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THE GEOMETRY OF NONLINEAR WAVE EQUATIONS IN TWO INDEPENDENT VARIABLES: METHODS FOR EXACT SOLUTION

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This thesis represents a mathematical investigation of certain wide classes of coupled systems of nonlinear wave equations in two dependent and two independent variables. Equations of this type occur in a broad range of problems in mathematical physics and in the differential geometry of 2-dimensional surfaces. The principal aim of the investigation is to develop general results and techniques for the explicit exact integration of the nonlinear systems under study. To this end we make use of the geometric theory of differential equations originally formulated by Élie Cartan early this century and later developed and applied by Ernest Vessiot in his study of the Darboux integrability of second order partial differential equations in one dependent and two independent variables.

The most important new result of this thesis is the discovery of the so called *separable Vessiot distributions* which are shown to be in correspondence with the *six*-dimensional simply transitive Lie algebras. Each such distribution is associated with one or more systems of coupled nonlinear wave equations and each of these systems are formally integrable by the classical method of Darboux. Furthermore, the correspondence

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between nonlinear systems and six-dimensional Lie algebras allows us to make use of an integration algorithm first discovered by Vessiot to reduce the integration of the original nonlinear wave equations to the integration of special systems of ordinary differential equations, the so called *Lie systems*.

We begin our investigation with a study of general coupled systems of nonlinear partial differential equations in two variables. A general theorem is obtained which allows us to recognise a given distribution of vector fields as the *Vessiot distribution* of some system of coupled partial differential equations in two independent and two dependent variables. We construct the singular subdistributions of the Vessiot distribution associated with any coupled system of nonlinear wave equations and these results are used to show that a certain wide class of nonlinear wave equations can be divided into a number of equivalence classes according to the number of the independent invariants admitted by each of the singular subdistributions of the associated Vessiot distribution. These classes can be said to be formally Darboux integrable if their singular subdistributions possess a certain minimum number of independent invariants. Thus, of the 27 distinct classes discovered, 14 of these are Darboux integrable [from the equivalence relation it follows that if an equation of a given class is Darboux integrable then so is every equation of the class, thus, the class itself is said to be Darboux integrable.]

As a first step in the detailed investigation of these Darboux integrable classes, we study in detail the class of nonlinear wave equations whose singular subdistributions each possess *three* independent invariants. It is demonstrated that these equations are in correspondence with the *six-dimensional simply transitive Lie algebras*. This fact allows us to reduce the integration of the original nonlinear wave equations to Lie systems. These systems are the widest class of ordinary differential equations known to possess nonlinear superposition principles.

To make explicit all the theoretical results of the thesis, we select a number of six-parameter Lie groups and construct their six-dimensional reciprocal simply transitive Lie algebras. We then use a method given by Vessiot to construct the associated nonlinear wave equations, and by integration to construct in the case of the nilpotent groups the exact general solution in terms of four arbitrary functions. We deal with three distinct group structures. Two of our groups are nilpotent. The first one leads to a generalisation of a canonical Darboux integrable wave equation originally discovered by Goursat and the second leads to a generalisation of the Euler-Poisson-Darboux equation which arises in a number of physical problems. We also consider the simple Lie group $SL(2,C)$ regarded as a six-parameter real Lie group locally isomorphic to the homogeneous Lorentz group $SO(3,1,R)$. In this case the equations constructed feature exponential nonlinearities and there is reason to believe that the finite non-periodic Toda lattice in the case $n = 2$ may be constructed and solved from $SL(2,C)$ using the results of this thesis. We also deal with the concept of *exact integrability*. We give a general theorem (based on a conjecture) that the equations of the class of non-linear wave equations associated with the seperable Vessiot distributions are *exactly integrable* and that their solutions may always be written in finite terms of at most four arbitrary functions and a finite number of their derivatives.

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