ON DIAGRAMS OF VECTOR SPACES

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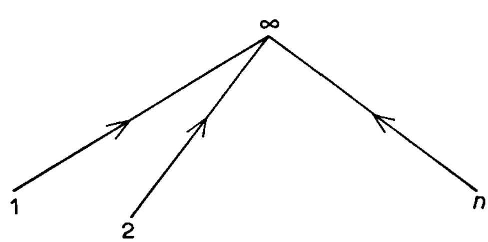
We record here two further remarks about the systems, studied in [1] and [2], consisting of a vector space U and a set $\boldsymbol{K}$ of subspaces of U. In § 1, we show that such a system may be viewed as a module over a suitable artinian ring; the results of [1] and [2] thus serve to illustrate the complexity of structure of these modules. The main idea, a little wider than one introduced by Mitchell in Chapter IX of [3], is to view a diagram of vector spaces, with a small category as the scheme of the diagram, as a module over the 'category ring' of the category.

In § 2, we answer negatively the question, raised in [1], as to whether each associative algebra E with identity, over a field $\boldsymbol{\Phi}$, can be represented as the endomorphism algebra of a $\boldsymbol{\Phi}$-vector space system U, $\boldsymbol{K}$ with $|\boldsymbol{K}| = 4$. Specifically, we show that the ring Δ_n of 'hollow triangular n-th order matrices over $\boldsymbol{\Phi}$' is so representable if and only if $n \leqq 5$.

1. Vector space systems as modules

Let Σ be a small category, $\boldsymbol{\Phi}$ an associative ring with identity, and $\mathscr{M}_{\Phi}$ the category of right $\boldsymbol{\Phi}$-modules. A covariant functor $D : \Sigma \to \mathscr{M}_{\Phi}$ will be called a *Σ-diagram of $\boldsymbol{\Phi}$-modules.* These diagrams are the objects of a category $\mathscr{D} = \mathscr{D}(\Sigma, \boldsymbol{\Phi})$, the morphisms of $\mathscr{D}$ being the natural transformations between diagrams. Since $\mathscr{M}_{\Phi}$ is abelian, so also is $\mathscr{D}$.

Consider the category Σ_{n+1} associated in the normal way with the partially ordered set depicted in the figure



Thus Σ_{n+1} has $n+1$ objects $1, 2, \cdots, n, \infty$, and morphisms $e_{ii} : i \to i$, $e_{\infty\infty} : \infty \to \infty$, and $e_{i,\infty} : i \to \infty$ for $1 \leqq i \leqq n$. Let $\boldsymbol{\Phi}$ be a field. Then a

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diagram $D : \Sigma_{n+1} \to \mathscr{M}_\Phi$ *in which each* $D(e_{i,\infty})$ *is injective* may be regarded as a vector space $U = D(\infty)$ and an *indexed* family $\boldsymbol{K} = \{\operatorname{Im} D(e_{i,\infty})\}$ of n subspaces of U. In particular, End D is just the endomorphism algebra, in the sense of [1], of the system U, $\boldsymbol{K}$.

In [3, Chapter IX], Mitchell shows that the category $\mathscr{D} = \mathscr{D}(\Sigma, \boldsymbol{\Phi})$ is equivalent to the category $\mathscr{M}_{\Phi(\Sigma)}$, where Σ is the category associated with a finite partially ordered set, and $\boldsymbol{\Phi}(\Sigma)$ is a suitable ring of matrices over $\boldsymbol{\Phi}$. This identification may be made for any small category Σ. Let I be its object set and $\boldsymbol{M}$ its morphism set. The appropriate ring $\boldsymbol{\Phi}(\Sigma)$ may be taken to be the $\boldsymbol{\Phi}$-algebra having $\boldsymbol{M}$ as a free basis, the multiplication of basis elements e, e' being defined by the rule

$$ee' = \begin{cases} \text{their product in } \Sigma, \text{ if defined,} \\ 0 \text{ otherwise.} \end{cases}$$

This construction thus generalises that of the group ring of a group. Notice that each object i determines an idempotent e_{ii} in $\boldsymbol{\Phi}(\Sigma)$, and that $\boldsymbol{\Phi}(\Sigma)$ has an identity, namely $\sum_{i \in I} e_{ii}$, if and only if I is finite. It is easy to describe Mitchell's identification of $\mathscr{D}$ with $\mathscr{M}_{\Phi(\Sigma)}$. Let $D \in \mathscr{D}$; define $M(D) = \oplus_{i \in I} D(i)$ and, for $e : j \to k$, define the action of e on $M(D)$ to be 0 on summands $D(i)$ with $i \neq j$, and $D(e)$ on the summand $D(j)$. This yields a functor from $\mathscr{D}$ to $\mathscr{M}_{\Phi(\Sigma)}$. Conversely, for each $\boldsymbol{\Phi}(\Sigma)$-module M, define $D : \Sigma \to \mathscr{M}_\Phi$ to be the diagram with values $D(i) = Me_{ii} (i \in I)$, and $D(e) : D(j) \to D(k)$ to be the map induced by right multiplication by $e : j \to k$. These two functors give the required equivalence of categories.

In the case of the category Σ_{n+1} depicted above, $\boldsymbol{\Phi}(\Sigma_{n+1})$ is generated by the $2n+1$ morphisms e_{ij}, and these satisfy the usual matrix identities $e_{ij}e_{kl} = \delta_{jk}e_{il}$. We call $\boldsymbol{\Phi}(\Sigma_{n+1})$ the ring $\Lambda_{n+1} = \Lambda_{n+1}(\boldsymbol{\Phi})$ of *open hollow triangular* $(n+1)$*-th order matrices over* $\boldsymbol{\Phi}$.

Let $\boldsymbol{\Phi}$ be a field. Then Λ_{n+1} is artinian, and of quite simple type. The results of [1] may be interpreted as statements about Λ_{n+1}-modules *in which all the morphisms in the associated vector space diagrams are injective.* In fact, it is easy to see that each Λ_{n+1}-module is the direct sum of one of this type and of an injective module.

We draw attention to the module versions of two results in [1] and [2].

(1) Let $n \geqq 5$. Each associative $\boldsymbol{\Phi}$-algebra E with identity may be represented as the endomorphism ring of a $\Lambda_{n+1}(\boldsymbol{\Phi})$-module, of $\boldsymbol{\Phi}$-dimension at most $7(\dim E)^2$.

(2) Let $n \geqq 5$, and let c be any finite or infinite cardinal. There is a $\Lambda_{n+1}(\boldsymbol{\Phi})$-module of $\boldsymbol{\Phi}$-dimension greater than or equal to c with endomorphism ring isomorphic to $\boldsymbol{\Phi}$.

We show in § 2 that (1) fails for $n = 4$. The Λ_2, Λ_3, Λ_4, and Λ_5-modules of finite $\boldsymbol{\Phi}$-dimension and endomorphism ring $\boldsymbol{\Phi}$ are listed (in vector space form) in [1]. We do not know whether Φ can be realised as endomorphism ring of a Λ_5-module of *infinite* dimension.

A modification of (2) may be obtained for an arbitrary ring Φ, in the following form. Let c be a finite (countable) cardinal, and $n \geqq 4$ ($n \geqq 5$). Then, there exists a $\Lambda_{n+1}(\Phi)$-module which has the opposite ring of Φ as endomorphism ring, and is free as a Φ-module, on a basis of cardinality $\geqq c$. Indeed, we can give an explicit presentation of such a module, or more conveniently, of the corresponding Φ-module system $U, \boldsymbol{K}$. If c is countable, take $U, K_1, \cdots, K_5$ to be the free Φ-modules on the following bases:

$$\begin{aligned} &U \text{ has basis } \{x_r\}_{r\geqq 1} \cup \{y_r\}_{r\geqq 1} \\ &K_1 \text{ has basis } \{x_r\}_{r\geqq 1} \\ &K_2 \text{ has basis } \{y_r\}_{r\geqq 1} \\ &K_3 \text{ has basis } \{x_r+y_r\}_{r\geqq 1} \\ &K_4 \text{ has basis } \{x_r+y_{r+1}\}_{r\geqq 1} \\ &K_5 \text{ has basis } \{x_1\}. \end{aligned}$$

A very easy computation shows that the endomorphisms of $U, \boldsymbol{K}$ are induced by maps of the form $x_r \to x_r\phi$, $y_r \to y_r\phi$ ($r \geqq 1$), for $\phi \in \Phi$; so the endomorphism ring of $U, \boldsymbol{K}$ is isomorphic to the opposite ring of Φ. For c finite, similar presentations of suitable systems $U, \boldsymbol{K}$, with $|\boldsymbol{K}| = 4$, are contained in the Appendix to [1]. One of their essential features is that the matrices expressing the given bases of the submodules K_i in terms of the given basis of U contain zeros and ones only.

2. Non-representability of some algebras

As in [1], let $\mathscr{E}(U, \boldsymbol{K})$ denote the ring of all endomorphisms of the Φ-vector space U which leave invariant each member of the set $\boldsymbol{K}$ of subspaces of U. It was shown in [1] that, if $|\boldsymbol{K}| = 5$, and E is any associative Φ-algebra with identity, of finite Φ-dimension, then there exist a finite dimensional space U, and $\boldsymbol{K}$, such that $\mathscr{E}(U, \boldsymbol{K}) \simeq E$. In case $|\boldsymbol{K}| = 4$, it was shown that this result could fail for some basefields Φ. We shall now show that it fails for *any* field Φ.

By a *hollow triangular n-th order matrix over the field* Φ, we mean an n-th order matrix (ϕ_{ij}) with entries in Φ such that $\phi_{ij} = 0$ unless either $i = j$, or $i = 1$, or $j = n$. The set of all such matrices forms a ring Δ_n. We assert that

there exists a pair $U, \boldsymbol{K}$ with $\dim U$ finite, $|\boldsymbol{K}| = 4$, and $\mathscr{E}(U, \boldsymbol{K}) \simeq \Delta_n$ if and only if $n \leqq 5$.

Nevertheless, if Φ is infinite, it may be shown that, for all n, Δ_n can be represented as the endomorphism ring of some Λ_5-module; of course, for $n > 5$, such a module cannot correspond to a pair U, $\boldsymbol{K}$. However, a modification of the argument below shows that the ring direct sum of Δ_6 and Φ cannot be the endomorphism ring of a Λ_5-module.

The proof of the assertion above involves much tedious and elementary case checking, and we merely outline it. Let U be a Φ-space and $\boldsymbol{K}$ a set of subspaces of U such that $\mathscr{E}(U, \boldsymbol{K}) \cong \Delta_n$. Let d_{rs} be the element of $\mathscr{E}(U, \boldsymbol{K})$ corresponding to the matrix in Δ_n with 1 at the place (r, s) and 0 elsewhere ($r = s$, or $r = 1$, or $s = n$). The elements d_{rs} form a Φ-basis of $\mathscr{E}(U, \boldsymbol{K})$, and $d_{rs}d_{tu} = \delta_{st}d_{ru}$.

Write $U_r = Ud_{rr}$ and $\boldsymbol{K}_r = \{Kd_{rr} : K \in \boldsymbol{K}\}$. Then $\boldsymbol{K}_r$ is a set of subspaces of U_r, and $U = \oplus_{r=1}^{n} U_r$. Let

$$\begin{aligned} H_{rs} &= \mathrm{Hom}\ ((U_r, \boldsymbol{K}_r), (U_s, \boldsymbol{K}_s)) \\ &= \{h \in \mathrm{Hom}\ (U_r, U_s) : \forall K \in \boldsymbol{K}, Kd_{rr}h \subseteqq Kd_{ss}\}. \end{aligned}$$

Each element h of H_{rs} may be extended to an element of $\mathscr{E}(U, \boldsymbol{K})$ by defining it to be 0 on U_t, for $t \neq r$. However, the only element of $\mathscr{E}(U, \boldsymbol{K})$ which maps U_r into U_s is 0 unless $r = s$, or $r = 1$, or $s = n$, in which cases it must be a scalar multiple of d_{rs}. Hence

$$(*) \quad \dim H_{rs} = \begin{cases} 1 \text{ if } r = s, \text{ or } r = 1, \text{ or } s = n, \\ 0 \text{ otherwise.} \end{cases}$$

In particular, $H_{rr} = \mathscr{E}(U_r, \boldsymbol{K}_r) \cong \Phi$.

Now let $\dim U$ be finite and $|\boldsymbol{K}| = 4$. The possible systems U_r, $\boldsymbol{K}_r$ with $H_{rr} \cong \Phi$ are listed in the Appendix to [1], and an examination of the homomorphisms between them shows that the conditions (*) cannot be satisfied if $n > 5$. On the other hand, for $n \leqq 5$, the conditions (*) may be satisfied in such a way that there exist $h_{rs} \in H_{rs}$ such that $h_{rs}h_{tu} = \delta_{st}h_{ru}$. So Δ_n is representable in the form $\mathscr{E}(U, \boldsymbol{K})$ provided $n \leqq 5$.

The condition that $\dim U$ be finite could be omitted if it is true that $\mathscr{E}(V, \boldsymbol{L}) \cong \Phi$ and $|\boldsymbol{L}| = 4$ implies that $\dim V$ is finite.

References

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