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Stability of Equilibrium Solutions in Planar Hamiltonian Difference Systems

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Abstract. In this paper, we study the stability in the Lyapunov sense of the equilibrium solutions of discrete or difference Hamiltonian systems in the plane. First, we perform a detailed study of linear Hamiltonian systems as a function of the parameters. In particular we analyze the regular and the degenerate cases. Next, we give a detailed study of the normal form associated with the linear Hamiltonian system. At the same time we obtain the conditions under which we can get stability (in linear approximation) of the equilibrium solution, classifying all the possible phase diagrams as a function of the parameters. After that, we study the stability of the equilibrium solutions of the first order difference system in the plane associated with mechanical Hamiltonian systems and Hamiltonian systems defined by cubic polynomials. Finally, we point out important differences with the continuous case.

1 Introduction

The importance of difference equations of the first order has grown significantly in recent years as evidenced by the large number of publications existing in the literature (see for example, [1, 15, 19]). In this paper we study particular systems of difference equations of second order called discrete or difference Hamiltonian systems, which were introduced, for example, in [2, 3, 9].

In order to describe our main contributions, we will recall the definition of discrete Hamiltonian systems. We start with linear difference Hamiltonian systems. These systems were formulated in [3], where some properties can be found.

Definition 1.1 Let $A(n), B(n), C(n) \in M_{N\times N}(\mathbb{R})$ be matrices. A *difference* (or *discrete*) *linear Hamiltonian system* is defined as the second order difference system of the form

$$\begin{aligned} \Delta x_1(n) &= A(n)x_1(n+1) + B(n)x_2(n),\\ \Delta x_2(n) &= C(n)x_1(n+1) - A^T(n)x_2(n), \end{aligned} \tag{1.1}$$

where $\Delta x(n) = x(n+1) - x(n)$, $B(n)$, and $C(n)$ are Hermitian matrices of order $N \times N$ in a domain $D \subset \mathbb{Z}$ and $I - A(n)$ is non singular in D. The system (1.1) is

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equivalent to

$$\begin{pmatrix} \Delta x_1(n) \\ \Delta x_2(n) \end{pmatrix} = M(n) \begin{pmatrix} x_1(n+1) \\ x_2(n) \end{pmatrix},$$

where

$$M(n) = \begin{pmatrix} A(n) & B(n) \\ C(n) & -A^T(n) \end{pmatrix}.$$

Bohner [6] studied the eigenvalues of the matrix $M(n)$ when this depends on one parameter and gave conditions to have a lower bound for the eigenvalues. In [2] he made an analysis of the linear Hamiltonian as a second order difference system.

Hereafter, we will assume that $D = \mathbb{Z}$. The system (1.1) naturally presents the inconvenience of defining a second order difference system, because we are interested in studying the stability of the null solution, and the majority of the strong background of stability theory is associated with difference systems of first order. It is easily verified that the system (1.1) can be reduced to a first order system, namely,

$$\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix} = S(n) \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix}, \tag{1.2}$$

where

$$S(n) = \begin{pmatrix} E(n) & F(n) \\ G(n) & H(n) \end{pmatrix},$$

with

$$E(n) = (I - A(n))^{-1}, \qquad G(n) = C(n)(I - A(n))^{-1}$$
$$F(n) = (I - A(n))^{-1}B(n), \qquad H(n) = C(n)(I - A(n))^{-1}B(n) + I - A^T(n).$$

According to [3, p. 83], the matrix

$$S(n) = \begin{pmatrix} E(n) & F(n) \\ G(n) & H(n) \end{pmatrix},$$

associated with the system (1.2) is symplectic for every n. For this reason, the study of linear Hamiltonian difference systems (1.1) is reduced to the study of linear difference systems of first order. In [2, 3] the authors studied the linear Hamiltonian systems in order to find a parallel between the continuous and discrete cases; they study the symplectic difference systems in particular. The study consists of obtaining general properties of such systems. In [14] a preliminary study of the symplectic matrix associated with the linear Hamiltonian system is made. In [17] general qualitative properties of the linear Hamiltonian systems in differences are studied; the methods that are used involve the Riccati type matrix. The authors proceed to extend the qualitative properties of the continuous and nonautonomous linear Hamiltonian systems. Zhang et al. in [26, 27] established several inequalities of the Lyapunov type for Hamiltonian linear systems in the planar case. Properties of disconjugacy for the discrete Hamiltonian systems are considered in [3,5,7,16,23]. Other studies of the linear Hamiltonian systems with time-scales are considered in [4,8]. According to [5], Hamiltonian systems in difference equations are defined as follows.

Definition 1.2 Let $H \in C^1(\mathbb{R}^{2N}, \mathbb{R})$, and denote by $\nabla H(z)$ the gradient of H in z. The difference Hamiltonian system for H is defined by

$$\Delta x(n) = J\nabla H(Lx(n)), \qquad n \in \mathbb{Z}, \tag{1.3}$$

where $x(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix}$, with $x_i(n) \in \mathbb{R}^N$, $i = 1, 2$, L is defined by $Lx(n) = \begin{pmatrix} x_1(n+1) \\ x_2(n) \end{pmatrix}$, and $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ is the standard symplectic matrix with I_N the identity matrix of order N.

Zheng [29] studied the existence of multiple periodic solutions using Morse theory. In [11] the Hamiltonian system

$$\begin{aligned} \Delta x_1(n) &= -H_{x_2}\big(n, x(n)\big), \\ \Delta x_2(n) &= H_{x_1}\big(n, x(n)\big), \end{aligned} \tag{1.4}$$

is considered, where $H(n, x(n))$ is periodic in n and superlinear when $\|x\| \to \infty$. The existence of homoclinic orbits is proved using critical point theorems.

In this paper our main objective is to study the stability of the equilibrium solution, which we suppose is $(0, 0)$, for the linear Hamiltonian system (1.1) and for the nonlinear Hamiltonian systems (1.3) in the planar and autonomous cases. Our strategy is to reduce the study of the stability of the equilibrium solution $(0, 0)$ of the two previous situations to one difference system of first order associated with the Hamiltonian system. Thus, we can apply the standard theory of difference equations in order to get our results. The analysis of the stability of the null solution of the associated Hamiltonian system is relatively new. Although there are several results in the literature about asymptotic behavior for solutions of difference equations, little is known for Hamiltonian difference systems.

To get our results, we have organized the work as follows. In Section 2 we study the stability of the null solution of a symplectic difference linear system as a function of the parameters associated to the matrix. Next, we relate the conditions of stability with the parameters of the associated linear Hamiltonian system. In particular, we characterize all the possible phase portraits of the linear system in the bi-dimensional case as a function of the involved parameters. Also, we analyze all the possible normal forms associated with the symplectic matrix and associated with the Hamiltonian linear system. Here, we emphasize that given the spectra of a symplectic matrix, where we are in the critical case; that is, the product of the eigenvalues is 1. In Section 3 we study the Lyapunov stability for the null solution of a particular, but interesting, Hamiltonian system, which is associated with a mechanical system. Different cases are analyzed and important differences are observed when they are compared with the continuous case. Finally, we analyze Hamiltonian difference systems defined by polynomials of degree three.

The proofs of our main results are achieved by the convenient use of Lyapunov theorems (see [1,15,20]) and the Chetaev Theorem (see [10,12]).

Zhang [28] studied the stability of the null solution in systems which are perturbations of linear Hamiltonian systems, but the perturbation are not necessarily Hamiltonian, as in system (1.4).

An important point is that in the study of stability in the Lyapunov sense of the null solution in a Hamiltonian system for the continuous case, the existence of resonance, the theory of normal form, and the existence of a first integral each play an important role (see details in [24, 25]).

2 Analysis of the Type of Stability of the Null Solution for Linear Hamiltonian Difference Systems in the Bidimensional Case

We intend to prove particular properties of system (1.2) coming from the system (1.1) when it is compared with the linear difference system in the general case.

It is evident that the point $(0,0)$ is a solution of (1.2). Next we are going to study the type of stability (linear) of the null solution of (1.2) for the autonomous case.

Let a linear Hamiltonian system of 2×2 be given by

$$\Delta x(n) = M \begin{pmatrix} x_1(n+1) \\ x_2(n) \end{pmatrix}, \tag{2.1}$$

$$M = \begin{pmatrix} a & c \\ b & -a \end{pmatrix}, \tag{2.2}$$

where $a, b, c \in \mathbb{R}$ and $1 - a \neq 0$. Then the associated linear symplectic system (1.2) has the form

$$x(n+1) = Sx(n), \quad n \geq n_0, \tag{2.3}$$

where the symplectic matrix is

$$S = \begin{pmatrix} \frac{1}{1-a} & \frac{b}{1-a} \\ \frac{c}{1-a} & \frac{(1-a)^2+bc}{1-a} \end{pmatrix}. \tag{2.4}$$

In [15] the existence of 11 possible phase portraits for the bidimensional linear difference system of first order are shown. In our case, the fact that the matrix S of the linear system (2.3) is symplectic implies that $\det S = 1$. Thus, if λ_1, λ_2 are its eigenvalues, then $\lambda_1\lambda_2 = 1$, and so $|\lambda_1|\,|\lambda_1| = 1$. Therefore, for the symplectic system (2.1), there are only the following possibilities:

(a) Saddle type, *i.e.,* $\lambda_1 > 1$ and $\lambda_2 < 1$. In this case the phase portrait is topologically given in Figure 1; therefore, the origin is unstable.

(b) Degenerate type, *i.e.,* $\lambda_1 = \lambda_2 = 1$ (or $\lambda_1 = \lambda_2 = -1$). In general, if the system with eigenvalues λ_1 and λ_2 is diagonalizable, it is verified that the phase portrait is given in the left side of Figure 2. Thus, the phase space is full of equilibrium solutions, and in particular, each of them is stable.

On the other hand, if the system is nondiagonalizable, we have that its phase portrait is topologically given in the right side of Figure 2. Therefore, the origin is unstable.

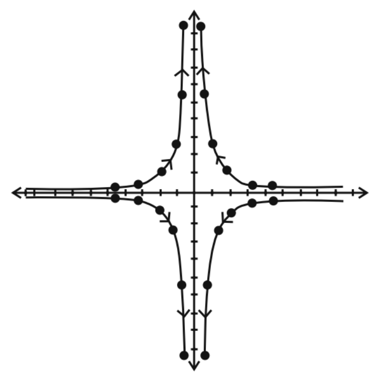

Figure 1: Saddle point (unstable) with $\lambda_1 > 1$ and $\lambda_2 < 1$.

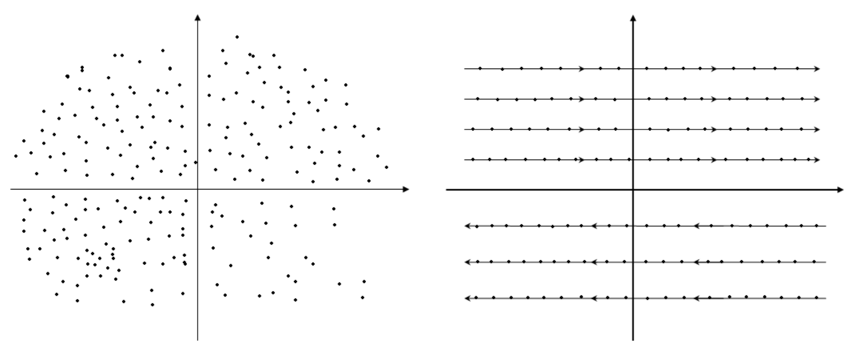

Figure 2: Degenerate cases. Left: diagonalizable, stable; Right: nondiagonalizable, unstable.

(c) Center type. In this case the eigenvalues are given by $\lambda_{1,2} = \alpha \pm i\beta$, with $\beta \neq 0$. Set $\omega = \arctan \frac{\beta}{\alpha}$. Then the solution of the system

$$\begin{pmatrix} y_1(n+1) \\ y_2(n+1) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} y_1(n) \\ y_2(n) \end{pmatrix}$$

is given by

$$y_1(n) = |\lambda_1|^n \big(y_{10}\cos(n\omega) + y_{20}\sin(n\omega) \big)$$
$$y_2(n) = |\lambda_1|^n \big(-y_{10}\sin(n\omega) + y_{20}\cos(n\omega) \big).$$

In this case $|\lambda_1| = 1$, and we obtain an equilibrium that is a center where all the orbits are circular with radius $r_0 = \sqrt{y_{10}^2 + y_{20}^2}$, and the phase portrait is topologically given in Figure 3. Therefore, the origin is stable. We should emphasize that for a symplectic 2×2 matrix, it is not possible to have the situation $\lambda \in \mathbb{C}$ with

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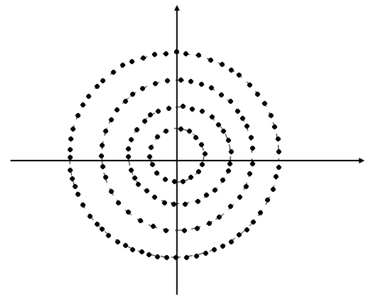

Figure 3: Center with $|\lambda_1| = 1$.

$|\lambda| \neq 1$. In fact, the nature of the symplectic matrices implies that if λ is complex, then $1/\lambda$ is an eigenvalue and $\overline{\lambda}$ and $1/\overline{\lambda}$ are also eigenvalues.

We will denote by $\mu_{1,2} = \pm\sqrt{a^2 + bc}$ the eigenvalues associated with matrix (2.2). On the other hand, the eigenvalues associated with the symplectic matrix (2.4) are

$$\lambda_{1,2} = \frac{1 + (1-a)^2 + bc \pm \sqrt{(1 + (1-a)^2 + bc)^2 - 4(1-a)^2}}{2(1-a)}. \tag{2.5}$$

Considering $\Delta_M = a^2 + bc$, it is verified that

$$\Delta_S = \Delta_M \left[\Delta_M + 4(1-a)\right].$$

By virtue of the previous properties we have the following result.

Proposition 2.1 *If $\Delta_M = 0$, then $\lambda_{1,2} = 1$. Moreover, if $b = 0$, then the following hold:*

(i) *If $c = 0$, then the null solution of (2.3) is stable and is of degenerate type and diagonalizable (see Figure 2).*

(ii) *If $c \neq 0$, then the null solution of (2.3) is unstable and is of the degenerate type and nondiagonalizable (see Figure 2).*

In the case $b \neq 0$, the null solution of (2.3) is unstable and is of degenerate type and non diagonalizable (see Figure 2).

Proof Since $\Delta_M = 0$, it can be verified from (2.5) that $\lambda_{1,2} = 1$. To prove the other items, we note that if $\Delta_M = 0$ and $b = 0$, then $a = 0$. Thus, the matrix S in (2.4) assumes the form

$$S = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.$$

Then, under the condition $c = 0$, proof of item (i) follows. In the case $c \neq 0$, S is not diagonalizable, and therefore we obtain the conclusion of item (ii).

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The condition $b \neq 0$ implies that $c = -\frac{a^2}{b}$, and then we have that the symplectic matrix S in (2.4) has the form

$$S = \begin{pmatrix} \frac{1}{1-a} & \frac{b}{1-a} \\ -\frac{a^2}{b(1-a)} & \frac{1}{1-a} \end{pmatrix}.$$

Then the eigenspace associated with the eigenvalue 1 has dimension one, which implies that the matrix is not diagonalizable, and therefore the null solution is unstable. Thus, we conclude the proof of the proposition. ∎

Now consider the following situation.

Proposition 2.2 (i) *Assume that $\Delta_M > 0$ and that $1 - a > 0$. Then the equilibrium solution $(0,0)$ of (2.3) is unstable and is of saddle type (see Figure 1).*

(ii) *Assume that $\Delta_M > 0$ and $1 - a < 0$.*

(a) *If $\Delta_M + 4(1-a) > 0$, then the equilibrium solution $(0,0)$ of (2.3) is unstable and is of saddle type (see Figure 1).*

(b) *If $\Delta_M + 4(1-a) < 0$, then the equilibrium solution $(0,0)$ of (2.3) is stable and is of center type (see Figure 3).*

(c) *If $\Delta_M + 4(1-a) = 0$, then the equilibrium solution $(0,0)$ of (2.3) is unstable if $b \neq 0$. For $b = 0$ and $c = 0$ the equilibrium point is stable, and if $c \neq 0$ the equilibrium solution is unstable.*

Proof In (i) it is verified that

$$\lambda_{1,2} = 1 + \frac{\Delta_M}{2(1-a)} \pm \frac{\sqrt{\Delta_M\left[\Delta_M + 4(1-a)\right]}}{2(1-a)},$$

and clearly

$$\lambda_1 = 1 + \frac{\Delta_M}{2(1-a)} + \frac{\sqrt{\Delta_M\left[\Delta_M + 4(1-a)\right]}}{2(1-a)} > 1.$$

So, (i) is proved.

The proof of the case (ii)(a) is obtained by observing that the eigenvalue

$$\lambda_2 = 1 + \frac{\Delta_M}{2(1-a)} - \frac{\sqrt{\Delta_M\left[\Delta_M + 4(1-a)\right]}}{2(1-a)} > 1.$$

For case (ii)(b) we have that the eigenvalues are given by

$$\lambda_{1,2} = 1 + \frac{\Delta_M}{2(1-a)} \pm \frac{i\sqrt{-\Delta_M\left[\Delta_M + 4(1-a)\right]}}{2(1-a)},$$

which satisfy $|\lambda_{1,2}| = 1$. Thus, we verify that the matrix S is semisimple, and therefore the null solution must be stable.

For case (ii)(c) we have that $\lambda_{1,2} = -1$. Then the type of stability of the null solution is decided according to the diagonalization of the matrix S. It follows that $(a-2)^2 = -bc$; that is, $a = 2 \pm \sqrt{-bc}$, so there are two possibilities for the matrix S in (2.5), namely,

$$S = \begin{pmatrix} \frac{-1}{1+\sqrt{-bc}} & \frac{-b}{1+\sqrt{-bc}} \\ \frac{-c}{1+\sqrt{-bc}} & -2+\frac{1}{1+\sqrt{-bc}} \end{pmatrix} \quad \text{or} \quad S = \begin{pmatrix} \frac{-1}{1-\sqrt{-bc}} & \frac{-b}{1-\sqrt{-bc}} \\ \frac{-c}{1-\sqrt{-bc}} & -2+\frac{1}{1-\sqrt{-bc}} \end{pmatrix}.$$

It is verified that for $b \neq 0$ the eigenspace associated with the eigenvalue -1 (of algebraic multiplicity 2) has dimension 1, where the solution $(0,0)$ of (1.2) is unstable. For the case $b = 0$ (*i.e.*, $a = 2$) it is shown that if $c = 0$, the solution $(0,0)$ of (1.2) is stable, and in the case $c \neq 0$, the solution $(0,0)$ of (1.2) is unstable. This concludes the proof. ∎

Remark 2.3 It is important to call the attention to the fact that condition

$$\Delta_M + 4(1-a) < 0,$$

could be true, since

$$\Delta_M + 4(1-a) = a^2 + bc + 4 - 4a = (a-2)^2 + bc,$$

which holds if $bc < 0$ and $|bc| > (a-2)^2$.

Now, we will proceed to the analysis of the case $\Delta_M < 0$.

Proposition 2.4 *Assume that $\Delta_M < 0$.*

(i) *If $1-a<0$, then the null solution $(0,0)$ of (2.3) is unstable and is of saddle type (see Figure 1).*

(ii) *If $1-a>0$ and*

(a) *$\Delta_M + 4(1-a) > 0$, then the equilibrium $(0,0)$ of (2.3) is stable and is of the center type (see Figure 3);*

(b) *$\Delta_M + 4(1-a) < 0$, then the equilibrium $(0,0)$ of (2.3) is unstable and is of saddle type (see Figure 1);*

(c) *$\Delta_M + 4(1-a) = 0$, then the equilibrium $(0,0)$ of (2.3) is unstable and is of degenerate type and nondiagonalizable (see Figure 2).*

Proof To prove item (i), we observe that $\Delta_M[\Delta_M + 4(1-a)] > 0$ and

$$\lambda_2 = 1 + \frac{\Delta_M}{2(1-a)} - \frac{\sqrt{\Delta_M[\Delta_M + 4(1-a)]}}{2(1-a)} > 1.$$

Therefore, the origin $(0,0)$ is unstable and is of saddle type.

For (ii)(a), from the hypotheses, we have that $\Delta_M[\Delta_M + 4(1-a)] < 0$, and so its eigenvalues are of the form

$$\lambda_{1,2} = 1 + \frac{\Delta_M}{2(1-a)} \pm i\frac{\sqrt{-\Delta_M[\Delta_M + 4(1-a)]}}{2(1-a)}.$$

They satisfy $|\lambda_{1,2}| = 1$ and are distinct. In conclusion, the solution $(0,0)$ of (1.2) is stable and is of center type.

For (ii)(b) from the hypotheses, we note that $1 + \frac{\Delta_M}{2(1-a)} < -1$, and so

$$\lambda_2 = 1 + \frac{\Delta_M}{2(1-a)} - \frac{\sqrt{\Delta_M\left[\Delta_M + 4(1-a)\right]}}{2(1-a)} < -1.$$

Thus, $|\lambda_1| > 1$, where we conclude that the equilibrium $(0,0)$ of (1.2) is unstable and is of saddle type.

Finally, for the case (ii)(c) we know that $\Delta_M = -4(1-a)$, and so $\lambda_{1,2} = -1$. Analogously to the proof of Proposition 2.2(ii)(c), we prove that the equilibrium $(0,0)$ of (1.2) is unstable, degenerate, and nondiagonalizable. ∎

Taking into account the expression of Δ_M and of Δ_S, we can point out the following properties.

(a) If $\Delta_M = 0$, then $\Delta_S = 0$.
(b) If $\Delta_S = 0$, then $\Delta_M = 0$ or $\Delta_M = 4(a-1)$.
(c) If $\Delta_M = 0$, then $\lambda_{1,2} = 1$ or system (1.2) is degenerate.
(d) If $\Delta_M > 0$, then $\mu_1 \in \mathbb{R}^+$ and $\mu_2 \in \mathbb{R}^-$.

In synthesis, the stability of the null solution for a planar and autonomous linear symplectic difference system associated to a linear Hamiltonian difference system is summarized as shown in Figure 4.

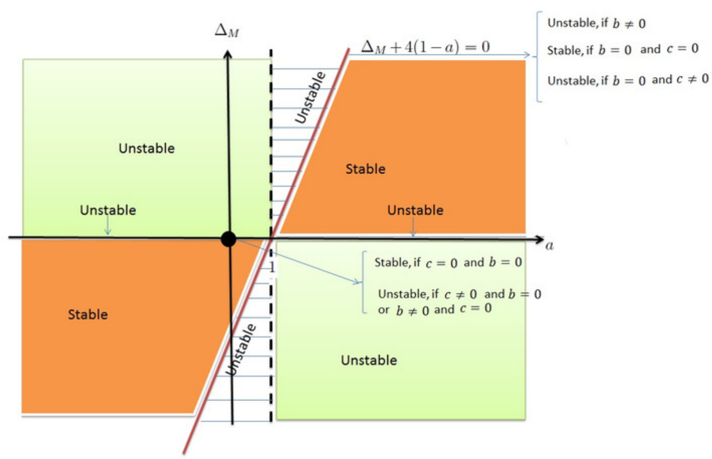

Figure 4: Type of stability for the null solution for planar and autonomous linear symplectic difference system.

2.1 Analysis of the Stability of the Null Solution of the Planar System (2.3) through Normal Form of Symplectic Matrices

The study of stability of the equilibrium solution $(0,0)$ can be also considered through the process of normalization of symplectic matrices applied to S defined by equation

(2.3). The study of the normal form has been considered, for example, in [21, 22, 24]. Here, all the possible scenarios according to the eigenvalues of the matrix S are analyzed. In our study, we will focus our attention to the case $S \in M_{2\times 2}(\mathbb{R})$. The details of our arguments are discussed with depth in [25] and [22].

Let $x(n+1) = Sx(n)$, $x(n) \in \mathbb{R}^2$ be a difference system with S a symplectic matrix as in (2.4). We characterize the possible normal form as follows:

Case 1. $\lambda_1 \in \mathbb{R} - \{0\}$ and $\lambda_2 = \frac{1}{\lambda_1}$, where $\lambda_1 \neq 1$. According to the theory of normal forms, the possible normal forms are:

$$\mathcal{S}_1 = \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda^{-1} \end{pmatrix} \qquad \text{or} \qquad \mathcal{S}_2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}.$$

For these symplectic matrices, according to (2.4), we obtain that the associated Hamiltonian matrices are

$$\mathcal{H}_1 = \begin{pmatrix} \frac{\lambda_1 - 1}{\lambda_1} & 0 \\ \frac{1}{\lambda} & \frac{1-\lambda_1}{\lambda_1} \end{pmatrix} \qquad \text{or} \qquad \mathcal{H}_2 = \begin{pmatrix} \frac{\lambda_1 - 1}{\lambda_1} & 0 \\ 0 & \frac{1-\lambda_1}{\lambda_1}, \end{pmatrix}$$

respectively.

Case 2. $\lambda_1 = \lambda_2 = 1$. Here the possible normal forms for S are

$$\mathcal{S}_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{or} \qquad \mathcal{S}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then according to (2.4), the Hamiltonian matrices are

$$\mathcal{H}_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{or} \qquad \mathcal{H}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0, \end{pmatrix}$$

respectively.

Case 3. If $\lambda_1 = \lambda_2 = -1$, then normal forms of S are

$$\mathcal{S}_1 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \qquad \text{or} \qquad \mathcal{S}_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then according (2.4), the Hamiltonian matrices are

$$\mathcal{H}_1 = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \qquad \text{or} \qquad \mathcal{H}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2, \end{pmatrix}$$

respectively.

Case 4. If $\lambda_1 = i\beta$ and $\lambda_2 = -i\beta$, then the normal form of the matrix S is given by

$$\mathcal{S} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}.$$

Notice that if we consider the previous expression for S, it is not possible to have this form $\mathcal{S}$, because in this case we obtain that $\frac{1}{1-a} = 0$, which is impossible.

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3 Study of Nonlinear Planar Hamiltonian Systems

Here we will point out some results that characterize the stability in the sense of Lyapunov of one equilibrium point of a Hamiltonian system, which are associated with mechanical systems. Moreover, we study the stability of the equilibrium solutions defined by Hamiltonian functions defined by cubic polynomials. Initially, we remember that a mechanical system corresponds to the differential equation of second order

$$\ddot{x}_1 = \nabla U(x_1), \tag{3.1}$$

where $U: \mathbb{R}^N \setminus S \to \mathbb{R}$ is a differentiable function (at least of class C^1) to real values and S denotes the set of singularities of U. In this case, U is called a potential function and the associated Hamiltonian function is given by

$$H = H(x_1, x_2) = \frac{1}{2}\|x_2\|^2 - U(x_1), \tag{3.2}$$

where the variable x_2 is called conjugate variable.

In order to compare our results with the continuous case for mechanical systems of the form (3.1), we first recall the Dirichlet Theorem. It was proved by Dirichlet [13] and formulated by Lagrange [18].

Theorem 3.1 *Let x_1^* be an isolated critical point that is a local maximum of U. Then the equilibrium solution $(x_1^*, 0)$ of the Hamiltonian system associated with (3.2) is stable.*

Proof This result is a consequence of Lyapunov's Theorem, defining the Lyapunov function as $V(x_1, x_2) = H(x_1, x_2) + U(x_1^*)$. For more details of the proof, see [25]. ∎

Considering the Hamiltonian function given in (3.2), the difference Hamiltonian system is given by

$$\begin{aligned} \Delta x_1(n) &= x_2(n), \\ \Delta x_2(n) &= \nabla U\big(x_1(n+1)\big). \end{aligned} \tag{3.3}$$

Since $\Delta x(n) = x(n+1) - x(n)$, it follows that system (3.3) is of first order, which assumes the form

$$\begin{aligned} x_1(n+1) &= x_1(n) + x_2(n), \\ x_2(n+1) &= x_2(n) + \nabla U\big(x_1(n) + x_2(n)\big). \end{aligned} \tag{3.4}$$

Next, we are going to analyze the planar case $H = \frac{1}{2}x_2^2 - U(x_1)$, where U is a differentiable function. In particular, system (3.4) is reduced to

$$\begin{aligned} x_1(n+1) &= x_1(n) + x_2(n), \\ x_2(n+1) &= x_2\big(n\big) + U'\big(x_1(n) + x_2(n)\big) \end{aligned} \tag{3.5}$$

The characterization of the equilibrium solution of the system (3.5) is the following proposition.

Proposition 3.2 *The equilibrium solutions of the system (3.5) are characterized by (x_1^*, x_2^*) where x_1^* is a critical point of U and $x_2^* = 0$.*

Proof Set $f(x_1, x_2) = (x_1 + x_2, x_2 + U'(x_1 + x_2))$. Then $f(x_1, x_2) = (x_1, x_2)$ if and only if $x_1 + x_2 = x_1$, $x_2 + U'(x_1 + x_2) = x_2$. Then $x_2 = 0$ y $U'(x_1) = 0$. ■

From now on we will assume without loss of generality that $(0,0)$ is an equilibrium solution for the nonlinear system (3.5), *i.e.,* 0 is a critical point of U. We are interested in determining the type of stability of the solution $(0,0)$. We are going to analyze an analogous result to the Dirichlet Theorem in the discrete case for the planar system (3.5).

Initially we propose the linear stability of the equilibrium solution $(0,0)$ of the mechanical system (3.5). The linearized system (3.5) around the point $(0,0)$ and denoting $\mu = U''(0)$, it follows that the linear part is the symplectic matrix $\begin{pmatrix} 1 & 1 \\ \mu & 1+\mu \end{pmatrix}$. Thus the associated eigenvalues are

$$\lambda_{1,2} = 1 + \frac{\mu}{2} \pm \frac{\sqrt{\mu(\mu+4)}}{2}.$$

According to the notation for the associated symplectic matrix S in (2.4) we must have that $a = 0$, $b = 1$, and $c = 1 + \mu$.

Theorem 3.3 (i) *If $\mu > 0$, then the null solution of* (3.5) *is linearly unstable (of saddle type) and is also unstable in the Lyapunov sense.*

(ii) *If $\mu = 0$, then the null solution of* (3.5) *is linearly unstable (degenerate case).*

(iii) *If $\mu \in (-4, 0)$, then the null solution of* (3.5) *is linearly stable (center type). If $\mu \in (-\infty, -4)$, then the null solution of* (3.5) *is linearly unstable (saddle type) and is also unstable in the Lyapunov sense. If $\mu = -4$, then the null solution of* (3.5) *is linearly unstable (degenerate case).*

Proof The proof of item (i) is clear, because if $\mu > 0$, then $\lambda_1 > 1$.

In item (ii) we have that $\Delta_M = 0$, so $\lambda_{1,2} = 1$, and by Proposition 2.1 it follows that $(0,0)$ is linearly unstable (degenerate case, the linear part is non diagonalizable).

For the item (iii) and $\mu \in (-4, 0)$, by Proposition 2.4(i) it follows that $(0,0)$ is linearly stable, since the eigenvalues are

$$\lambda_{1,2} = 1 + \frac{a}{2} \pm \frac{i\sqrt{-a(a+4)}}{2},$$

i.e., of the center type. For $\mu \in (-\infty, -4)$, by Proposition 2.4(ii), we have that $(0,0)$ is linearly unstable (saddle type), and thus is unstable in the Lyapunov sense. Finally, if $\mu = -4$, it follows that $\lambda_{1,2} = -1$, so by Proposition 2.4(iii) we have that $(0,0)$ is linearly unstable (degenerate, nondiagonalizable linear part). ■

Remark 3.4 The condition $\mu > 0$ in the previous proposition in particular tells us that 0 is a local minimum of U. Thus our result shows that there is a great difference between the continuous and the discrete case, because in the continuous case the equilibrium $(0,0)$ is always stable in the Lyapunov sense.

To complete the study of the nonlinear stability of the solution $(0,0)$ of (3.5) by Theorem 3.3, we need to analyze the case $\mu \in [-4, 0]$.

Theorem 3.5 *If $\mu \in (-4,0)$ with U such that $U(0) = 0$ and analytic in a neighborhood of 0, then the origin of system (3.5) is unstable in the Lyapunov sense.*

Proof We consider the quadratic form $V(x_1,x_2) = \alpha x_1^2 + \beta x_1^2$, with α and β chosen conveniently. Then if we take $\mu = U''(0)$ and calculating the difference of V, we have

$$\begin{aligned}
\Delta V(x_1,x_2) &= V\big(x_1+x_2, x_2+U'(x_1+x_2)\big) - V(x_1,x_2)\\
&= \alpha(x_1+x_2)^2 + \beta\big(x_2+U'(x_1+x_2)\big)^2 - \alpha x_1^2 - \beta x_2^2\\
&= 2\alpha x_1x_2 + \alpha x_2^2 + 2\beta x_2 U'(x_1+x_2) + \beta\big(U'(x_1+x_2)\big)^2\\
&= 2\alpha x_1x_2 + \alpha x_2^2 + 2\beta x_2 \mu(x_1+x_2) + \beta\big(\mu(x_1+x_2)\big)^2\\
&= 2\alpha x_1x_2 + \alpha x_2^2 + 2\beta\mu x_1x_2 + 2\beta\mu x_2^2 + \beta\mu^2 x_1^2 + 2\beta\mu^2 x_1x_2\\
&\quad + \beta\mu^2 x_2^2 + O(\|x\|^3)\\
&= \beta\mu^2 x_1^2 + 2(\alpha+\beta\mu+\beta\mu^2)x_1x_2 + (\alpha+\beta\mu+\beta\mu^2)x_2^2 + O(\|x\|^3).
\end{aligned}$$

For the case $\mu \in (-1,0)$ we take $\alpha > 0$ and $\beta > 0$ such that $-\frac{1}{\mu} < \frac{\beta}{\alpha} < -\frac{1}{\mu(1+\mu)}$. With this election, we verify that V is definite positive in a neighborhood of the origin and the quadratic form $\beta\mu^2x_1^2 + 2(\alpha+\beta\mu+\beta\mu^2)x_1x_2 + (\alpha+\beta\mu+\beta\mu^2)x_2^2$ is positive definite by the Hurwitz criterion. In fact, $\beta\mu^2 > 0$ and

$$\det\begin{pmatrix} \beta\mu^2 & \alpha+\beta\mu+\beta\mu^2\\ \alpha+\beta\mu+\beta\mu^2 & \alpha+\beta\mu+\beta\mu^2\end{pmatrix} = -(\alpha+\beta\mu)(\alpha+\beta\mu+\beta\mu^2),$$

which is positive because of the hypotheses. In the case $\mu \in (-4,-1)$ we take $\beta < -\frac{\alpha}{\mu(1+\mu)}$, and in particular, β is negative. Thus, in a neighborhood of the origin, ΔV is negative definite and V takes positive values. Finally, by the Theorem of Instability of Lyapunov, we conclude that the origin of the system (3.5) is unstable in the Lyapunov sense. ∎

3.1 Hamiltonian Polynomials of Degree 3

We consider the Hamiltonian function

$$H = \frac{\beta}{2}x_2^2 + \alpha x_1x_2 + \frac{\gamma}{2}x_1^2 + \frac{e}{3}x_2^3.$$

Thus, the associated Hamiltonian system is given by

$$\begin{aligned}
\Delta x_1(n+1) &= \beta x_1(n+1) + \gamma x_2(n) + e x_2^2(n),\\
\Delta x_2(n+1) &= -\alpha x_1(n+1) - \beta x_2(n).
\end{aligned}$$

It can be reduced to the first order difference system

$$\begin{aligned}
x_1(n+1) &= \frac{1}{1-\beta}x_1(n) + \frac{\gamma}{1-\beta}x_2(n) + \frac{e}{1-\beta}x_2^2(n)\\
x_2(n+1) &= -\frac{\alpha}{1-\beta}x_1(n) + \Big(1-\beta-\frac{\alpha\gamma}{1-\beta}\Big)x_2(n) - \frac{\alpha e}{1-\beta}x_2^2(n).
\end{aligned} \tag{3.6}$$

It is clear that $(x_1(n), x_2(n)) = (0,0)$ is an equilibrium solution of (3.6).

Theorem 3.6 *For $\beta \neq 1$, $\alpha\gamma < 0$, and $e \in \mathbb{R} - \{0\}$ the null solution of (3.6) is unstable in the Lyapunov sense.*

Proof Let $V(x_1, x_2) = x_1 x_2$ be a Chetaev's function, which is positive in the first and third quadrants, and is negative in the second and fourth quadrants. It is verified that

$$\begin{aligned}\Delta V(x_1,x_2) &= V(f(x_1,x_2)) - V(x_1,x_2)\\ &= -\frac{\alpha}{(1-\beta)^2}x_1^2 + \Big(\gamma - \frac{\alpha\gamma^2}{(1-\beta)^2}\Big)x_2^2 - \frac{2\alpha\gamma}{(1-\beta)^2}x_1x_2\\ &\quad + \Big(e + \frac{2\alpha\gamma e}{(1-\beta)^2}\Big)x_2^3 + \frac{2\alpha e}{(1-\beta)^2}x_1x_2^2 + \frac{\alpha e^2}{(1-\beta)^2}x_2^4.\end{aligned}$$

Considering only the first quadrant, it is enough to consider $\beta \neq 1$, $\alpha < 0$, $\gamma > 0$ and $e > 0$. In the third quadrant, we take $\beta \neq 1$, $\alpha < 0$, $\gamma > 0$, and $e < 0$. Therefore, in both cases, in the region $V > 0$, we must have that ΔV is positive definite, so by the Theorem of Chetaev, we conclude that the null solution of system (3.6) is unstable.

For the second quadrant, the convenient parameters are $\beta \neq 1$, $\alpha > 0$, $\gamma < 0$, and $e < 0$. In the fourth quadrant we take $\beta \neq 1$, $\alpha > 0$, $\gamma < 0$, and $e > 0$. Then in both cases, in the region $V < 0$, we have that ΔV is negative definite, so by the Theorem of Chetaev, we conclude that the null solution of the system (3.6) is unstable. ■

Next, we study the Hamiltonian system, where the Hamiltonian is given by

$$H(x_1,x_2) = \frac{\alpha}{2}x_1^2 + \beta x_1x_2 + \gamma x_2^2 + p(x_2),$$

where $p(x_2)$ is a polynomial starting with terms of order greater or equal to 3 in the variable x_2. We will assume that $\beta \neq 1$. Thus, the associated Hamiltonian system is given by

$$\begin{aligned}x_1(n+1) &= \frac{1}{1-\beta}x_1(n) + \frac{\gamma}{1-\beta}x_2(n) + \frac{1}{1-\beta}p'(x_2(n)),\\ x_2(n+1) &= -\frac{\alpha}{1-\beta}x_1(n) + \Big(-\frac{\alpha\gamma}{1-\beta} + 1 - \beta\Big)x_2(n) - \frac{\alpha}{1-\beta}p'(x_2(n)).\end{aligned}\tag{3.7}$$

It is clear that the origin is an equilibrium solution of system (3.7), since $p'(0) = 0$. In order to study the type of stability of the solution $(0,0)$ we will analyze the Chetaev function $V(x_1,x_2) = x_1x_2$. It is clear that this function is positive in the first and third quadrants. Moreover, through the solutions of (3.7), we have that

$$\begin{aligned}\Delta V(x_1,x_2) &= -\frac{\alpha}{(1-\beta)^2}x_1^2 - \frac{2\alpha\gamma}{(1-\beta)^2}x_1x_2 + \Big(\frac{-\alpha\gamma^2}{(1-\beta)^2} + \gamma\Big)x_2^2\\ &\quad - \frac{2\alpha}{(1-\beta)^2}x_1p'(x_2) + \Big(-\frac{2\alpha\gamma}{(1-\beta)^2} + 1\Big)x_2p'(x_2) - \frac{\alpha}{(1-\beta)^2}\big(p'(x_2)\big)^2.\end{aligned}\tag{3.8}$$

To decide the sign of ΔV, we will first analyze the particular case where $p(x_2) = \frac{e}{3}x_2^3 + \frac{f}{5}x_2^5$, so we can formulate the following result.

Theorem 3.7 *Assume $\beta \neq 1$, $\alpha\gamma < 0$, and $p(x_2) = \frac{e}{3}x_2^3 + \frac{f}{5}x_2^5$ such that $e \cdot f > 0$. Then the null solution of the system (3.7) is unstable in the Lyapunov sense.*

Proof By the form of p_2 it is verified in (3.8) that ΔV assumes the form

$$\begin{aligned}\Delta V(x_1,x_2) = &-\frac{\alpha}{(1-\beta)^2}x_1^2 - \frac{2\alpha\gamma}{(1-\beta)^2}x_1x_2 + \Big(-\frac{\alpha\gamma^2}{(1-\beta)^2}+\gamma\Big)x_2^2\\ &-\frac{2\alpha}{(1-\beta)^2}x_1(ex_2^2+fx_2^4) + \Big(\frac{-2\alpha\gamma}{(1-\beta)^2}+1\Big)x_2(ex_2^2+fx_2^4)\\ &-\frac{\alpha}{(1-\beta)^2}(ex_2^2+fx_2^4)^2.\end{aligned}$$

In the first quadrant, considering the restrictions $\alpha < 0$, $\gamma > 0$, $e > 0$, and $f > 0$, we have that ΔV is positive definite. For the third quadrant, taking $\alpha < 0$, $\gamma > 0$, $e < 0$ and $f < 0$, we get that ΔV is positive definite.

In the second quadrant it is enough to consider $\alpha > 0$, $\gamma < 0$, $e < 0$, and $f < 0$, while in the fourth quadrant we take $e > 0$, $f > 0$, and then we will have that ΔV is negative definite.

The conclusion of the theorem follows from Chetaev's Theorem. ∎

Remark 3.8 The previous theorem can be generalized if we take $p(x_2)$ such that

$$p(x_2) = \sum_{j=1}^{n}\frac{\alpha_j x_2^{2j+1}}{2j+1},$$

where $p(x_2)$ is an arbitrary polynomial at least of degree 3 without variations of sign. Under these conditions it is verified that the null solution of system (3.7) is unstable. In a more general way, we can take a polynomial $p(x_2)$ such that $p'(x)$ is positive or negative depending on the sign of the parameters α and γ.

Now, we consider the Hamiltonian function

$$H(x_1,x_2) = \frac{\alpha}{2}x_2^2 + \beta x_1x_2 + \frac{\gamma}{2}x_1^1 + \frac{f}{3}x_1^3 + \frac{e}{3}x_2^3 + gx_1^2x_2 + hx_1x_2^2.$$

Thus, the Hamiltonian system is given by

$$\begin{aligned}\Delta x_1(n) &= \beta x_1(n+1) + \gamma x_2(n) + gx_1^2(n+1) + 2hx_1(n+1)x_2(n) + ex_2^2(n),\\ \Delta x_2(n) &= -\alpha x_1(n+1) - \beta x_2(n) - fx_1^2(n+1) - 2gx_1(n+1)x_2(n) - hx_2^2(n).\end{aligned}$$

Assuming that $g = 0$, the associated system of first order assumes the form

$$\begin{aligned}x_1(n+1) &= \frac{x_1(n)+cx_2(n)+ex_2^2(n)}{1-\beta-2hx_2(n)},\\ x_2(n+1) &= -\alpha\Big[\frac{x_1(n)+\gamma x_2(n)+ex_2^2(n)}{1-\beta-2hx_2(n)}\Big] + [1-\beta-hx_2(n)]x_2(n)\\ &\quad - f\Big[\frac{x_1(n)+cx_2(n)+ex_2^2(n)}{1-\beta-2hx_2(n)}\Big]^2,\end{aligned}\tag{3.9}$$

where clearly $x_2(n)$ must be different than $\frac{1-\beta}{2h}$. By virtue of this restriction we must assume that $\beta \neq 1$.

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For the case $g \neq 0$, the associated difference system of first order is

(3.10)

$$\begin{aligned} x_1(n+1) &= \frac{1-\beta-2hx_2(n)}{2g} \\ &\quad \pm \sqrt{-\frac{x_1(n)}{g} - \frac{\gamma x_2(n)}{g} - \frac{ex_2^2(n)}{g} + \frac{[1-\beta-2hx_2(n)]^2}{4g^2}} \\ x_2(n+1) &= \Big[-\alpha - 2gx_2(n) - \frac{f(1-\beta-2hx_2(n))}{g}\Big] \\ &\quad\times \Bigg[\frac{1-\beta-2hx_2(n)}{2g} \pm \sqrt{-\frac{x_1(n)}{g} - \frac{\gamma x_2(n)}{g} - \frac{ex_2^2(n)}{g} + \frac{[1-\beta-2hx_2(n)]^2}{4g^2}}\Bigg] \\ &\quad + \frac{f}{g}x_1(n) + \Big(1-\beta+\frac{\gamma f}{g}\Big)x_2(n) + \Big(\frac{ef}{g} - h\Big)x_2^2(n). \end{aligned}$$

It is verified that $(0,0)$ corresponds to an equilibrium solution of system (3.9). To analyze system (3.10) we must be very careful, because extra conditions are needed in order to have an equilibrium solution at the point $(0,0)$. In fact, if $x_1(n+1) = f_1(x_1(n), x_2(n))$ y $x_2(n+1) = f_2(x_1(n), x_2(n))$, then

$$\begin{aligned} f_1(0,0) &= \frac{1-\beta}{2g} \pm \sqrt{\Big(\frac{1-\beta}{2g}\Big)^2} = \frac{1-\beta}{2g} \pm \Big|\frac{1-\beta}{2g}\Big|, \\ f_2(0,0) &= \Big[-\alpha - \frac{f}{g}(1-\beta)\Big]\Big[\frac{1-\beta}{2g} \pm \sqrt{\Big(\frac{1-\beta}{2g}\Big)^2}\Big] \\ &= \Big[-\alpha - \frac{f}{g}(1-\beta)\Big]\Big[\frac{1-\beta}{2g} \pm \Big|\frac{1-\beta}{2g}\Big|\Big]. \end{aligned}$$

Thus, if $\frac{1-\alpha}{g} > 0$, the origin is an equilibrium solution of system (3.10). It is verified that the linearization of system (3.9) with $\beta \neq 1$ around the equilibrium $(0,0)$ is given by the symplectic matrix

$$S = \begin{pmatrix} \frac{1}{1-\beta} & \frac{\gamma}{1-\beta} \\ -\frac{\alpha}{1-\beta} & -\frac{\alpha\gamma}{1-\beta} - \beta + 1 \end{pmatrix},$$

whose eigenvalues have the form

$$\begin{aligned} \lambda_1 &= -\frac{1}{2(\beta-1)}\Big[-\alpha\gamma + \beta^2 - 2\beta + 2 + \sqrt{(-\alpha\gamma + \beta^2 - 2\beta + 2)^2 - 4(\beta-1)^2}\Big], \\ \lambda_2 &= -\frac{1}{2(\beta-1)}\Big[-\alpha\gamma + \beta^2 - 2\beta + 2 - \sqrt{(-\alpha\gamma + \beta^2 - 2\beta + 2)^2 - 4(\beta-1)^2}\Big]. \end{aligned}$$

The following theorem establishes the conditions that must be verified in order to have stability in the case when $g = 0$.

Theorem 3.9 *Assume that in system (3.9) it is verified that $\beta \neq 1$, $(\beta^2 - 2\beta - \alpha^2) > 0$ and moreover that*

$$\begin{aligned}
&13\beta^4 + 4\beta^2 + \alpha^2 + \alpha^2\beta^2 + \alpha^2\beta^2\gamma^2 + 8\alpha\beta^3\gamma + \gamma^2 + \beta^2\gamma^2 + \beta^6 \\
&\quad + e^2 + \beta^2 e^2 + 4\gamma e + \alpha^2\gamma^2 + 4\alpha\beta e + 2\alpha^2\gamma e \\
&< 12\beta^3 + 2\alpha^2\beta + 2\alpha^2\beta\gamma^2 + 4\beta\gamma e + 2\beta\gamma + \gamma^2 e^2 + 2\alpha\gamma^2 + 12\alpha^3 + 14\alpha\beta e \\
&\quad + 12\alpha\beta^2\gamma + 2\alpha\beta^4\gamma + 2\alpha\gamma + 2\beta e + 6\beta^5 + 2\beta\gamma^2 + 2\alpha\beta^2 e + \alpha^2 e^2 + 2\alpha e.
\end{aligned}$$

Then the null solution of the system (3.9) is asymptotically stable in the Lyapunov sense.

Proof Let $V(x_1, x_2) = x_1^2 + x_2^2$ be a Lyapunov function. It is clear that this function is positive definite in a neighborhood of the origin. Now we evaluate ΔV:

$$\begin{aligned}
\Delta V(x_1, x_2) &= \Big(\frac{x_1 + \gamma x_2 + e x_2^2}{1 - \beta - 2hx_2}\Big)^2 + \Big[\frac{-\alpha x_1 - \alpha\gamma x_2 - \alpha e x_2^2}{1 - \beta - 2hx_2} + \frac{(1 - \beta - 2hx_2)^2 x_2}{1 - \beta - 2hx_2} \\
&\qquad - \frac{f(x_1 + \gamma x_2 + e x_2^2)^2}{(1 - \beta - 2hx_2)^2}\Big]^2 - x_1^2 - x_2^2 = \\
&= \frac{(x_1 + \gamma x_2 + e x_2^2)^2}{(1 - \beta - 2hx_2)^2} + \Big[\frac{(1 - \beta - 2hx_2)(-\alpha x_1 - \alpha\gamma x_2 - \alpha e x_2^2)}{(1 - \beta - 2hx_2)^2} \\
&\quad + \frac{(1 - \beta - 2hx_2)^3 x_2}{(1 - \beta - 2hx_2)^2} - \frac{f(x_1 + \gamma x_2 + e x_2^2)^2}{(1 - \beta - 2hx_2)^2}\Big]^2 - x_1^2 - x_2^2 \\
&= \frac{(x_1 + \gamma x_2 + e x_2^2)^2}{(1 - \beta - 2hx_2)^2} + \frac{1}{(1 - \beta - 2hx_2)^4}\Big[(1 - \beta - 2hx_2)(-\alpha x_1 - \alpha\gamma x_2 - a e x_2^2) \\
&\quad + (1 - \beta - 2hx_2)^3 x_2 - f(x_1 + \gamma x_2 + e x_2^2)^2\Big]^2 - x_1^2 - x_2^2.
\end{aligned}$$

Multiplying by $(1 - \alpha - 2hx_2)^4$ on both sides we obtain the equality

$$\begin{aligned}
&(1 - \beta - 2hx_2)^4 \Delta V(x_1, x_2) \\
&\qquad = (1 - \beta - 2hx_2)^2 (x_1 + \gamma x_2 + e x_2^2)^2 \\
&\qquad\quad + \big[(1 - \beta - 2hx_2)(-\alpha x_1 - \alpha\gamma x_2 - \alpha e x_2^2) \\
&\qquad\quad + (1 - \beta - 2hx_2)^3 x_2 - f(x_1 + \gamma x_2 + e x_2^2)^2\big]^2 \\
&\qquad\quad - (1 - \beta - 2hx_2)^4 x_1^2 - (1 - \beta - 2hx_2)^4 x_2^2 \\
&\qquad = (1 - \beta - 2hx_2)^2 (x_1 + \gamma x_2 + e x_2^2)^2 \\
&\qquad\quad + (1 - \beta - 2hx_2)^2 (-\alpha x_1 - \alpha\gamma x_2 - \alpha e x_2^2)^2 \\
&\qquad\quad + (1 - \beta - 2hx_2)^6 x_2^2 + f^2 (x_1 + \gamma x_2 + e x_2^2)^4 \\
&\qquad\quad + 2x_2 (1 - \beta - 2hx_2)^4 (-\alpha x_1 - \alpha\gamma x_2 - \alpha e x_2^2) \\
&\qquad\quad - 2f(1 - \beta - 2hx_2)(-\alpha x_1 - \alpha\gamma x_2 - \alpha e x_2^2)(x_1 + \gamma x_2 \\
&\qquad\quad + e x_2^2) - 2f x_2 (1 - \beta - 2hx_2)^3 (x_1 + \gamma x_2 + e x_2^2)^2 \\
&\qquad\quad - (1 - \beta - 2hx_2)^4 x_1^2 - (1 - \beta - 2hx_2)^4 x_2^2.
\end{aligned}$$

After some manipulation and simplification the expression on the right-hand side is equivalent to

$$\begin{aligned}
S &= (2\beta + 4\beta^3 - 4\beta^4 - 5\beta^2 + \alpha^2 + \alpha^2\beta^2 - 2\alpha^2\beta)x_1^2 \\
&\quad + (2\beta^2\gamma + 2\beta^2 e - 4\beta\gamma - 4\beta e + 2\alpha^2\gamma + 8\alpha\beta + 8\alpha\beta^3 - 2\alpha\beta^4 - 12\alpha\beta^2 + 2e \\
&\quad + 2\gamma - 2\alpha - 4\alpha^2\beta\gamma + 2\alpha^2\beta^2\gamma)x_1x_2 \\
&\quad + (14\beta^4 - 2\beta - 16\beta^3 - 6\beta^5 + \beta^6 + 9\beta^2 - 4\beta\gamma e + 2\beta^2\gamma e - 2\alpha^2\beta\gamma^2 + \alpha^2\beta^2\gamma^2 \\
&\quad + 8\alpha\beta\gamma + 8\alpha\beta^3\gamma - 2\alpha\beta^4\gamma - 12\alpha\gamma\beta^2 + \gamma^2 + e^2 + \beta^2\gamma^2 + 2\beta e^2 + \beta^2 e^2 \\
&\quad + 2\gamma e - 2\beta\gamma^2 + \alpha^2\gamma^2 - 2\alpha\gamma)x_2^2 + O(\|x\|^3) \\
&= -(\beta-1)^2(\beta^2 - 2\beta - \alpha^2)x_1^2 + 2(\beta-1)(2\alpha\beta + \alpha^2\gamma e + e + \gamma - \alpha - \alpha\beta^2)x_1x_2 \\
&\quad + (\beta-1)^2(\beta^4 - 4\beta^3 - 2\alpha\gamma\beta^2 + 5\beta^2 + 4\alpha\beta\gamma - 2\beta \\
&\quad + \alpha^2\gamma^2 + \gamma^2 + e^2 - 2\alpha\gamma + 2\gamma e)x_2^2 + O(\|x\|^3),
\end{aligned}$$

where $O(\|x\|^3)$ denotes terms of order equal and greater than 3. We observe that ΔV is negative definite when $S < 0$. In fact, $(\beta^2 - 2\beta - \alpha^2)$ is positive, and furthermore

$$\begin{aligned}
&13\beta^4 + 4\beta^2 + \alpha^2 + \alpha^2\beta^2 + \alpha^2\beta^2\gamma^2 + 8\alpha\beta^3\gamma + \gamma^2 + \beta^2\gamma^2 + \beta^6 + e^2 + \beta^2 e^2 \\
&\qquad + 4\gamma e + \alpha^2\gamma^2 + 4\alpha\beta e + 2\alpha^2\gamma e \\
&\quad < 12\beta^3 + 2\alpha^2\beta + 2\alpha^2\beta\gamma^2 + 4\beta\gamma e + 2\beta\gamma + \gamma^2 e^2 + 2\alpha\gamma^2 + 12\alpha^3 \\
&\qquad + 14\alpha\beta e + 12\alpha\beta^2\gamma + 2\alpha\beta^4\gamma + 2\alpha\gamma + 2\beta e + 6\beta^5 \\
&\qquad + 2\beta\gamma^2 + 2\alpha\beta^2 e + \alpha^2 e^2 + 2\alpha e.
\end{aligned} \tag{3.11}$$

Now, we note that $-(\beta-1)^2(\beta^2 - 2\beta - \alpha^2) < 0$ is negative and that $\det\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22}\end{pmatrix} > 0$, where

$$\begin{aligned}
s_{11} &= -(\beta-1)^2(\beta^2 - 2\beta - \alpha^2), \\
s_{12} &= (\beta-1)(2\alpha\beta + \alpha^2\gamma e + e + \gamma - \alpha - \alpha\beta^2), \\
s_{21} &= (\beta-1)(2\alpha\beta + \alpha^2\gamma e + e + \gamma - \alpha - \alpha\beta^2), \\
s_{22} &= (\beta-1)^2(\beta^4 - 4\beta^3 - 2\alpha\gamma\beta^2 + 5\beta^2 + 4\alpha\beta\gamma \\
&\qquad\qquad - 2\beta + \alpha^2\gamma^2 + \gamma^2 + e^2 - 2\alpha\gamma + 2\gamma e).
\end{aligned}$$

In fact,

$$\begin{aligned}
\det\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22}\end{pmatrix} &= -(\beta-1)^4\Big(13\beta^4 + 4\beta^2 + \alpha^2 + \alpha^2\beta^2 + \alpha^2\beta^2\gamma^2 + 8\alpha\beta^3\gamma \\
&\quad + \gamma^2 + \beta^2\gamma^2 + \beta^6 + e^2 + \beta^2 e^2 + 4\gamma e + \alpha^2\gamma^2 + 4\alpha\beta e - 12\beta^3 \\
&\quad - 2\alpha^2\beta - 2\alpha^2 bc^2 - 4\beta\gamma e - 2\beta\gamma + 2\alpha^2\gamma e - \gamma^2 e^2 - 2\alpha\gamma^2 \\
&\quad - 12\alpha^3 - 14\alpha\beta e - 12\alpha\gamma\beta^2 - 2\alpha\beta^4\gamma - 2\alpha\gamma - 2\beta e - 6\beta^5 \\
&\quad - 2\beta\gamma^2 - 2\alpha\beta^2 e - (\alpha^2 e^2 - 2\alpha e)\Big).
\end{aligned}$$

By inequality (3.11) we have that the factor at the right-hand side of the determinant, is less than zero and $-(\beta-1)^4 < 0$, we conclude that the quadratic form S is negative

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definite. Therefore, ΔV is negative definite. By Lyapunov's Theorem for asymptotic stability we conclude that the null solution of system (3.9) is asymptotically stable. ■

Example 3.10 Observe in system (3.9) that if $\alpha = e = 1$ and $\beta = \frac{5}{2}$, the condition $\beta^2 - 2\beta - \alpha^2 = (\frac{5}{2})^2 - 5 - 1 = \frac{1}{4} > 0$ is satisfied, and substituting in (3.11), we verify that $\frac{61629}{64} < \frac{15585}{16}$, which implies that the null solution of the system (3.9), given by

$$
\begin{aligned}
x_1(n+1) &= \frac{x_1(n) + x_2(n) + x_2^2(n)}{-\frac{3}{2} - 2hx_2(n)} \\
x_2(n+1) &= -\Big[\frac{x_1(n) + x_2(n) + x_2^2(n)}{-\frac{3}{2} - 2hx_2(n)}\Big] + \Big[-\frac{3}{2} - hx_2(n)\Big]x_2(n) \\
&\quad - f\Big[\frac{x_1(n) + x_2(n) + x_2^2(n)}{-\frac{3}{2} - 2hx_2(n)}\Big]^2.
\end{aligned}
$$

is stable in the Lyapunov sense.

Finally, we point out that the value of f is arbitrary and is not relevant at the moment of analyzing the type of stability of the null solution.

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