i=0}^{\infty} T_i : x_n \in T_n, \forall n$, then $x_n \longrightarrow v(\delta)$.

https://doi.org/10.4153/CJM-1997-033-8 Published online by Cambridge University Press



(a) (b)

Figure 1

PROOF. We first prove the lemma in the case the dimension $n = 2$, since the general case is reduced to it. By construction of S_δ, it is easy to see that A has at most two accumulation points in $\partial\mathbb{H}^2$. Denote them by ξ and η. If $\eta \neq \xi$, we may write $A = \{a_m \mid m = 1, 2, \ldots\} \cup \{b_j \mid j = 1, 2, \ldots\}$, such that a_{m+1} (resp. b_{j+1}) belongs to the subarc of $\partial\mathbb{H}^2$ defined by a_m and ξ (resp. b_j and η) and not containing η (resp. ξ)—(see Figure 1a). For each triangle T_i^2 in S_δ let C_i denote the unique inscribed circle in T_i^2. Let μ_i, λ_i be the points of tangency of C_i, C_{i+1}, respectively, to the common side of T_i^2, T_{i+1}^2 (see Figure 1a). By choosing, if necessary, a subsequence we may assume that $\mu_i \longrightarrow \eta$ and $\lambda_i \longrightarrow \xi$. As a consequence $d(\mu_i, \lambda_i) \longrightarrow \infty$. The distance $d(\mu_i, \lambda_i)$ (called the *gluing weight*, *cf.* [3]) is determined by the isometry used to glue together T_i^2, T_{i+1}^2. Since X is a finite polyhedron the set

$$\{d(\mu_i, \lambda_i) : i = 1, 2, \ldots \infty\}$$

is finite. It follows that $\eta = \xi$.

We next consider arbitrary dimension n. Let $\{a_m\}$, $\{b_j\}$ be (infinite) sequences in A converging to $\eta, \xi \in \partial\mathbb{H}^2$ respectively, with $\eta \neq \xi$. By passing if necessary to subsequences of $\{a_m\}, \{b_j\}$, we may choose an ordered sequence $\{T_k^2\}_{k\in\mathbb{N}}$ of ideal 2-faces in S_δ such that

- T_k^2, T_{k+1}^2 are glued together along (a unique) common 1-face of S_δ
- each a_m, b_j, $m, j = 1, 2, \ldots$ is an ideal vertex of some T_k^2

Set $S_{\xi,\eta} = \bigcup_k T_k^2$. The embedding $S_{\xi,\eta} \hookrightarrow S_\delta$ induces a map $\partial S_{\xi,\eta} \longrightarrow \partial S_\delta$ which is a homeomorphism onto its image. By the previous case, *i.e.* when dimension $n = 2$, $\partial S_{\xi,\eta}$ has exactly one accumulation point. It follows that the set $\{a_m, b_j \mid m, j = 1, 2, \ldots\}$ has exactly one accumulation point, contradicting the assumption $\eta \neq \xi$.

Assume next that $v \neq v(\delta)$ is an ideal point in $\partial\mathbb{H}^n$ which is the ideal vertex of infinitely many polytopes T_i. By construction of S_δ, these polytopes are, necessarily, consecutive. Let $\gamma_{vv(\delta)}$ be the geodesic determined by v and $v(\delta)$. Pick a point x on the image of

https://doi.org/10.4153/CJM-1997-033-8 Published online by Cambridge University Press

$\gamma_{vv(\delta)}$ and choose a sequence $\{x_i\}$ such that x_i lies on a common 1-face of T_i and T_{i+1} and $x_i \longrightarrow x$. We will be viewing $\{x_i\}$ as a sequence in $\mathbb{H}^n, S_\delta$ and $\tilde{X}$. $\{x_i\}$ is a Cauchy sequence in $\mathbb{H}^n$ and, hence, it is Cauchy in $S_\delta \subset \tilde{X}$. Since $\tilde{X}$ is complete $\{x_i\}$ converges to some point $y \in \tilde{X}$. Let $\widetilde{V_y}$ be a neighborhood of y in $\tilde{X}$. Then $\widetilde{V_y} \cap \tilde{X}^{(1)}$, where $\tilde{X}^{(1)}$ denotes the 1-skeleton of $\tilde{X}$, has infinitely many components. Figure 1b demonstrates this situation in the hyperbolic disc $\mathbb{H}^2$ with polytopes drawn using triangles. Then $p(\widetilde{V_y})$ is a neighborhood of $p(y)$ in X. Moreover, $p(\widetilde{V_y}) \cap X^{(1)}$ has infinitely many components. This is a contradiction since X is a finite polyhedron.

Finally, let $\{x_n\}$ be a sequence S_δ with $x_n \in T_n, \forall n$. Then, the distance of each x_n from the $(n-1)$-skeleton $S_\delta^{(n-1)}$ of S_δ is bounded above for all n. Hence, we may choose a sequence $\{x_n'\} \subset S_\delta^{(n-1)}$ such that $d(x_n, x_n')$ is bounded for all n and the x_n''s belong to pairwise distinct $(n-1)$-faces of S_δ. The sequence $\{x_n'\}$ converges, necessarily, to a boundary point which, by the first part of this lemma, is $v(\delta)$. Consequently, $x_n \longrightarrow v(\delta)$. ∎

PROPOSITION 8. *There exists a continuous surjective map* $f\colon \partial\tilde{\Delta} \longrightarrow \partial\tilde{X}$.

PROOF. Let $\xi \in \partial\tilde{\Delta}$ and $\delta\colon [0,\infty) \longrightarrow \tilde{\Delta}$ a geodesic ray representing ξ. δ can be viewed as a piece-wise geodesic ray in $\tilde{X}$ and, as explained above, the developing hypersurface $S_\delta = \bigcup_{i=0}^\infty T_i$ of δ exists with the base point $x_0 \in T_0$. Denote by c_i the center of each polytope T_i. By Lemma 7, ∂S_δ is homeomorphic to $A \cup \{v(\delta)\}$. Identifying ∂S_δ with $A \cup \{v(\delta)\}$ by means of this homeomorphism, the sequence $\{c_i\}$ represents a unique element in ∂S_δ, namely $v(\delta)$. By Theorem 2.2 of [6, Chapter 3], the isometric embedding $S_\delta \hookrightarrow \tilde{X}$ induces a homeomorphism of ∂S_δ onto its image in $\partial\tilde{X}$. This homeomorphism simply maps a sequence in S_δ to itself in $\tilde{X}$. Hence $\{c_i\}$ determines an element in $\partial\tilde{X}$ which we define to be the image of ξ under f. To see that this map f is well defined, let δ' be an other geodesic ray representing ξ and defining a sequence of centers $\{c_i'\}$. Since $\operatorname{Im}\delta$ and $\operatorname{Im}\delta'$ intersect the same sequence of fundamental domains of $\tilde{\Delta}$ (which are of finite diameter), we may assume (by choosing, if necessary, subsequences) that for each i, both c_i and c_i' belong to the same fundamental domain of $\tilde{\Delta}$. Then, $d_{\tilde{X}}(c_i, c_i')$ is bounded for all i and, hence, represent the same element in $\tilde{X}$.

We next show that f is continuous. Since $\tilde{X} \cup \partial\tilde{X}$ is metrizable (see [6, page 134]), it suffices to consider sequences. Assume $\{\xi_n\}$ is a sequence in $\partial\tilde{\Delta}$ converging to a point ξ such that $\{f(\xi_n)\}$ does not converge to $f(\xi)$. By considering, if necessary, a subsequence we may assume that $f(\xi_n) \longrightarrow \eta \neq f(\xi)$ for some $\eta \in \partial\tilde{X}$.

Let $r_n, r\colon [0,\infty) \longrightarrow \tilde{\Delta}$, $n \in \mathbb{N}$ be geodesic rays representing ξ_n, ξ, respectively. Viewing again each r_n, r as a piece-wise geodesic in $\tilde{X}$, each intersects an infinite sequence of polytopes $\{T_{nj}\}_{j=0}^\infty$, $\{T_j\}_{j=0}^\infty$ of $\tilde{X}$, respectively. Let c_{nj} and d_j be the centers of the ideal polytopes T_{nj} and T_j, respectively. Apparently, as $j \longrightarrow \infty$

$$c_{nj} \longrightarrow \xi_n \text{ and } d_j \longrightarrow \xi \quad \text{in } \tilde{\Delta} \cup \partial\tilde{\Delta} \tag{3}$$

and, by definition of f

$$c_{nj} \longrightarrow f(\xi_n) \text{ and } d_j \longrightarrow f(\xi) \quad \text{in } \tilde{X} \cup \partial\tilde{X} \tag{4}$$

https://doi.org/10.4153/CJM-1997-033-8 Published online by Cambridge University Press

as $j \longrightarrow \infty$. For each $n \in \mathbb{N}$, $\exists j_n \in N : d_{\tilde{X}\cup\partial\tilde{X}}\big(c_{nj_n}, f(\xi_n)\big) < 1/n$ and $d_{\tilde{\Delta}\cup\partial\tilde{\Delta}}(c_{nj_n}, \xi_n) < 1/n$. Set $b_n = c_{nj_n}$. It follows that

$$b_n \longrightarrow \eta \quad \text{in } \tilde{X}\cup\partial\tilde{X} \text{ and } b_n \longrightarrow \xi \quad \text{in } \tilde{\Delta}\cup\partial\tilde{\Delta} \tag{5}$$

By (3) and (5) the sequences $\{b_n\}$ and $\{d_j\}$ define the same point in $\partial\tilde{\Delta}$ and, therefore,

$$(b_n, d_j)_{\tilde{\Delta}} \longrightarrow \infty \quad \text{as } n, j \longrightarrow \infty \tag{6}$$

where $(\cdot\,,\cdot)$ denotes the hyperbolic product in the sense of Gromov. Denote by $\gamma^{\tilde{\Delta}}_{b_n d_j}$ (resp. $\gamma^{\tilde{X}}_{b_n d_j}$) the geodesic segment in $\tilde{\Delta}$ (resp. $\tilde{X}$) joining b_n with d_j.

It is easy to see that the infinite graph $\tilde{\Delta}$ has the following property:

(7) any two paths in $\tilde{\Delta}$ with the same endpoints and which are length minimizing in their homotopy class (with end points fixed) intersect the same ordered sequence of fundamental domains of $\tilde{\Delta}$.

The ordered sequence of polytopes intersected by $\gamma^{\tilde{X}}_{b_n d_j}$ gives rise to a (not necessarily unique) segment with endpoints b_n, d_j in $\tilde{\Delta}$, denoted by $\delta^{\tilde{\Delta}}_{b_n d_j}$. This segment is length minimizing in its homotopy class with endpoints fixed and intersects, in the same order, all polytopes (maybe more) intersected by $\operatorname{Im}\gamma^{\tilde{X}}_{b_n d_j}$. Properties (6) and (7) imply that

$$d_{\tilde{\Delta}}(x_0, \operatorname{Im}\gamma^{\tilde{\Delta}}_{b_n d_j}) \longrightarrow \infty \quad \text{as } n, j \longrightarrow \infty$$

Since $\gamma^{\tilde{\Delta}}_{b_n d_j}$ and $\delta^{\tilde{\Delta}}_{b_n d_j}$ are uniformly bounded it follows

$$d_{\tilde{\Delta}}(x_0, \operatorname{Im}\delta^{\tilde{\Delta}}_{b_n d_j}) \longrightarrow \infty \quad \text{as } n, j \longrightarrow \infty \tag{8}$$

Let x_{nj} be the projection of x_0 on $\gamma^{\tilde{X}}_{b_n d_j}$ and let $\gamma^{\tilde{X}}_{x_0 x_{nj}}$ be the geodesic segment joining x_{nj} with x_0. As before, denote by $\delta^{\tilde{\Delta}}_{x_0}$ the geodesic in $\tilde{\Delta}$ joining x_0 with the center c_{nj} of the polytope containing x_{nj}. Finally, denote by $\gamma^{\tilde{\Delta}}_{x_0}$ the geodesic in $\tilde{\Delta}$ which projects x_0 on $\delta^{\tilde{\Delta}}_{b_n d_j}$. Then (8) implies that

$$\#(\text{of fundamental domains intersected by } \operatorname{Im}\gamma^{\tilde{\Delta}}_{x_0}) \longrightarrow \infty$$

as $n, j \longrightarrow \infty$. Property (7) implies that c_{nj} and the projection of x_0 on $\delta^{\tilde{\Delta}}_{b_n d_j}$ lie in the same fundamental domain of $\tilde{\Delta}$. Hence,

$$\#(\text{of fundamental domains intersected by } \operatorname{Im}\delta^{\tilde{\Delta}}_{x_0}) \longrightarrow \infty$$

$n, j \longrightarrow \infty$. Apparently the same holds true for the geodesic segment $\gamma^{\tilde{X}}_{b_n d_j}$ and, therefore,

$$d_{\tilde{X}}(x_0, \operatorname{Im}\gamma^{\tilde{X}}_{b_n d_j}) \longrightarrow \infty \quad \text{as } n, j \longrightarrow \infty \tag{9}$$

Since $\tilde{X}$ is a (-1)-hyperbolic geodesic space, Lemma 2.7 of [6, Chapter 3] asserts that

$$d_{\tilde{X}}(x_0, \operatorname{Im} \gamma^{\tilde{X}}_{b_n d_j}) \le (b_n, d_j)_{\tilde{X}} + 4(-1) \tag{10}$$

Hence, by (9) and (10), we have that

$$(b_n, d_j)_{\tilde{X}} \longrightarrow \infty \quad \text{as } n, j \longrightarrow \infty$$

and, hence, $\{b_n\}$ and $\{d_j\}$ represent the same point in $\partial \tilde{X}$. By (4) and (5) this is a contradiction, since $\eta \neq f(\xi)$. Hence f is continuous.

We next show that f is onto. Let $\xi \in \partial \tilde{X}$ and $\gamma_{x_0\xi}$ the unique geodesic ray from x_0 to ξ. Its image $\gamma_{x_0\xi}\big([0,\infty)\big)$ either intersects an infinite sequence of polytopes $\bigcup_{i=0}^{\infty} T_i = S_{\gamma_{x_0\xi}}$ or, there exists a polytope T_{i_0} and a positive real M such that $\gamma_{x_0\xi}\big([M,\infty)\big) \subset T_{i_0}$. In the first case, pick a sequence $\{z_i\}$ representing $\xi \in \partial \tilde{X}$ with $z_i \in \overset{\circ}{T}_i \cap \gamma_{x_0\xi}\big([0,\infty)\big)$. Join the centers c_i of the polytopes T_i to obtain a broken geodesic δ. In S_δ, the sequences $\{c_i\}$ and $\{z_i\}$ determine, by Lemma 7, the same boundary element, namely $v(\delta)$. So do their images under the topological embedding $\partial S_\delta \hookrightarrow \partial X$. Since $\{z_i\}$ represents $\xi \in \partial \tilde{X}$, δ represents a pre-image of ξ.

In the latter case, *i.e.*, when there exists a polytope T_{i_0} and $0 < M \in \mathbb{R}$ such that $\gamma_{x_0\xi}\big([M,\infty)\big) \subset T_{i_0}$, denote by v the ideal vertex of T_{i_0} determined by $\gamma_{x_0\xi}$ and let T_{i_0+1} be an ideal polytope of $\tilde{X}$ having one $(n-1)$-face in common with T_{i_0} so that this common face contains v as an ideal vertex. Continuing this way we obtain an infinite sequence of polytopes whose centers $\{c_i\}$ determine a broken geodesic δ. v is as an ideal vertex of all polytopes T_i, $i \ge i_0$. Hence, by Lemma 7, $v = v(\delta)$. As in the previous case, pick sequence $\{z_i\}$ representing $\xi \in \partial \tilde{X}$ with $z_i \in \gamma_{x_0\xi}\big([0,\infty)\big)$. In S_δ, the sequences $\{c_i\}$ and $\{z_i\}$ both represent the boundary point $v(\delta) \in \partial S_\delta$. Therefore, $\{c_i\}$ and $\{z_i\}$ represent the same element in $\partial \tilde{X}$. Since $\{z_i\}$ represents ξ, δ represents a pre-image of ξ. ∎

COROLLARY 9. *Let X be a finite n-dimensional ideal polyhedron and $\tilde{X}$ its universal covering. Then the limit set of the action of $\pi_1(X)$ on $\tilde{X}$ is the whole $\partial \tilde{X}$.*

PROOF. Let $\eta \in \partial \tilde{X}$. We will show that $\eta \in \overline{\mathrm{Fix}_h(\tilde{X})}$, where $\mathrm{Fix}_h(\tilde{X})$ is the set of boundary points of $\tilde{X}$ which are fixed by hyperbolic elements of Γ. Then $\eta \in \Lambda_{\tilde{X}}(\Gamma)$ since by Corollary 12 proven below and Proposition 3, $\overline{\mathrm{Fix}_h(\tilde{X})} = \Lambda_{\tilde{X}}(\Gamma)$. We need the following property

$$\zeta \in \mathrm{Fix}_h(\tilde{\Delta}) \Rightarrow f(\zeta) \in \mathrm{Fix}_h(\tilde{X}) \tag{11}$$

To see this assume that $\phi \in \Gamma$ is a hyperbolic isometry with $\phi(+\infty) = \zeta$. Pick a geodesic ray $\delta_{x_0\zeta}\colon [0,+\infty) \longrightarrow \tilde{\Delta}$ representing ζ with the base point x_0 being a center of a polytope. ϕ may not translate $\delta_{x_0\zeta}$. However, we may choose the base point x_0 so that $\phi^n(x_0) \in \operatorname{Im} \delta_{x_0\zeta}$, $\forall n \in \mathbb{N}$. In particular, $\{\phi^n(x_0)\}$ is a subsequence of the sequence of the centers $\{c_i\}$ of the polytopes intersected by $\operatorname{Im} \delta_{x_0\zeta}$ (*cf.* proof of Proposition 8). Hence, by definition of f, the points $\phi^n(x_0)$, $n \in \mathbb{N}$ represent $f(\zeta)$.

To complete the proof, let $\xi \in \partial \tilde{\Delta} = \Lambda_{\tilde{\Delta}}(\Gamma)$ (*cf.* property (2) above) such that $f(\xi) = \eta$. By Proposition 3, there exists a sequence of hyperbolic isometries $\{\phi_k\} \subset \Gamma$ such that

https://doi.org/10.4153/CJM-1997-033-8 Published online by Cambridge University Press

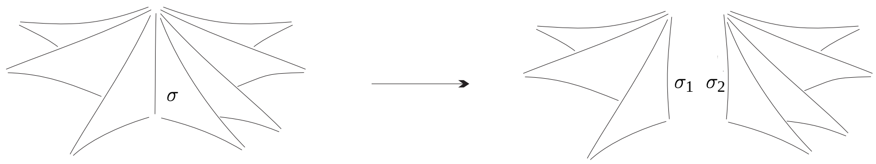


Figure 2

$\phi_k(+\infty) \longrightarrow \xi$. By property (11) $f\big(\phi_k(+\infty)\big) \in \mathrm{Fix}_h(\tilde{X})$ and by continuity of the map f, posited in Proposition 8, $f\big(\phi_k(+\infty)\big) \longrightarrow \eta$. ∎

Using Proposition 6 we obtain the following

COROLLARY 10. *Let X be a finite n-dimensional ideal polyhedron and Ω_X be the non-wandering set of the geodesic flow. Then $\Omega_X = GX$.*

4. **Proofs of theorems on** GX**.** In this section we give the proofs of Theorems 1 and 2. The following proposition is, in fact, a corollary of Theorem 2. However, we prove it first because it implies that the cardinality of the limit set $\Lambda(\Gamma)$ is infinite (see Corollary 12 below) which is necessary in order to use Propositions 3, 4, 5 and 6.

PROPOSITION 11. *GX contains infinitely many closed geodesics whose images in X are pair-wise distinct.*

PROOF. A $(n-1)$-dimensional ideal polytope σ of X is said to have *index* k_σ, $k_\sigma \in \mathbb{N}$, if k_σ $(n-1)$-faces of hyperbolic n-polytopes of X are glued together to form σ. Assume σ is an ideal $(n-1)$-polytope of X with index $k_\sigma \geq 4$. Then there exists a finite n-dimensional ideal hyperbolic space X' (not necessarily connected) and ideal $(n-1)$-polytopes σ_1, σ_2 of X' such that:

- $k_{\sigma_1} < k_\sigma$, $k_{\sigma_2} < k_\sigma$ and $k_{\sigma_1} + k_{\sigma_2} = k_\sigma$
- $\exists$ an isometry $\sigma_1 \longrightarrow \sigma_2$ such that X is obtained from X' by identifying σ_1 with σ_2 via this isometry.

Figure 2 displays (in dimension 2) how an 1-simplex σ of X gives rise to two 1-simplices σ_1, σ_2 in X' with $k_{\sigma_1} = 3$, $k_{\sigma_2} = 4$.

Applying this procedure for all ideal $(n-1)$-polytopes with index ≥ 4 repeatedly, we obtain a finite n-dimensional ideal hyperbolic space Y such that

$$k_\sigma = 2 \text{ or, } 3 \forall \text{ ideal } (n-1)\text{-polytope } \sigma \text{ of } Y$$

Denote by $\tau_1, \tau_2, \ldots, \tau_m$ the ideal $(n-1)$-polytopes of Y which have index 3. We will construct a n-dimensional manifold M out of Y, having negative curvature and finitely many cusps. Cutting along each τ_j, $j = 1, \ldots, m$, we obtain a (not necessarily connected) n-dimensional ideal hyperbolic space Z containing exactly $3m$ ideal $(n-1)$-polytopes, say $\tau_{1a}, \tau_{1b}, \tau_{1c}, \ldots, \tau_{ma}, \tau_{mb}, \tau_{mc}$, such that

https://doi.org/10.4153/CJM-1997-033-8 Published online by Cambridge University Press

- $\tau_{1a}, \tau_{1b}, \tau_{1c}, \ldots, \tau_{ma}, \tau_{mb}, \tau_{mc}$ are precisely the ideal $(n-1)$-polytopes which are free faces of some ideal n-polytopes (or polytope)
- Y is obtained from Z by gluing together appropriately each triad τ_{ja}, τ_{jb} and τ_{jc}.

Consider two copies of Z and label them Z, Z'. We will use primes to indicate to which copy (Z or Z') an ideal $(n-1)$-polytope belongs. Glue Z with Z' along their free ideal $(n-1)$-polytopes as follows:

- for each $j = 1, \ldots, m$ identify τ_{ja} with τ'_{jb}, τ_{jb} with τ'_{jc} and τ_{jc} with τ'_{ja} using the isometry which identifies τ_{ja} with $\tau_{jb} (\equiv \tau'_{jb})$ in X

Let M be the resulting space. All $(n-1)$-polytopes of M have index 2. This means that M is a finite volume manifold of negative curvature with cusps.

It is clear from the above construction that each closed geodesic γ^M in M determines a closed curve γ^X in X which locally minimizes length. Hence, γ^X is a closed geodesic in X. Moreover, if γ_1^M, γ_2^M are two closed geodesics in M, then

$$\operatorname{Im}\gamma_1^M \neq \operatorname{Im}\gamma_2^M \quad \text{in } M \Leftrightarrow \operatorname{Im}\gamma_1^X \neq \operatorname{Im}\gamma_2^X \quad \text{in } X$$

Proposition 11 now follows from the fact (see [7]) that there exist infinitely many closed geodesics in M which are pair-wise non-homotopic or, equivalently, whose images are pair-wise distinct. ∎

Let X be a n-dimensional ideal polyhedron and Γ its fundamental group acting properly by isometries on the universal cover $\tilde{X}$. Each closed geodesic g in GX determines two points in $\partial\tilde{X}$, namely $\tilde{g}(+\infty)$ and $\tilde{g}(-\infty)$. Since g is closed, there exists a hyperbolic isometry $\phi \in \Gamma$ which translates $\tilde{g}$. Hence, $\tilde{g}(+\infty), \tilde{g}(-\infty) \in \Lambda(\Gamma)$. Moreover, any two closed geodesics in X with distinct images in X are non-homotopic, hence, they determine distinct points in the boundary $\partial\tilde{X}$. Therefore, we have the following

COROLLARY 12. *Let X be an n-dimensional ideal polyhedron and Γ its fundamental group acting properly by isometries on the universal cover $\tilde{X}$. Then the cardinality of the limit set $\Lambda(\Gamma)$ is infinite.*

Set $\partial^2\tilde{X} = \{(\xi,\eta) \in \partial\tilde{X} \times \partial\tilde{X} : \xi \neq \eta\}$ and let $\rho\colon G\tilde{X} \longrightarrow \partial^2\tilde{X}$ be the fiber bundle with fiber $\mathbb{R}$ given by $\rho(g) = \big(g(-\infty), g(+\infty)\big)$. This bundle is trivial (see [9, Section 8.3]) and let $H\colon G\tilde{X} \xrightarrow{\approx} \partial^2\tilde{X} \times \mathbb{R}$ be the trivilization of ρ with respect to a base point x_0 defined by

$$H(g) = \big(g(-\infty), g(+\infty), s\big) \tag{12}$$

where s is the real number such that $d\big(x_0, g(\mathbb{R})\big) = d\big(x_0, g(-s)\big)$. We will be calling $g(-s)$ the *projection* of x_0 on the image $g(\mathbb{R})$ of the geodesic g, and denote it by $\mathrm{pr}_g(x_0)$. Denote by $\tilde{\Phi}$ the geodesic flow $\mathbb{R} \times G\tilde{X} \longrightarrow G\tilde{X}$ and let $\Psi_t\colon \partial^2\tilde{X} \times \mathbb{R} \longrightarrow \partial^2\tilde{X} \times \mathbb{R}$ be the composite of $\tilde{\Phi}$ with H, given by the formula

$$\Psi_t(\xi_1, \xi_2, s) = (\xi_1, \xi_2, s+t), \quad \text{for all } (\xi_1, \xi_2) \in \partial^2\tilde{X} \text{ and } s \in \mathbb{R}. \tag{13}$$

https://doi.org/10.4153/CJM-1997-033-8 Published online by Cambridge University Press

PROOF OF THEOREM 1. By Proposition 4 and Corollary 12, there exists an element

$$(\xi, \eta) \in \Lambda \times \Lambda \quad \text{whose } \Gamma\text{-orbit is dense in } \Lambda \times \Lambda. \tag{14}$$

Let $\tilde{\delta} = H^{-1}(\xi, \eta, s_{\tilde{\delta}})$ be the geodesic in $G\tilde{X}$ determined by ξ and η and some $s_{\tilde{\delta}} \in \mathbb{R}$—(*cf.* equation (12)). Let δ be its projection to X. We will show that the $\mathbb{R}$-orbit of δ is dense in GX.

Let $\beta \in GX$ be arbitrary and $\tilde{\beta}$ a lifting in $G\tilde{X}$. Let $H(\tilde{\beta}) = \big(\tilde{\beta}(-\infty), \tilde{\beta}(+\infty), s_{\tilde{\beta}}\big)$ for some $s_{\tilde{\beta}} \in \mathbb{R}$,—(*cf.* equation (12)). By Corollary 10, β belongs to the non-wandering set Ω_X of the flow, and therefore, by Proposition 6, the endpoints $\tilde{\beta}(-\infty)$, $\tilde{\beta}(+\infty)$ of $\tilde{\beta}$ lie in Λ. Then by (14),

$$\exists\{\phi_n\} \subseteq \Gamma : \big(\phi_n(\xi), \phi_n(\eta)\big) \longrightarrow \big(\tilde{\beta}(-\infty), \tilde{\beta}(+\infty)\big)$$

Choose $\{t_n\} \subseteq \mathbb{R}$ such that $t_n + s_{\tilde{\delta}} \longrightarrow s_{\tilde{\beta}}$. Then

$$\big(\phi_n(\xi), \phi_n(\eta), t_n + s_{\tilde{\delta}}\big) \longrightarrow \big(\tilde{\beta}(-\infty), \tilde{\beta}(+\infty, s_{\tilde{\beta}}\big)$$

as $n \longrightarrow +\infty$ and, hence, $H^{-1}\big(\phi_n(\xi), \phi_n(\eta), t_n + s_{\tilde{\delta}}\big) \longrightarrow \tilde{\beta}$. It follows that $\tilde{\Phi}_{t_n}(\phi_n\tilde{\delta}) \longrightarrow \tilde{\beta}$, where $\tilde{\Phi}$ denotes the geodesic flow on $\tilde{X}$. Thus, $\Phi_{t_n}\big(\pi_X(\phi_n\tilde{\delta})\big) \longrightarrow \pi_X(\tilde{\beta})$ and, hence $\Phi_{t_n}(\delta) \longrightarrow \beta$. ∎

PROOF OF THEOREM 2. Let $\beta \in GX$ and $\tilde{\beta} \in G\tilde{X}$ a lifting. Set $\eta = \tilde{\beta}(-\infty)$ and $\xi = \tilde{\beta}(+\infty)$. By Corollary 10 we may assume that β belongs to the non-wandering set Ω_X of the flow, and therefore, by Proposition 6, the endpoints ξ, η of $\tilde{\beta}$ lie in Λ. By Proposition 5 and Corollary 12 there exist sequences $\{x_n\}$, $\{y_n\} \subset \Lambda$ and $\{\phi_n\} \subset \Gamma$: $x_n \longrightarrow \eta, y_n \longrightarrow \xi$ and, for each $n \in N$, ϕ_n is a hyperbolic element of Γ with $\phi_n(x_n) = x_n$ and $\phi_n(y_n) = y_n$. Let $H(\tilde{\beta}) = (\xi, \eta, s)$ for some $s \in \mathbb{R}$. Denote by $\widetilde{\beta_n}$ the geodesic $H^{-1}(x_n, y_n, s_n)$ where s_n is chosen so that $s_n \longrightarrow s$. Then $\widetilde{\beta_n} \longrightarrow \tilde{\beta}$ in the $G\tilde{X}$-metric. Since ϕ_n translates $\widetilde{\beta_n}$, $\beta_n = \pi_X(\widetilde{\beta_n}) = p_X \circ \widetilde{\beta_n}$ are closed geodesics in GX, where p_X is the covering projection $p_X: \tilde{X} \longrightarrow X$. By continuity of π_X, $\beta_n \longrightarrow \beta$ with β_n being closed geodesics as required. ∎

REMARK. It is shown in [5] that Γ-conform measures exist on the boundary $\partial\tilde{\Delta}$ of (any tree) $\tilde{\Delta}$. Moreover, the product measure $\mu \times \mu$, appropriately adjusted, gives rise to a Γ-invariant measure ν' on $\partial^2\tilde{\Delta}$. If ℓ is the Lebesque measure on $\mathbb{R}$, $\nu' \times \ell$ is a Γ-invariant and geodesic invariant flow measure on $G\tilde{\Delta} \approx \partial^2\tilde{\Delta} \times \mathbb{R}$. The map f defined by Proposition 8, gives rise to a map $G\tilde{\Delta} \approx \partial^2\tilde{\Delta} \times \mathbb{R} \longrightarrow \partial^2\tilde{X} \times \mathbb{R} \approx G\tilde{X}$, which can be used to define a Borel measure on $G\tilde{X}$ which is geodesic flow invariant and Γ-invariant. Its projection to GX turns out to be a locally finite and geodesic flow invariant measure. It remains to be examined whether there exists a flow invariant *probability* measure on GX. Such a measure would be important to a further study of the geodesic flow on n-dimensional ideal polyhedra.

ACKNOWLEDGMENT. We would like to thank the referee for very helpful suggestions and observations which improved this work significantly in many ways. One of them was to generalize this work from $n = 2$ to any dimension.

REFERENCES

1. M. Bourdon, *Structure conforme au bord et flot géodésique d'un* CAT(-1) *espace*, Enseign. Math. **41**(1995), 63–102.
2. M. Bridson, *Geodesics and curvature in metric simplicial complexes*. In: Group Theory from a Geometrical Viewpoint, ICTP, Trieste, Italy, March 26–April 6, 1990, (eds. E. Ghys, A. Haefliger and A. Verjovsky), 1991, 373–463.
3. Ch. Charitos, *Closed geodesics on ideal polyhedra of dimension 2*, Rocky Mountain J. Math. (1) **26**(1996), 507–521.
4. M. Coornaert, *Sur les groupes proprement discontinus d'isométries des espaces hyperboliques au sens de Gromov*, Thèse U.L.P., Publication de l'IRMA.
5. ———, *Measures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov*, Pacific J. Math. (2) **159**(1993), 241–270.
6. M. Coornaert, T. Delzant and A. Papadopoulos, *Géométrie et théorie des groupes*, Lecture Notes in Math. **1441**, Springer-Verlag, 1990.
7. P. Eberlein, *Geodesic flows on negatively curved manifolds I*, Ann. of Math. **95**(1972), 151–170.
8. P. Eberlein and B. O'Neil, *Visibility manifolds*, Pacific J. Math. **46**(1973), 45–109.
9. M. Gromov, *Hyperbolic groups*. In: Essays in Group Theory, MSRI Publ. **8**, Springer Verlag, 1987, 75–263.
10. F. Paulin, *Constructions of hyperbolic groups via hyperbolization of polyhedra*. In: Group Theory from a Geometrical Viewpoint, ICTP, Trieste, Italy, March 26–April 6, 1990, (eds. E. Ghys and A. Haefliger), 1991.
11. J. Ratcliffe, *Foundations of hyperbolic geometry*, Graduate Texts in Math., Springer-Verlag, 1994.
12. W. P. Thurston, *The Geometry and Topology of Three-manifolds*, Lecture Notes, Princeton University, Princeton, New Jersey, 1979.

Agricultural University of Athens
Department of Mathematics
75 Iera Odos
Athens 11855
Greece
e-mail: gmat2xax@auadec.aua.ariadne-t.gr

University of the Aegean
Department of Mathematics
Karlovassi
Samos 83200
Greece
e-mail: gtsap@aegean.gr