

BOUNDS FOR THE SEPARATION OF REAL ZEROS OF POLYNOMIALS

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(Received 29 October, 1992)

Communicated by P. C. Fenton

Abstract

We measure the separation of the zeros of the polynomial $f(x) = \prod_1^n (x - a_i)$, $a_1 < a_2 < \dots < a_n$ by $\delta(f) = \min_{i \neq j} (a_{i+1} - a_i)$. We establish a bound for the amount by which the ratio $\delta(f' - kf)/\delta(f)$ exceeds 1.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 26C10.

1. Introduction

The problem of the location of the zeros of a polynomial of arbitrary degree has a long history in mathematics. The number of zeros on the real axis may be investigated by Descartes' rule of signs [3, Part 5, Ch. 1], or in greater detail by means of Sturm sequences [1, Ch 6.3]. In the case of complex zeros, methods for the location of zeros in specified regions of the plane are studied in [2]. Numerical aspects of the problem can be found in [1, Ch 6].

In this paper we propose to study the separation $(a_i - a_j)$ of the zeros of a polynomial $f(x) = \prod_1^n (x - a_i)$. We shall measure the separation both by the minimum separation $\delta(f) = \min_{i \neq j} |a_j - a_i|$ and by the maximum or total separation $\Delta(f) = \max_{i,j} |a_j - a_i|$. The quantity $\delta(f)$ gives us a measure of the closeness of the zeros so that we can detect when they are near to coincidence.

In the case when the zeros are real and distinct, we showed in [4] that $\delta(f)$ is strictly increased by the differential operator $D - kI$, that is, that $\delta(f' - kf) > \delta(f)$, and we shall show in Section 3 of this paper that the same is true for $\Delta(f)$. In Section 2 we sharpen the result for $\delta(f)$ to give an explicit lower bound for the ratio $\delta(f' - kf)/\delta(f)$; in particular, there is a constant $c_n > 1$ such that $\delta(f') \geq c_n \delta(f)$.

Much more remains to be done in respect of improved constants, extension of the results to the complex plane and exploitation of possible symmetries among the zeros.

It is a pleasure to acknowledge the contributions of Professor R. E. Scraton for Theorem 2(i) and for extensive numerical testing.

2. Minimum Separation

We suppose throughout that $f(x) = \prod_1^n (x - a_i)$ is a polynomial of degree n with distinct real zeros which we shall assume to be in increasing order: $a_1 < a_2 < \dots < a_n$. The principal result of this section is as follows:

THEOREM 1. *For all $k \in \mathbb{R}$ and all polynomials f of degree $n \geq 2$, ($n \geq 3$ if $k = 0$)*

$$\delta(f' - kf)/\delta(f) > 1 + \frac{4n}{(n+2)^2} \cdot \frac{1}{\pi^2 + (|k|\delta(f) + \rho + 2/n)^2},$$

where $\rho = \log(4(\pi + 1)/(\pi - 1)) = 2.05\dots$

From Rolle's theorem the $n - 1$ zeros of f' lie one in each interval (a_i, a_{i+1}) ; more generally it is easy to see by considering the graph of f'/f that for given real k , the polynomial $f' - kf$ has one zero, which we shall denote by b_i , in each interval (a_i, a_{i+1}) . In addition if $k > 0$ there is one further zero $b_n > a_n$, while if $k < 0$ there is one further zero $b_0 < a_1$.

We shall say that the separation of the points a_i is *almost equal* when either a_1, \dots, a_{n-1} are equally separated and a_n ($\geq a_{n-1}$) is otherwise unrestricted, or similarly when a_2, \dots, a_n are equally separated and a_1 ($\leq a_2$) is unrestricted. It may happen that when the points a_i vary, some of them tend to $\pm\infty$, in which case the corresponding terms are omitted from the summations.

Our plan is as follows. Firstly, we fix j and find the configuration of a_i 's which minimises $r_j = b_{j+1} - b_j$; this is done in Lemma 1. In this configuration we estimate the position of b_j and find lower estimates for r_j , which are stated in Lemma 2. Finally, in the proof of Theorem 1 we minimise this lower estimate over j .

LEMMA 1. *For given j and arbitrary a_i , subject to a given minimum spacing $\delta(f)$, the minimum value of $r_j = b_{j+1} - b_j$ is attained when all a_i not at infinity are equally or almost equally spaced.*

PROOF OF LEMMA 1. For given j the function r_j depends continuously on the values of a_i ; this includes the possibility that some a_i may tend to $\pm\infty$, as already noted. Since $\delta(f)$ is fixed and the results are unaffected by translation, we could assume for instance that some $a_i = 0$, and $a_{i+1} = \delta(f)$, and so all the a_i cannot go to $\pm\infty$.

Hence for given j we can assume that a_1, a_2, \dots, a_n assume the configuration which minimises r_j , and we have to show that in this configuration all a_i not at infinity are equally or almost equally spaced.

It follows from the equation $\sum_{i=1}^n 1/(b_j - a_i) = k$ that for fixed $k \in \mathbb{R}$

$$\frac{\partial b_j}{\partial a_i} = \frac{1}{(b_j - a_i)^2 S_j} \quad \text{where} \quad S_j = \sum_{m=1}^n 1/(b_j - a_m)^2.$$

Note that $0 < \partial b_j / \partial a_i < 1$ for all i, j . Hence

$$\frac{\partial r_j}{\partial a_i} = \frac{1}{(b_{j+1} - a_i)^2 S_{j+1}} - \frac{1}{(b_j - a_i)^2 S_j},$$

and this is > 0 (respectively $= 0, < 0$) accordingly as

$$\left(\frac{b_j - a_i}{b_{j+1} - a_i} \right)^2 > (\text{or } =, \text{ or } <) \frac{S_{j+1}}{S_j}.$$

But the ratio S_{j+1}/S_j is independent of i , while the left side, as a function of i , is strictly decreasing and < 1 for $1 \leq i \leq j$, and strictly decreasing and > 1 for $j + 2 \leq i \leq n$. Hence for a given j we must have one of the following three cases for the sequence of signs of $\partial r_j / \partial a_i, 1 \leq i \leq n$:

- (i) if $S_{j+1}/S_j > 1$, the sequence of signs has the form $- - \dots - (?) (+ + \dots +)(0)(- - \dots -)$ where the $(?)$ is in the $(j + 1)$ st position, and either the (0) or the blocks of $+$ or $-$ terms may be absent on the right,
- (ii) if $S_{j+1}/S_j = 1$, the sequence is $- - \dots - (?) + + \dots +$, with $(?)$ in the $(j + 1)$ st position,
- (iii) if $S_{j+1}/S_j < 1$, the sequence is $(+ + \dots +)(0)(- - \dots -)(?) + + \dots +$, similarly.

Notice also that if instead of varying a single a_i , we move simultaneously a block of consecutive a_i , all having the same sign for $\partial r_j / \partial a_i$, then the effect on r_j of either an increase or a decrease will be the same as for a move in the individual terms; this also holds if one of the signs is zero.

We now consider case (i) in detail and suppose initially that $j \geq 1$ so that there is at least one $\partial r_j / \partial a_i$ with $i < j + 1$, and all these derivatives must be < 0 . Then all intervals (a_i, a_{i+1}) with $i \leq j$ must be of length $\delta(f)$, since otherwise r_j could be decreased by a simultaneous increase in a_1, \dots, a_i . Now look at the signs to the right of $j + 1$. These cannot end with a $-$ since if they did, then r_j could be decreased by an increase in a_n ($\delta(f)$ being already attained to the left of a_{j+1}). Hence the sign sequence must be $- - \dots - (?) (+ + \dots +)(0)$. If there is zero at a_n we have no control on the length of (a_{n-1}, a_n) . Apart from this, all intervals (a_i, a_{i+1}) with a $+$ at

a_{i+1} must be of length $\delta(f)$ since otherwise r_j could be decreased by a simultaneous decrease in a_{i+1}, \dots, a_n . This proves equal or almost equal spacing in this case.

In the case $j = 0$ the sign sequence is $(?)(+ + \dots +)(0)(- - \dots -)$. Now if the last sign is $-$, then the last interval (a_{n-1}, a_n) must have length $\delta(f)$, otherwise r_j is decreased by an increase in a_n . Then there can be no preceding 0 or $-$ signs, since if there were, then r_j could be decreased by a simultaneous increase of all terms with a 0 or $-$ sign. Hence the only remaining possibilities are either $(?) + + \dots + (0)$, which we showed above leads to equal or almost equal spacing, or $(?) + + \dots + -$, which we shall show cannot occur. We can assume that $n \geq 4$ since equal or almost equal spacing is automatic if $n = 2$ or 3 . Then $\partial r_j / \partial a_2 > 0$, and we know that $\delta(f)$ is already attained on (a_{n-1}, a_n) where $n - 1 \geq 3$. Hence if $a_2 - a_1 > \delta(f)$ then a decrease in a_2 (only!) will decrease r_j . Hence $\delta(f)$ is attained also on (a_1, a_2) and so as above there can be no $-$ sign at a_n .

Cases (ii) and (iii) are treated similarly, and Lemma 1 is proved.

NOTE. Numerical examples show that at least three possibilities can occur. For instance if $n = 4$ and $a_i = i, 1 \leq i \leq 3, a_4 = 4 + t$, and $k = -1$, then r_0 is a decreasing, and r_2 is an increasing function of t , while r_1 is firstly increasing and then decreasing with a maximum near $t = 6$.

From now on we shall study the case of equal separation in detail. Almost equal separation will not be considered again until the latter part of the proof of Theorem 1.

Our next objective is to estimate the positions of the points b_j . To find these we write $b_j = a_j + t_j$ for all $j \geq 1$; we can simplify by putting $\delta(f) = 1$ and $a_i = i$ for $1 \leq i \leq n$. Then we have $b_j = j + t_j$ and we can also put $t_0 = b_0$. Then we have the following estimates for t_j and $t_{j+1} - t_j$ in the normalised case.

LEMMA 2. Let $E(t) = \pi \cot \pi t$ and let E^{-1} be the branch of the inverse with values in the interval $(0, 1)$: $E^{-1}(y) = 1/2 - \tan^{-1}(y/\pi)/\pi$. Let h_n denote the harmonic number $1 + 1/2 + \dots + 1/n$. Then we have the following estimates for t_j :

- (i)(a) $t_j > E^{-1}(k + h_{n-j} - h_{j-1}) = p_j$ say, for $1 \leq j \leq n$,
- (b) $t_j < E^{-1}(k + h_{n-j-1} - h_j) = q_j (= p_{j+1})$ say, for $0 \leq j \leq n - 1$.
- (ii) We have $t_{j+1} - t_j > n / [(n - j)(j + 1)(\pi^2 + k^2)]$ where

$$k' = \begin{cases} k + h_{n-2} - 1, & \text{for } j = 0 \\ \max(|k + h_{n-j} - h_{j-1}|, |k + h_{n-j-2} - h_{j+1}|), & \text{for } 1 \leq j \leq n - 2 \\ k + 1 - h_{n-2}, & \text{for } j = n - 1. \end{cases}$$

Note that (ii) gives us a proof that $t_{j+1} > t_j$, which is the special case of $\delta(f' - kf) > \delta(f)$ for equal spacing.

PROOF OF LEMMA 2. (i) We know that t_j is defined as a solution of the equation $\psi_j(t) = \sum_{r=1-j}^{n-j} 1/(t-r) = k$. Write $E(t) = \pi \cot \pi t = \psi_j(t) + \Psi_j(t)$. For sufficiently large N , $\Psi_j(t)$ can be written

$$\Psi_j(t) = \sum_{r=-N}^{-j} \frac{1}{t-r} + \sum_{r=n-j+1}^N \frac{1}{t-r} + \sum_{r=N+1}^{\infty} \left(\frac{1}{t-r} + \frac{1}{t+r} \right),$$

and from the periodicity of $E(t)$ it follows that

$$\Psi_j(t+1) = \Psi_j(t) + \frac{1}{t+j-n} - \frac{1}{t+j}.$$

In particular $\Psi_j(0) = h_{n-j} - h_{j-1}$ and $\Psi_j(1) = h_{n-j-1} - h_j$ whenever these are defined.

But $E(t_j) = k + \Psi_j(t_j)$ and Ψ_j is decreasing on its intervals of continuity. Hence since $t_j > 0$ for $1 \leq j \leq n$ we have $E(t_j) < k + \Psi_j(0)$, or $t_j > E^{-1}(k + h_{n-j} - h_{j-1})$. Similarly for $0 \leq j \leq n - 1$ we have $t_j < 1$ and so $k + \Psi_j(1) < E(t_j)$ and $t_j < E^{-1}(k + h_{n-j-1} - h_j)$.

(ii) Note that

$$\sum_{i=1}^n \frac{1}{b_j - a_i} = \sum_{i=1}^n \frac{1}{j + t_j - i} = k = \sum_{i=1}^n \frac{1}{b_{j+1} - a_i} = \sum_{i=1}^n \frac{1}{j + 1 + t_{j+1} - i}.$$

Let $\phi_j(t) = \sum_{i=1}^{n-1} 1/(j + t - i)$, so that ϕ_j is decreasing on each interval of continuity, and the above can be written

$$\begin{aligned} \frac{1}{j-n+t_j} + \phi_j(t_j) &= k = \frac{1}{j+t_{j+1}} + \phi_j(t_{j+1}), \quad \text{or} \\ \phi_j(t_{j+1}) - \phi_j(t_j) &= -\frac{n+t_{j+1}-t_j}{(n-j-t_j)(j+t_{j+1})}, \end{aligned}$$

which is negative, since $t_{j+1} > t_j$ as already noted. In particular we have $|\phi_j(t_{j+1}) - \phi_j(t_j)| > n/[(n-j)(j+1)]$ for $0 \leq j \leq n-1$.

But from the mean value theorem we have $t_{j+1} - t_j = (\phi_j(t_{j+1}) - \phi_j(t_j))/\phi'_j(\theta)$ for some $\theta \in (t_j, t_{j+1})$ and

$$|\phi'_j(\theta)| = \sum_{i=1}^{n-1} \frac{1}{(j+\theta-i)^2} < \frac{\pi^2}{\sin^2 \pi \theta},$$

so that for $1 \leq j \leq n-2$,

$$\begin{aligned} t_{j+1} - t_j &> \frac{n}{\pi^2(n-j)(j+1)} \inf \{ \sin^2 \pi \theta : t_j \leq \theta \leq t_{j+1} \} \\ &= \frac{n}{\pi^2(n-j)(j+1)} \min \{ \sin^2 \pi t_j, \sin^2 \pi t_{j+1} \} \\ &> \frac{n}{\pi^2(n-j)(j+1)} \min \{ \sin^2 \pi p_j, \sin^2 \pi q_{j+1} \}. \end{aligned}$$

Since $E(p_j) = \pi \cot \pi p_j = k + h_{n-j} - h_{j-1}$ it follows that

$$\sin^2 \pi p_j = \frac{\pi^2}{\pi^2 + (k + h_{n-j} - h_{j-1})^2}, \quad \text{and similarly}$$

$$\sin^2 \pi q_{j+1} = \frac{\pi^2}{\pi^2 + (k + h_{n-j-2} - h_{j+1})^2}.$$

Hence

$$t_{j+1} - t_j > \frac{n}{(n-j)(j+1)} \frac{1}{\pi^2 + k'^2}$$

where $k' = \max(|k + h_{n-j} - h_{j-1}|, |k + h_{n-j-2} - h_{j+1}|)$, as required, if $1 \leq j \leq n - 2$.

When $j = n - 1$ we have $|\phi'_j(\theta)| = \sum_{i=0}^{n-2} 1/(\theta + i)^2$ which is positive and decreasing on $(0, \infty)$. Hence since $t_n > \theta > t_{n-1} > p_{n-1} > 0$ we have

$$|\phi'_j(\theta)| < |\phi'_j(t_{n-1})| < |\phi'_j(p_{n-1})| < \sin^2(\pi p_{n-1})/\pi^2 = 1/(\pi^2 + (k + 1 - h_{n-2})^2)$$

as required. The case $j = 0$ is similar, and the proof of Lemma 2 is complete.

Before we can go on to the proof of Theorem 1, we need the following technicality.

LEMMA 3. Let $G(t) = t(1-t)[\pi^2 + \{k + \log((1-t)/t)\}^2]$, for $t \in (0, 1)$, and $G(0) = G(1) = 0$.

Then $G(t) \leq (\pi^2 + (|k| + c)^2)/4$ where $c = \log((\pi + 1)/(\pi - 1))$.

PROOF OF LEMMA 3. Suppose that $k \geq 0$ without loss of generality. Then $G(t)$ is continuous and non-negative on $[0, 1]$ and $G(t) \geq G(1-t)$ for $0 \leq t \leq 1/2$, so that in looking for a maximum we need consider only the interval $[0, 1/2]$. We have $G'(t) = 0$ when

$$0 = \frac{1}{t} - \frac{1}{1-t} + \frac{2\theta}{\pi^2 + \theta^2} \left(-\frac{1}{1-t} - \frac{1}{t} \right)$$

where $\theta = k + \log((1-t)/t)$, or $1 - 2t = 2\theta/(\pi^2 + \theta^2)$. But $2\theta/(\pi^2 + \theta^2) \leq 1/\pi$, so we must have $1/2 \geq t \geq t_0 = (1 - 1/\pi)/2$.

Hence we can get an upper bound for $G(t)$ by maximising $t(1-t)$ at $t = 1/2$, and by maximising $\pi^2 + \theta^2$ at $t = t_0$, which gives $G(t) \leq (\pi^2 + (k + \log((\pi + 1)/(\pi - 1)))^2)/4$ as required.

PROOF OF THEOREM 1. The proof is in two parts. In the first we assume equal separation and use Lemmas 2 and 3 to obtain the result. In the second we consider what modifications are needed to deal with the case of almost equal separation.

We can suppose that $k \geq 0$ since the result is the same for k and $-k$. We have to minimise over j the various estimates for $t_{j+1} - t_j$ which appear in Lemma 2. More precisely we have to find upper bounds for

$$f_1(j) = (n - j)(j + 1)(\pi^2 + (k + h_{n-j-2} - h_{j+1})^2) \quad \text{for } 0 \leq j \leq n - 2, \quad \text{and}$$

$$f_2(j) = (n - j)(j + 1)(\pi^2 + (k + h_{n-j} - h_{j-1})^2) \quad \text{for } 1 \leq j \leq n - 1.$$

We consider f_1 firstly and deduce the results for f_2 . We use repeatedly the elementary inequalities

$$0 \leq \log \frac{m + 1}{n + 1} \leq h_m - h_n \leq \log \frac{m + \frac{1}{2}}{n + \frac{1}{2}}$$

which are valid for all integers $m \geq n \geq 0$.

In dealing with f_1 we first consider the values of j for which $k + h_{n-j-2} - h_{j+1} \geq 0$, and we bound this above by

$$k + \log \frac{n - j - \frac{3}{2}}{j + \frac{3}{2}} \quad \text{or} \quad k + \log \frac{n - j - 1}{j + 2}$$

accordingly as $j + 1 \leq n - j - 2$ or not. But

$$\begin{aligned} & (n - j)(j + 1) \left(\pi^2 + \left(k + \log \frac{n - j - \frac{3}{2}}{j + \frac{3}{2}} \right)^2 \right) \\ & \leq (n - j) \left(j + \frac{3}{2} \right) \left(\pi^2 + \left(k + \log \frac{n - j}{j + \frac{3}{2}} \right)^2 \right) \\ & \leq \left(n + \frac{3}{2} \right)^2 (\pi^2 + (k + c)^2) / 4, \end{aligned}$$

by Lemma 3 with $t = (j + 3/2)/(n + 3/2)$; similarly

$$(n - j)(j + 1) \left(\pi^2 + \left(k + \log \frac{n - j - 1}{j + 2} \right)^2 \right) \leq (n + 2)^2 (\pi^2 + (k + c)^2) / 4$$

by Lemma 3 with $t = (j + 2)/(n + 2)$.

For values of j with $k + h_{n-j-2} - h_{j+1} < 0$ (which certainly requires $j + 1 > n - j - 2$) we have $|k + h_{n-j-2} - h_{j+1}| < k + h_{j+1} - h_{n-j-2} < k + \log \left[(j + \frac{3}{2}) / (n - j - \frac{3}{2}) \right]$, and so

$$f_1(j) < (n - j)(j + 1) \left(\pi^2 + \left(k + \log \frac{j + \frac{3}{2}}{n - j - \frac{3}{2}} \right)^2 \right)$$

$$\begin{aligned} &= (n - j)(j + 1) \left(\pi^2 + \left(k + \log \frac{j + \frac{3}{2}}{n - j} + \log \frac{n - j}{n - j - \frac{3}{2}} \right)^2 \right) \\ &\leq (n - j)(j + 1) \left(\pi^2 + \left(k + \log 4 + \log \frac{j + \frac{3}{2}}{n - j} \right)^2 \right) \\ &< \left(n + \frac{3}{2} \right)^2 (\pi^2 + (k + \log 4 + c)^2) / 4, \end{aligned}$$

using Lemma 3 with $k + \log 4$ for k and $t = (n - j)/(n + 3/2)$. The consideration of f_2 is reduced to that of f_1 by putting $n - j - 1$ for j and $-k$ for k . This completes the proof of Theorem 1 in the case of equal separation when $d = \delta(f) = 1$, giving a slightly stronger result in which the $2/n$ term is absent in the denominator. For the general case in which $d > 0$ is unrestricted, we rescale, replacing $\delta(f' - kf)$ by $\delta(f' - kf)/d$ and k by kd , and the result follows for this case.

To show how the proof must be modified in the case of almost equal separation we use the following lemma which refers back to the initial situation in which the points a_i are unrestricted.

LEMMA 4. For $1 \leq j \leq n$ write $b_j = a_j + t_j$ and $b_{j-1} = a_j - u_j$, so that t_1, \dots, t_{n-1} and u_1, \dots, u_{n-1} are defined for all k , while t_n exists only for $k > 0$, and u_1 only for $k < 0$. Let $\sigma_j = \sum_{k \neq j} 1/(a_j - a_k)$, $\sigma_j^* = \sum_{k \neq j} 1/(j - k)d = (h_{j-1} - h_{n-j})/d$ where $d = \delta(f)$, and let $F_j(t) = \sum_{i=1}^n 1/(t + (j - i)d)$ for $0 < t < d$, so that F_j is a continuous decreasing function from $(0, d)$ onto \mathbb{R} .

Then (i) $t_j \geq F_j^{-1}(k + \sigma_j^* - \sigma_j)$, and (ii) $u_j \geq d - F_{j-1}^{-1}(k + \sigma_j^* - \sigma_j)$.

Note that if we add the results of (i) and (ii) and use the result that $F_j^{-1} > F_{j-1}^{-1}$ which follows from Lemma 2(i), we obtain $t_j + u_j > d$; that is, we have obtained a new proof of $\delta(f' - kf) > \delta(f)$ along completely different lines from the one in [4].

PROOF OF LEMMA 4. The equation $\sum_1^n 1/(x - a_i) = k$ is satisfied when $x = a_j + t_j$, so

$$\sum_1^n 1/(t_j + a_j - a_i) = k \text{ and consequently } -dk/dt_j = \sum_1^n 1/(t_j + a_j - a_i)^2.$$

Hence for $0 < t_j < d$,

$$-\frac{dk}{dt_j} \leq \sum_1^n \frac{1}{(t_j + (j - i)d)^2},$$

$$\frac{d}{dt_j} \left(\frac{1}{t_j} - k \right) \leq \frac{d}{dt_j} \left(\sum_{i \neq j} \frac{-1}{t_j + (j-i)d} \right), \quad \text{which gives}$$

$$\frac{1}{t_j} - k + \sigma_j \leq \sum_{i \neq j} \left(\frac{1}{(j-i)d} - \frac{1}{t_j + (j-i)d} \right)$$

on integration from 0 to t_j . But the last equation gives

$$k + \sigma_j^* - \sigma_j \geq \sum_1^n 1/(t_j + (j-i)d) = F_j(t_j)$$

as required for (i); the proof of (ii) is similar.

The proof of Theorem 1 can now be completed quickly. Suppose that in the case of almost equal separation we have $a_i = id$ for $1 \leq i \leq n - 1$ and $a_n \geq nd$. Then

$$0 > \sigma_j^* - \sigma_j = \frac{1}{(j-n)d} - \frac{1}{jd - a_n} > \frac{-1}{(n-j)d}.$$

But in the proof of Lemma 3 the maxima of f_1 and f_2 are attained when $1 \leq j \leq n/2$, and so we have $0 > \sigma_j^* - \sigma_j > -2/nd$. Putting this into (i) and (ii) of Lemma 4 shows that we can use the estimates for the equally spaced configuration (given by the functions F_j^{-1}) with a value of k which is altered by at most $2/(nd)$ and this proves the stated result in full.

Concerning the best value of the ratio in Theorem 1, we can make the following observations. For the case of equal spacing and $\delta(f) = 1$, we define $c_n(k) = \min_j r_j = \min_j (b_{j+1} - b_j)$ for any $k \in \mathbb{R}$. It follows from $\delta(f' - kf) > \delta(f)$ that $c_n(k) > 1$ and we conjecture that $c_n(kd)$ gives the best value of the ratio in Theorem 1; that is that equal separation always gives the configuration which minimises r_j .

Since each b_j is a real-analytic function of k , $c_n(k)$ is piecewise analytic. Simple asymptotic calculations show both that $r_j \simeq 1 + n/(k^2 j(n-j))$ for fixed j as $k \rightarrow +\infty$ (and a similar result at $-\infty$), and that for $k = 0$, $c_n(0) \simeq 1 + 4/(\pi^2 n)$ as $n \rightarrow \infty$. This shows that Theorem 1 gives at least the correct orders of magnitude.

However it is certainly not the case that the minimum of r_j is always attained for the same value of j ; it happens even in the smallest interesting cases ($n = 4, 5$) that the location of the minimum changes from one branch to another as k varies. Hence if, as we conjecture, $c_n(k)$ is really the best value of the ratio in Theorem 1, then this cannot be given by a single real-analytic function at all points.

There can be no upper bound of the form $\delta(f' - kf) \leq (\text{constant})\delta(f)$ since $\delta(f)$ may tend to 0 while $\delta(f' - kf)$ remains bounded away from 0. However for $k \neq 0$ we have trivially $\delta(f' - kf) \leq \delta(f) + n/|k|$; the following argument shows

slightly more, namely, that $\delta(f' - kf)^2 \leq \delta(f)^2 + (n/k)^2$. A stronger result, that $\delta(f' - kf) < \delta(f) + O(\log n \log \log n/|k|)$ will appear in [5].

Suppose then that $k > 0$ is given and that j is chosen so that $\delta(f) = a_{j+1} - a_j$: say $a_j = 0, a_{j+1} = d = \delta(f)$. Then we have $0 = a_j < b_j < d = a_{j+1} < b_{j+1}$ and $\delta(f' - kf) \leq b_{j+1} - b_j$.

However when we fix the positions of a_j, a_{j+1} while allowing j and the other a_i to vary unrestrictedly, then the maximum of b_{j+1} occurs when $j + 1 = n$, while $a_1 < \dots < a_{n-1}$ are all close to zero, so that

$$b_{j+1} < \frac{1}{2} \left\{ (d + n/k) + \sqrt{(d + n/k)^2 - 4(n - 1)d/k} \right\}.$$

Similarly the minimum of b_j is when $j = 1$ and all $a_2 < \dots < a_n$ are close to d , so that $b_j > \left\{ (d + n/k) - \sqrt{(d + n/k)^2 - 4d/k} \right\} / 2$, and we deduce that

$$\begin{aligned} \delta(f' - kf) &\leq \frac{1}{2} \left\{ \sqrt{(d + n/k)^2 - 4(n - 1)d/k} + \sqrt{(d + n/k)^2 - 4d/k} \right\} \\ &\leq \sqrt{d^2 + (n/k)^2}. \end{aligned}$$

Similar estimates occur in the next section of the paper when we consider the total separation $\Delta(f)$.

3. Total Separation

For $\Delta(f) = a_n - a_1$ we have different results for f' and for $f' - kf, (k \neq 0)$. The restriction to polynomials having only real zeros still applies.

Notice that in this section we have to take account of repeated zeros of f , a possibility that obviously did not occur in Section 2. To handle this we denote the distinct zeros of f by $\alpha_i, 1 \leq i \leq m$, say where α_i is repeated n_i times so that $\sum_1^m n_i = n$, and let β_i be the zero of f' which lies in the interval (α_i, α_{i+1}) .

THEOREM 2. (i) *For all polynomials of degree $n \geq 3$ we have*

$$\sqrt{\frac{n - 2}{n}} \Delta(f) \leq \Delta(f') \leq \Delta(f).$$

There is equality on the left if and only if $f(x) = x(x - 1/2)^{n-2}(x - 1)$, taking $\alpha_1 = 0, \alpha_m = 1$. There is equality on the right if and only if both a_1 and a_n are repeated zeros of f .

(ii) If $k \neq 0$ we have on putting $\Delta(f) = d$ that

$$d = \Delta(f) < \frac{1}{2} \left\{ \sqrt{(d + n/|k|)^2 - 4d/|k|} + \sqrt{(d + n/|k|)^2 - 4(n - 1)d/|k|} \right\} < \Delta(f' - kf) < \frac{1}{2} \left\{ (d + n/|k|) + \sqrt{(d + n/|k|)^2 - 4d/|k|} \right\}.$$

PROOF. (i) The inequality $\Delta(f') \leq \Delta(f)$ is immediate from Rolle’s theorem; we shall not consider it further.

The other inequality is trivially true when both α_1 and α_m are repeated. If only one of them (say α_1) is repeated then $\Delta(f') = \beta_{m-1}$, and as in the proof of Lemma 1 we have $\partial\beta_{m-1}/\partial\alpha_i > 0$ for all $i, 2 \leq i \leq m - 1$, and it follows that no such α_i can exist in the minimum configuration. This means that we need only consider $f(x) = x^{n-1}(x - 1)$ in which case $\Delta(f') = (n - 1)/n$ which is $\geq \sqrt{(n - 2)/n}$ as required.

The more interesting case is when neither α_1 nor α_m are repeated. Then f has the form $f(x) = x \prod_2^{m-1} (x - \alpha_i)^{n_i} (x - 1)$, which gives

$$\frac{f'(x)}{f(x)} = \frac{1}{x} + \sum_2^{m-1} \frac{n_i}{x - \alpha_i} + \frac{1}{x - 1}$$

which is equal to zero if $x = \beta_j$ and so

$$\frac{\partial\beta_j}{\partial\alpha_i} = \frac{n_i}{(\beta_j - \alpha_i)^2 S_j} \quad \text{with} \quad S_j = \frac{1}{\beta_j^2} + \sum_2^{m-1} \frac{n_i}{(\beta_j - \alpha_i)^2} + \frac{1}{(\beta_j - 1)^2}.$$

Consequently $\partial(\beta_{m-1} - \beta_1)/\partial\alpha_i$ is > 0 (or $= 0$ or < 0) according to whether

$$\left(\frac{\beta_1 - \alpha_i}{\beta_{m-1} - \alpha_i} \right)^2 > (\text{or } =, \text{ or } <) \frac{S_{m-1}}{S_1}.$$

But S_{m-1}/S_1 is independent of i , while the left side is a strictly increasing function of i for $2 \leq i \leq m - 2$. It follows that in the minimum configuration at most one of these derivatives can equal zero and so f must have the form $x(x - \alpha)^{n-2}(x - 1)$. In this case we find that $\beta_{1,2} = (2\alpha - 1 + n \pm \sqrt{(2\alpha - 1 + n)^2 - 4\alpha n})/(2n)$, so $\Delta(f') = \sqrt{(2\alpha - 1 + n)^2 - 4\alpha n}/n$ which is minimised when $\alpha = 1/2$ and (i) is proved.

(ii) We take $k > 0$ and $\alpha_1 = 0$ without loss of generality. Then $f' - kf$ has one further zero $\beta_m > \alpha_m$, and $\Delta(f' - kf) = \beta_m - \alpha_1$ if α_1 is repeated, $\Delta(f' - kf) = \beta_m - \beta_1$ otherwise.

In the first case, since $\partial\beta_m/\partial\alpha_i > 0$ for all i with $1 < i < m$, no such α_i can exist in either a minimum or a maximum configuration. Hence, writing $a_n = \alpha_m = \Delta(f) = d$,

β_m is minimised when $f(x) = x^{n-1}(x-d)$, and maximised when $f(x) = x^2(x-d)^{n-2}$, giving

$$(1) \quad \begin{aligned} \frac{1}{2} \{ (d + n/k) + \sqrt{(d + n/k)^2 - 4(n-1)d/k} \} &\leq \Delta(f' - kf) \\ &\leq \frac{1}{2} \{ (d + n/k) + \sqrt{(d + n/k)^2 - 8d/k} \} \end{aligned}$$

when α_1 is repeated.

The second case in which α_1 is not repeated is more difficult. The proof can proceed as in (i) above to the point where there can be at most one variable point α , $0 < \alpha < d = \Delta(f)$, so that $f(x) = x(x - \alpha)^{n-r}(x - d)^{r-1}$, for some r , $1 < r \leq n$. However the determination of β_1 and β_3 requires the explicit calculation of the roots of a cubic equation, and finding the value of α to give a stationary value of $\beta_3 - \beta_1$ turns out to be impossibly cumbersome.

We fall back on some approximations. The maximum and minimum values of β_3 are given respectively when $\alpha = d$ and when α is (arbitrarily close to) zero, with $r = 2$ in the latter case. This gives the range of values

$$\begin{aligned} \frac{1}{2} \{ (d + n/k) + \sqrt{(d + n/k)^2 - 4(n-1)d/k} \} &< \beta_3 \\ &\leq \frac{1}{2} \{ (d + n/k) + \sqrt{(d + n/k)^2 - 4d/k} \}. \end{aligned}$$

The same configurations give minimum and maximum values of β_1 :

$$0 < \beta_1 \leq \frac{1}{2} \{ (d + n/k) - \sqrt{(d + n/k)^2 - 4d/k} \}.$$

Combining these gives

$$(2) \quad \begin{aligned} \frac{1}{2} \{ \sqrt{(d + n/k)^2 - 4d/k} + \sqrt{(d + n/k)^2 - 4(n-1)d/k} \} &< \beta_3 - \beta_1 = \Delta(f' - kf) \\ &< \frac{1}{2} \{ (d + n/k) + \sqrt{(d + n/k)^2 - 4d/k} \}, \end{aligned}$$

and the result follows on combining (1) and (2).

NOTE. It is easy to check that the expression on the left of (2) is greater than d and less than the left side of (1), so we have shown in particular that if $k \neq 0$ then $\Delta(f' - kf) > \Delta(f)$ as claimed in the introduction.

References

- [1] P. Henrici, *Applied and computational complex analysis, I* (Wiley, New York, 1974).
- [2] M. Marden, *Geometry of polynomials* (Amer. Math. Soc., Providence, 1965).
- [3] G. Szegő and G. Pólya, *Problems and theorems in analysis, II* (Springer, New York, 1976).
- [4] P. L. Walker, 'Separation of the zeros of polynomials', *Amer. Math. Monthly*, **100** (1993), 272–273.
- [5] ———, 'Upper bounds for separation of real zeros of polynomials', *Proc. Edinburgh Math. Soc.*, to appear.

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