J. Austral. Math. Soc. 21 (Series A) (1976), 299–309.

TRIPLEABLENESS OF PRO-C-GROUPS

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(Received 6 May 1974, revised 20 November 1974)

1. Introduction

Let C be a nontrivial full subcategory of the category F of finite discrete groups and continuous homomorphisms, closed under subobjects, quotient and finite products. We consider the category PC of pro-C-groups and continuous homomorphisms (i.e. inverse limits of C-groups) which forms a variety in category PF of profinite groups and continuous homomorphisms. The study of pro-C-groups is motivated by their occurrence as Galois groups of field extensions in algebraic number theory (see Serre (1965)). The purpose of this paper is to study the tripleableness of the forgetful functors from PC to various underlying categories. It is also shown that PC is equivalent to the category of algebras of the theory of the forgetful functor from C to S (the category of sets and mappings).

2. Beck's Tripleableness Theorems

Let $F \dashv U : B \to A$ be an adjoint pair of functors. Eilenberg and Moore (1965) showed that this adjoint pair gives rise to a triple $\mathcal{T} = (T, \eta, \mu)$ in the category A. Let $A^{\mathcal{T}}$ denote the category of $\mathcal{T}$-algebras. There is a canonical functor $\Phi : B \to A^{\mathcal{T}}$. The adjoint pair $F \dashv U$ is called *tripleable* if Φ is an equivalence of categories. A functor $U : B \to A$ is called *tripleable* if U has a left adjoint F and the adjoint pair $F \dashv U$ is tripleable.

A pair $X \overset{d_0}{\underset{d_1}{\rightrightarrows}} Y$ of morphisms in B is called *U-split* if there is a split coequalizer diagram in A:

$$UX \overset{Ud_0}{\underset{Ud_1}{\rightrightarrows}} UY \overset{d}{\underset{s}{\rightleftarrows}} Z$$

$$\rightleftarrows$$

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with the properties

(i) $d(Ud_0) = d(Ud_1)$ (ii) $ds = 1_Z$

(iii) $(Ud_1)t = sd$ (iv) $(Ud_0)t = 1_{UY}$.

A *split pair* is an id_B-split pair.

$\boldsymbol{B}$ has *coequalizers of U-split pairs* if each U-split pair of morphisms in $\boldsymbol{B}$ has a coequalizer in $\boldsymbol{B}$.

U preserves coequalizers of U-split pairs if whenever $X \overset{d_0}{\underset{d_1}{\rightrightarrows}} Y$ is U-split and has a coequalizer $y \xrightarrow{f} W$ in $\boldsymbol{B}$, then

$$Uf = \text{coeq}(Ud_0, Ud_1).$$

We say that *U reflects coequalizers of U-split pairs* if $X \rightrightarrows Y \rightarrow W$ being mapped into a split coequalizer diagram by U implies that $X \rightrightarrows Y \rightarrow W$ is a coequalizer diagram in $\boldsymbol{B}$.

Beck (1966, 1967) proved the following:

THEOREM 2.1. *$F \dashv U : \boldsymbol{B} \rightarrow \boldsymbol{A}$ is tripleable if and only if it satisfies the following conditions:*

PTT(1) : $\boldsymbol{B}$ has coequalizers of U-split pairs
PTT(2) : U preserves coequalizers of U-split pairs
{ *PTT(3) : U reflects coequalizers of U-split pairs*
{ *or PTT(3)′ : U reflects isomorphisms.*

COROLLARY (*Crude Tripleability Theorem*). *$F \dashv U : \boldsymbol{B} \rightarrow \boldsymbol{A}$ is tripleable if it satisfies the following conditions:*

CTT(1) : $\boldsymbol{B}$ has coequalizers
CTT(2) : U preserves coequalizers
{ *CTT(3) : U reflects coequalizers*
{ *or CTT(3)′ : U reflects isomorphisms.*

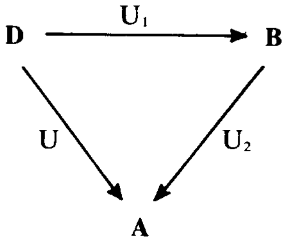
THEOREM 2.2. *$F \dashv U : \boldsymbol{B} \rightarrow \boldsymbol{A}$ is tripleable if it satisfies*

VTT(1): Every U-split pair of morphisms in $\boldsymbol{B}$ splits and also one of PTT(3), PTT(3)′.

A functor $U : \boldsymbol{B} \rightarrow \boldsymbol{A}$ is called *PTT*, *CTT* or *VTT* if it satisfies the conditions of Theorem 2.1, Corollary of Theorem 2.1, or Theorem 2.2 respectively.

3. Some Results on Tripleableness

PROPOSITION 3.1. *In the following commutative diagram of categories and functors*



if (i) *U is tripleable*

(ii) *U_2 satisfies PTT*(3)

(iii) *U_1 has a left adjoint*

then U_1 is tripleable.

PROOF. We shall show that U_1 is *PTT*. Since every U_1-split pair of maps in $\boldsymbol{D}$ is U-split, it follows that U_1 satisfies *PTT*(1), Let

$$X \overset{d_0}{\underset{d_1}{\rightrightarrows}} Y$$

be a U_1-split pair of maps in $\boldsymbol{D}$ with

$$X \overset{d_0}{\underset{d_1}{\rightrightarrows}} Y \xrightarrow{d} Z$$

as a coequalizer diagram in $\boldsymbol{D}$. Then

$$Ud = \text{coeq}(Ud_0, Ud_1)$$

since U preserves coequalizers of U-split pairs. The pair of maps

$$U_1X \overset{U_1d_0}{\underset{U_1d_1}{\rightrightarrows}} U_1Y$$

is U_2 split and since U_2 reflects coequalizers of U_2-split pairs, we have

$$U_1d = \text{coeq}(U_1d_0, U_1d_1)$$

which proves that U_1 satisfies *PTT*(2). Finally, U satisfies *PTT*(3)′ and so does U_1.

COROLLARY. *Assume $U_2: \boldsymbol{B} \to \boldsymbol{A}$ is tripleable. Then $U_1: \boldsymbol{D} \to \boldsymbol{B}$ is tripleable if U_2U_1 is tripleable and U_1 has a left adjoint.*

The converse of the above corollary is false, i.e. the composition of tripleable functors is *not* necessarily tripleable. For example, take $\boldsymbol{D}$ = category of torsion free abelian groups and homomorphisms; $\boldsymbol{B}$ = category of abelian groups and homomorphisms; $\boldsymbol{A}$ = category of sets and mappings, together with the forgetful functors.

PROPOSITION 3.2. *The composition* $\mathbf{D} \xrightarrow{U_1} \mathbf{B} \xrightarrow{U_2} \mathbf{A}$ *is tripleable if one of the following conditions is satisfied*:

(a) *U_1 is CTT and U_2 is tripleable*
(b) *U_1 is tripleable and U_2 is VTT.*

PROOF. Let $U = U_2U_1$, then U has a left adjoint if U_1 and U_2 have.

(a) We will prove that $U = U_2U_1$ satisfies *PTT*. Now, $\mathbf{D}$ has coequalizers and in particular $\mathbf{D}$ has coequalizers of U-split pairs. Let

$$X \underset{d_1}{\overset{d_0}{\rightrightarrows}} Y \xrightarrow{d} 3Z$$

be a diagram in $\mathbf{D}$, and suppose that (d_0, d_1) is U-split. If $d = \text{coeq}(d_0, d_1)$, then $U_1d = \text{coeq}(U_1d_0, U_1d_1)$. Also (U_1d_0, u_1d_1) is U_2 split, and so $U_2U_1d = \text{coeq}(U_2U_1d_0, U_2U_1d_1)$, which proves that U satisfies *PTT*(2). If $Ud = \text{coeq}(Ud_0, Ud_1)$, then $U_1d = \text{coeq}(U_1d_0, U_1d_1)$ since U_2 reflects coequalizers of U_2-split pairs. If we assume that U_1 satisfies *CTT*(3), then $d = \text{coeq}(d_0, d_1)$. However, if U_1 satisfies *CTT*(3)′, then by virtue of the fact that U_2 satisfies *PTT*(3)′, it is immediate that U satisfies *PTT*(3)′. This proves that U is tripleable.

(b) Let $X \underset{d_1}{\overset{d_0}{\rightrightarrows}} Y$ be U-split. Then (U_1d_0, U_1d_1) is U_2-split, and since U_2 satisfies *VTT*(1), (d_0, d_1) is U_1-split. It follows that (d_0, d_1) has a coequalizer. Let

$$X \underset{d_1}{\overset{d_0}{\rightrightarrows}} Y \xrightarrow{d} Z$$

be a coequalizer diagram in $\mathbf{D}$ with (d_0, d_1) U-split. Then (U_1d_0, U_1d_1) is U_2-split and hence splits by *VTT*(1) of U_2. By *PTT*(2) of U_1, $U_1d = \text{coeq}(U_1d_0, U_1d_1)$. U_2 satisfies *PTT*(2) and so $Ud = \text{coeq}(Ud_0, Ud_1)$. This proves that U satisfies *PTT*(2). Clearly U satisfies *PTT*(3) (respectively *PTT*(3)′).

PROPOSITION 3.3. *Let $F \dashv UU : \mathbf{B} \to \mathbf{A}$ be an adjoint pair. If the back adjunction $\varepsilon : FU \to id_{\mathbf{B}}$ is an isomorphism, then U is VTT.*

PROOF. Verification of *VTT*(1): Let $X \underset{d_1}{\overset{d_0}{\rightrightarrows}} Y$ be a U split pair of maps in $\mathbf{B}$. Then we have a split coequalizer diagram in $\mathbf{A}$

$$UX \underset{Ud_1}{\overset{Ud_0}{\rightrightarrows}} UY \underset{s}{\overset{d}{\rightleftarrows}} W$$

$$\underset{Ut}{\leftarrow}$$

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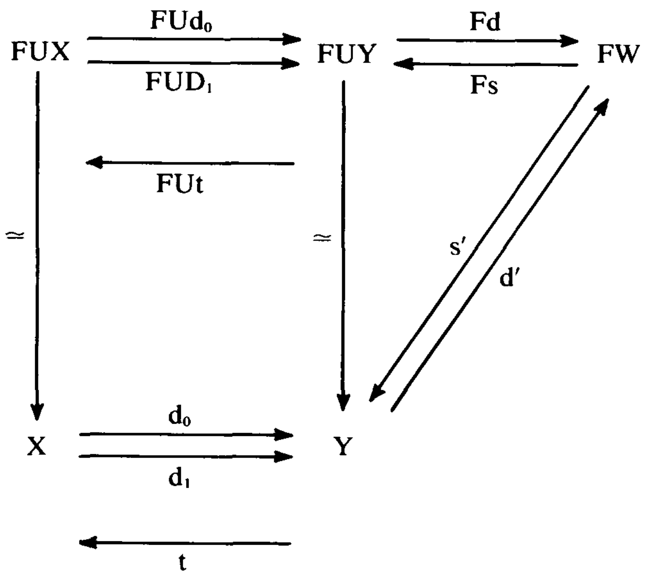
and hence a split coequalizer diagram in $\boldsymbol{B}$:

$$FUX \underset{FUd_1}{\overset{FUd_0}{\rightrightarrows}} FUY \underset{Fs}{\overset{Fd}{\rightleftarrows}} FW$$

$$\underset{FUt}{\rightarrow}$$

Moreover, $\varepsilon : FU \rightarrow id_{\boldsymbol{B}}$ is a natural isomorphism, and so we have the following commutative diagrams:



where

$$s' : (FW \overset{Fs}{\rightarrow} FUY \overset{\simeq}{\rightarrow} Y)$$

$$d' : (Y \overset{\simeq}{\rightarrow} FUY \overset{Fd}{\rightarrow} FW).$$

Clearly,

$$X \underset{d_1}{\overset{d_0}{\rightrightarrows}} Y \underset{s'}{\overset{d'}{\rightleftarrows}} FW$$

$$\underset{t}{\leftarrow}$$

is a split coequalizer diagram in $\boldsymbol{B}$ and so $d' = \text{coeq}(d_0, d_1)$.

Verification of $PTT(3)'$: Let $X \overset{f}{\rightarrow} Y$ in $\boldsymbol{B}$ be such that $Uf : UX \rightarrow UY$ is an isomorphism. Then $FUf : FUX \rightarrow FUY$ is an isomorphism. By naturality of $FU \overset{\varepsilon}{\rightarrow} 1d_{\boldsymbol{B}}$, we have the following commutative diagram

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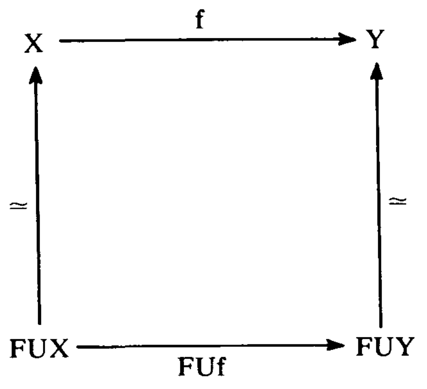
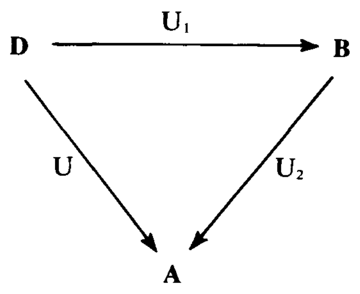


which implies that f is an isomorphism.

Corollary. *If $\boldsymbol{B}$ is a full reflective subcategory of $\boldsymbol{A}$, then the inclusion functor $U:\boldsymbol{B}\rightarrow\boldsymbol{A}$ is VTT.*

Proof. Let $F\dashv U:\boldsymbol{B}\rightarrow\boldsymbol{A}$. Then it is well-known that $\varepsilon: FU\rightarrow id_{\boldsymbol{B}}$ is an isomorphism. The result then follows from the above proposition.

Proposition 3.4. *Consider the following commutative diagram of categories and functors*:



Assume that U_2 has a left adjoint F_2 with the property that the back adjunction $\varepsilon: F_2U_2\rightarrow id_{\boldsymbol{B}}$ is an isomorphism. Then U_1 is tripleable if and only if U is tripleable.

Proof. The necessity part is a combination of Proposition 3.3 and Proposition 3.2. For the sufficiency, we only need to verify that U_1 has a left adjoint, and the rest follows from Proposition 3.1. Let $F\dashv U$. Then for $B\in\boldsymbol{B}$, $D\in\boldsymbol{D}$.

$$(FU_2B, D)\cong(U_2B, UD\cong(F_2U_2B, U_1D)\cong(B, U_1D)$$

which shows that $FU_2\dashv U_1$.

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COROLLARY. *Let $U_2: \boldsymbol{B} \to \boldsymbol{A}$ be the inclusion functor of a full reflective subcategory* $\boldsymbol{B}$ *of* $\boldsymbol{A}$, *then $U_1: \boldsymbol{D} \to \boldsymbol{B}$ is tripleable if and only if $U_2 U_1: \boldsymbol{D} \to \boldsymbol{A}$ is tripleable.*

4. Tripleableness of Pro-C-Groups

In what follows, $\boldsymbol{C}$ will denote a nontrivial full subcategory of the category of finite discrete groups and continuous homomorphisms, closed under subobjects, quotient objects and finite products.

PC = category of pro-C-groups and continuous homomorphisms
Top = category of topological spaces and continuous mappings
PTop = category of pointed topological spaces and continuous mappings preserving the base point
CHTop = category of compact Hausdorff topological spaces and continuous mappings
G = category of groups and homomorphisms
S = category of sets and mappings

A topological group G is called a pro-C-group if it satisfies the following equivalent conditions:

(a) G is an inverse limit of $\boldsymbol{C}$-groups.
(b) G is profinite and G/N is a $\boldsymbol{C}$-group for every open normal subgroup N of G.
(c) G is compact, Hausdorff and admits a family Φ of open normal subgroups of G, such that Φ is a functional system of neighbourhoods of the identity and has the property that for each $N \in \Phi$, G/N is a $\boldsymbol{C}$-group.

THEOREM 4.1. *The underlying set functor $U: \boldsymbol{PC} \to \boldsymbol{S}$ is tripleable.*

PROOF. The functor $U: \boldsymbol{PC} \to \boldsymbol{S}$ has a left adjoint $F: \boldsymbol{S} \to \boldsymbol{PC}$ given by

$$F(X) = \lim L/N$$

where L is the free discrete group on X and N ranges over all normal subgroups of L such that $L/N \in |\boldsymbol{C}|$ (Gildenhuys & Lim (1972)). We shall show that U satisfies PTT. Let $X \underset{d_1}{\overset{d_0}{\rightrightarrows}} Y$ be a U-split pair of maps in $\boldsymbol{PC}$. Then we have a split coequalizer diagram in $\boldsymbol{S}$.

$$X \underset{d_1}{\overset{d_0}{\rightrightarrows}} Y \underset{s}{\overset{d}{\rightleftarrows}} Z.$$

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Let

$$R = \{(d_0x, d_1x) \in Y \times Y : x \in X\}$$

then

$$R \circ R^{op} = \{(y_1, y_2) \in Y \times Y2{:}2\exists\, y_3 \in Y \text{ with } (y_1, y_3) \in R,\ (y_2, y_3) \in R\}.$$

We assert that $R \circ R^{op}$ is the equivalence relation generated by R. It suffices to show that

$$(y_1, y_2) \in R \circ R^{op} \quad \text{iff} \quad dy_1 = dy_2.$$

The necessity follows from the fact that $dd_0 = dd_1$. For the sufficiency, let $(y_1, y_2) \in Y \times Y$ with $dy_1 = dy_2$. Then

$$d_1ty = sdy_1 = sdy_2 = d_1ty_2; \quad d_0ty_1 = y_1; \quad d_0ty_2 = y_2.$$

It then follows that

$$(y_1, d_1ty_1) = (d_0ty_1, d_1ty_1) \in R$$

and

$$(y_2, d_1ty_1) = (d_0ty_2, d_1ty_2) \in R.$$

So we have $(y_1, y_2) \in R \circ R^{op}$. It is easily seen that $R \circ R^{op}$ is in fact a congruence relation on Y. In order for $Y/(R \circ R^{op})$ to be a pro-C-group, it remains to show that $R \circ R^{op}$ is closed. Let

$$D = \{(y_1, y_2, y_3) \in Y \times Y \times Y : (y_1, y_3) \in R,\ (y_2, y_3) \in R\}.$$

Then $D = p_1^{-1}R \cap p_2^{-1}R$ where $p_i = Y \times Y \times Y \to Y \times Y$ are the continuous maps defined by $p_i(y_1, y_2, y_3) = (y_i, y_3)$; $i = 1, 2$. R is closed, being the image of a compact set in a compact Hausdorff space under the continuous map $(d_0, d_1) : X \to Y \times Y$. Hence D is closed. But $R \circ R^{op}$ is the image of D under the closed map $p_3 : Y \times Y \times Y \to Y \times Y$ with $p_3(y_1, y_2, y_3) = (y_1, y_2)$ and so $R \circ R^{op}$ is closed. Hence $Y/R \circ R^{op}$ is a pro-C-group. We shall show that

$$X \overset{d_0}{\underset{d_1}{\rightrightarrows}} Y \xrightarrow{\pi} Y/(R \circ R^{op})$$

is a coequalizer diagram in $\boldsymbol{PC}$, where π is the quotient map. If $Y \xrightarrow{d'} W$ in $\boldsymbol{PC}$ has the property $d'd_0 = dd_1$, then the map

$$\psi : Y/(R \circ R^{op}) \to W$$

defined by

$$\psi[y] = d'y$$

is well-defined morphism in $\boldsymbol{PC}$ and is unique with the property $\psi \circ \pi = d'$. This proves that U satisfies $PTT(1)$.

Note that the map $d: Y \to Z$ induces an isomorphism

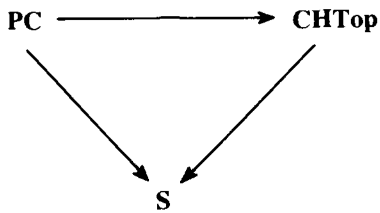
$$Y/(R \circ R^{op} \xrightarrow{\simeq} Z$$

in S, it is then easily seen that U satisfies $PTT(2)$ and $PTT(3)$.

REMARK. Theorem 4.1 can also be obtained as a consequence of a result by Manes (1969) (§3, Proposition 3.6 which states that every Birkhoff subcategory of a tripleable category over $\boldsymbol{S}$ is tripleable. This is because $\boldsymbol{PC}$ is a Birkhoff subcategory over $\boldsymbol{S}$ (see Linton (1965), page 90).

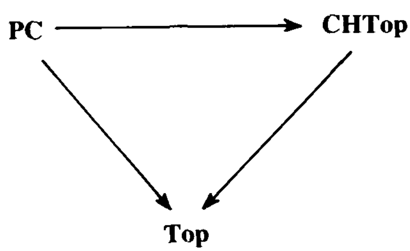
THEOREM 4.2. *The forgetful functors from* ***PC*** *to the categories* ***CHTop***, ***Top***, ***PTop***, ***G*** *are tripleable.*

PROOF. In the following, all functors in diagrams are forgetful. It is well-known that the forgetful functor from ***CHop*** to ***S*** is tripleable (see Linton (1965)), and so by applying the Corollary of Proposition 3.1 to the following commutative diagram:

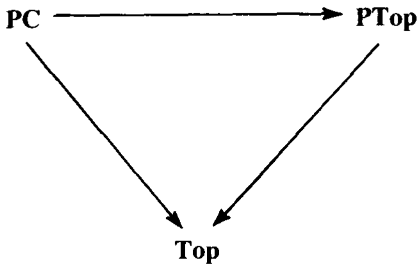


we conclude that the forgeful functor $U: \boldsymbol{PC} \to \boldsymbol{CHTop}$ is tripleable.

CHTop is a full reflective subcategory of ***Top***. The tripleableness of the forgetful functor $U: \boldsymbol{PC} \to \boldsymbol{Top}$ will then follow by applying the Corollary of Proposition 3.4 to the following commutative diagram:

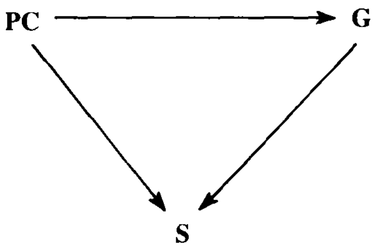


To show that the forgetful functor $U: \boldsymbol{PC} \to \boldsymbol{PTop}$ is tripleable, we consider the following commutative diagram:



Observe that the forgeful functor $\boldsymbol{PTop} \to \boldsymbol{Top}$ satisfies $PTT(3)$. It then follows from Proposition 3.1 that $U:\boldsymbol{PC} \to \boldsymbol{PTop}$ is tripleable.

It is well-known that the forgeful functor $\boldsymbol{GS}$ is tripleable. By applying the Corollary of Proposition 3.1 to the following commutative diagram:



we conclude that the inclusion functor $\boldsymbol{PC} \to \boldsymbol{G}$ is tripleable.

5. A Theorem of Equivalence

Let $U_0 : \boldsymbol{C} \to \boldsymbol{S}$ be the underlying set functor. This induces a theory T_{U_0} in $\boldsymbol{S}$. Let $(T_{U_0}:\boldsymbol{S})$ denote its category of algebras (see Linton (1965)).

THEOREM 5.1. $\boldsymbol{PC} \approx (T_{U_0}:\boldsymbol{S})$.

PROOF. Let $\mathcal{T}$ be the triple arising from the adjoint pair $F \dashv U : \boldsymbol{PC} \to \boldsymbol{S}$. This induces an adjoint pair $F^{\mathcal{T}} \to U^{\mathcal{T}} : \boldsymbol{S}^{\mathcal{T}} \to \boldsymbol{S}$ (see Eilenberg and Moore (1965)). It is well known that $(T_{U^{\mathcal{T}}}:\boldsymbol{S}) \approx \boldsymbol{S}^{\mathcal{T}}$ (see Linton (1965)). By Theorem 4.1.,

$$\boldsymbol{S}^{\mathcal{T}} \approx \boldsymbol{PC}.$$

Thus to show that

$$\boldsymbol{PC} \approx (T_{U_0}:\boldsymbol{S})$$

it is enough to show

$$T_{U_0} \cong T_{U^{\mathcal{T}}}.$$

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This amounts to showing that one has an isomorphism

$$UFX \cong \mathrm{Nat}(U_0^X, U_0)$$

where the latter represents the set of natural transformations from $\boldsymbol{C}$ to $\boldsymbol{S}$.

$UFX = U(\lim_N LX/N)$ by definition of FX,

$\cong \lim_N (U(LX/N))$ by the construction of $\lim_N LX/N$

$\cong \lim_N \mathrm{Nat}(\boldsymbol{C}(LX/N,\_), U_0)$ by Yoneda's Lemma,

$\cong \mathrm{Nat}(\mathrm{colim}_N \boldsymbol{C}(LX/N,\_), U_0)$ on commuting limits,

$\cong \mathrm{Nat}(\boldsymbol{PC}(\lim_N LX/N,\_), U_0)$ by construction of the free $\boldsymbol{PC}$-object on $X \in \boldsymbol{C}$,

$= \mathrm{Nat}(\boldsymbol{PC}(FX,\_), U_0)$ by definition of FX,

$\cong \mathrm{Nat}(\boldsymbol{S}(X, U\_), U_0)$ because $F \dashv U$,

$= \mathrm{Nat}(U_0^X, U_0$ as required.

The author would like to express his thanks to Dion Gildenhuys for useful discussion and to the referee for shortening the proof of Theorem 5.1.

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