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Abstract

Let C be a locally planar curve. Its versal deformation admits a stratification by the genera of the fibres. The strata are singular; we show that their multiplicities at the central point are determined by the Euler numbers of the Hilbert schemes of points on C. These Euler numbers have made two prior appearances. First, in certain simple cases, they control the contribution of C to the Pandharipande–Thomas curve counting invariants of three-folds. In this context, our result identifies the strata multiplicities as the local contributions to the Gopakumar–Vafa BPS invariants. Second, when C is smooth away from a unique singular point, a conjecture of Oblomkov and the present author identifies the Euler numbers of the Hilbert schemes with the 'U(∞)' invariant of the link of the singularity. We make contact with combinatorial ideas of Jaeger, and suggest an approach to the conjecture.

1. Introduction

Let $\mathcal{C} \to \Lambda$ be a projective flat family of integral, locally planar, complex algebraic curves over a smooth base. The fibres necessarily share the same arithmetic genus g, and it is known [DH88, Tes80] that the geometric genus gives a lower semicontinuous function $\tilde{g} : \Lambda \to \mathbb{Z}$. For $h \leqslant g$ we write

$$\Lambda_h = \{\lambda \in \Lambda \mid \mathcal{C}_\lambda \text{ is of geometric genus} \leqslant h\}.$$

This gives a stratification by closed subvarieties

$$\Lambda_0 \subset \cdots \subset \Lambda_g = \Lambda.$$

By semicontinuity, $\lambda \notin \Lambda_h$ unless $\tilde{g}(\lambda) \leqslant h$. By convention we take $\Lambda_h = \emptyset$ for any $h > g$.

We say the family is *locally versal at* $\lambda \in \Lambda$ when the induced deformations of the germs of the singular points of $\mathcal{C}_\lambda$ are versal. We recall properties of versal deformations of singularities in § 4 and refer to [GLS07] for a detailed treatment. At a locally versal point λ, in the range $\tilde{g}(\lambda) \leqslant h \leqslant g$, it is known that the stratum Λ_h is non-empty of pure codimension $g - h$ and is the closure of the locus Λ_h^+ of curves with $g - h$ nodes [DH88, Tes80]. While Λ_h^+ is smooth, Λ_h will generally be singular. We are interested in the multiplicities $\deg_\lambda \Lambda_h$, i.e., the number of points near λ in which Λ_h intersects a generic space of the appropriate codimension. For instance, since $\Lambda_g = \Lambda$ is smooth by assumption,

$$\deg_\lambda \Lambda_g = 1. \qquad (1)$$

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The multiplicities depend only on the singularities of $\mathcal{C}_\lambda$. Denote the germs of the singularities by c_i, their respective contributions to $g-\tilde{g}(\lambda)$ by $\delta(c_i)$, and the bases of their miniversal deformations by $\mathbb{V}(c_i)$. Fix $\pi:\Lambda\to\prod\mathbb{V}(c_i)$ compatible with the deformations of c_i induced by Λ; it is unique up to first order and smooth by local versality of Λ [GLS07, p. 237]. We write $\mathbb{V}_h^+(c_i)$ for the locus in $\mathbb{V}(c_i)$ where the fibres are smooth away from exactly $\delta(c_i)-h$ nodes, and $\mathbb{V}_h(c_i)$ for its closure. The stratifications are compatible: $\Lambda_{\tilde{g}+h}=\pi^{-1}(\bigcup_{h=\sum h_i}\prod_i\mathbb{V}_{h_i}(c_i))$. Since π is a smooth morphism,

$$\deg_\lambda\Lambda_{\tilde{g}+h}=\sum_{h=\sum h_i}\prod_i\deg_{[c_i]}\mathbb{V}_{h_i}(c_i). \tag{2}$$

In particular, the multiplicity of Λ_{g-1} is given by the sum of the multiplicities of the discriminant loci in the $\mathbb{V}(c_i)$. These are the Milnor numbers $\mu(c_i)$. If c_i be the germ of $f(x,y)=0$ at $(0,0)$, then, for sufficiently general $g(x,y)$, the function $f(x,y)+\epsilon g(x,y)$ has only simple critical points in a neighborhood of $(0,0)$ and, moreover, only one in each fibre $(f+\epsilon g)^{-1}(t)$. The total number of critical points is by definition $\mu(c_i)$. The family of curves $C_{s,t}=\{(x,y)\mid f+sg=t\}$ induces a map from the germ at zero in the (s,t)-plane to $\mathbb{V}(c_i)$, and the line $s=\epsilon$ intersects the discriminant locus in the $\mu(c_i)$ values of t for which $(f+sg)^{-1}(t)$ acquires a node.

Let $b(c_i)$ be the number of analytic local branches. Milnor has shown [Mil68, Theorem 10.5] that $\mu(c_i)=2\delta(c_i)+1-b(c_i)$. Therefore $\chi(\mathcal{C}_\lambda)=2-2\tilde{g}+\sum(1-b(c_i))=2-2g+\sum\mu(c_i)$, and so

$$\deg_\lambda\Lambda_{g-1}=\chi(\mathcal{C}_\lambda)+2g-2. \tag{3}$$

One expects that going to deeper strata will lead to increasingly difficult calculations. Nonetheless, Fantechi, Göttsche, and van Straten [FGS99] showed that the multiplicity of the deepest stratum is equal to the topological Euler number of the compactified Jacobian $\overline{\operatorname{Pic}}^0(\mathcal{C}_\lambda)$. This space is described in detail in [AK80]; it parameterizes torsion free, rank one, degree zero sheaves on $\mathcal{C}_\lambda$:

$$\deg_\lambda\Lambda_0=\chi(\overline{\operatorname{Pic}}^0(\mathcal{C}_\lambda)). \tag{4}$$

Unless $\mathcal{C}_\lambda$ is rational, both sides of (4) vanish; the left-hand side because $\lambda\notin V_0$ by semicontinuity, and the right-hand side because the compactified Jacobian is topologically a product of the Jacobian of the normalization of $\mathcal{C}_\lambda$ and factors coming from the singularities. However, if $\bar{c}_i$ is a rational curve smooth away from a singularity analytically isomorphic to c_i, then (2) and (4) imply

$$\deg_\lambda\Lambda_{\tilde{g}(\lambda)}=\prod_i\chi(\overline{\operatorname{Pic}}^0(\bar{c}_i)). \tag{5}$$

Our main result interpolates between (1), (3), (4) and (5). We will need the Hilbert schemes of points, $X^{[n]}=\{\text{zero-dimensional subschemes of } X \text{ of length } n\}$.

Theorem A. *Let $\mathcal{C}\to\Lambda$ be a family of complete, integral, locally planar curves of arithmetic genus g. If the family is locally versal at $\lambda\in\Lambda$, then*

$$\sum_{n=0}^{\infty}q^n\chi(\mathcal{C}_\lambda^{[n]})=\sum_{h=\tilde{g}}^{g}q^{g-h}(1-q)^{2h-2}\deg_\lambda\Lambda_h.$$

There is an equivalent local version. Let c be the germ of a plane curve singularity. Fix a plane curve C such that c is the germ of C at some point p. We write $c^{[n]}$ for the subvariety of $C^{[n]}$ whose closed points parameterize subschemes of C which are set-theoretically supported at p. This space depends only on the completion of C at p.

THEOREM A$'$. *Let c be the germ of a plane curve singularity which contributes δ to the arithmetic genus and has b analytic local branches. If $[c] \in \mathbb{V}$ is the central point in the base of a versal deformation, then*

$$\sum_{n=0}^{\infty} q^n \chi(c^{[n]}) = \sum_{h=0}^{\delta} q^{\delta-h}(1-q)^{2h-b} \deg_{[c]} \mathbb{V}_h.$$

Theorem A is restricted to (1) and (3); it implies (4) and (5) because $C^{[n]}$ is a $\mathbb{P}^{n-g}$ bundle over $\overline{\mathrm{Pic}}^{\,0}(C)$ for large n [AK80]. The proof combines the methods of Fantechi, Göttsche, and van Straten [FGS99], techniques of Pandharipande and Thomas [PT10], and the following smoothness result. For a morphism $X \to Y$, we denote the relative Hilbert scheme by $X_Y^{[k]} = \{(y \in Y, [Z] \in X_y^{[k]})\}$.

THEOREM B. *Let $\mathcal{C} \to \Lambda$ be a family of complete, reduced, locally planar curves. If the family is locally versal at $\lambda \in \Lambda$ and $\lambda \in \mathbb{D}^k \subset \Lambda$ is a generic, sufficiently small k-dimensional polydisc, then the total space of the relative Hilbert scheme $\mathcal{C}_{\mathbb{D}^k}^{[h]}$ is smooth if $h \leqslant k$.*

We recall facts about generating series of Euler numbers of Hilbert schemes in § 2, and prove Theorem A in § 3, assuming Theorem B, which we prove in § 4. We present formulas for the multiplicities in the case of ADE singularities in § 5. The final two sections discuss previous appearances of the series in the left-hand side of Theorem A. In § 6, we explain its relation to the contribution of $\mathcal{C}_\lambda$ to Gopakumar–Vafa invariants in Pandharipande–Thomas theory [PT10]. Section 7 suggests that Theorem A may relate a conjecture of Oblomkov and Shende [OS10], which compares Euler numbers of Hilbert schemes of points on singular curves to the HOMFLY polynomials of the links of the singularities, to work of Jaeger on state-sum formulae for the HOMFLY polynomial [Ja91]. The reader is warned that the final section, in the words of the anonymous reviewer, 'is rather speculative, unfinished, and at best has the status of a possible approach that might be tried.'

2. Background

We need the following properties of Euler numbers of complex varieties:

- $\chi(X \backslash Y) = \chi(X) - \chi(Y)$ for $X \subset Y$ a closed immersion;
- $\chi(A \times B) = \chi(A)\chi(B)$;
- $\chi(\mathbb{A}^1) = 1$.

The first property makes it natural to weight Euler numbers by constructible functions. For f a constructible function on a space Z, we write $\chi(Z, f) := \sum_i \chi(f^{-1}(i)) \cdot i$.

The remainder of the section collects for convenience facts about generating functions of Euler numbers of Hilbert schemes of locally planar curves. All results are extracted from the work of Pandharipande and Thomas [PT10, Appendix B]. Unless otherwise specified, C is an integral, Gorenstein curve of arithmetic genus g and geometric genus $\tilde{g}$.

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The following statement is elementary.

Proposition 1. *Let* $\mathbf{f} = (f_0, f_1, \ldots)$ *be an arbitrary sequence of integers. Then there is a unique sequence of integers* $\mathbf{n} = (n_g, n_{g-1}, \ldots)$ *giving an equality of formal power series*

$$\sum_{d=0}^{\infty} f_d q^d = \sum_{h=-\infty}^{g} n_h q^{g-h}(1-q)^{2h-2}.$$

The matrix $T = T(g)$ *such that* $\mathbf{n} = T\mathbf{f}$ *is lower triangular with ones on the diagonal; in particular,*

$$n_g = f_0,$$
$$n_{g-1} = f_1 + (2g-2)f_0.$$

The n_h *vanish for* $h < 0$ *if and only if* $f_d - f_{2g-2-d} = c \cdot (d+1-g)$ *for some* c*, in which case* $c = n_0$*.*

Definition 2. Let $n_h(C) \in \mathbb{Z}$ be defined by

$$\sum_{n=0}^{\infty} q^n \chi(C^{[n]}) = \sum_{h=-\infty}^{g} q^{g-h}(1-q)^{2h-2} n_h(C).$$

Lemma 3. *For* C *a smooth curve,* $\sum q^n \chi(C^{[n]}) = (1-q)^{-\chi(C)}$*. If, additionally,* C *is proper of genus* g*, then* $n_g(C) = 1$ *and* $n_h(C) = 0$ *for* $h \neq g$*.*

Proof. For a smooth curve, the Hilbert schemes and symmetric products agree. Thus the claim follows from Macdonald's calculation of the cohomology of symmetric products of curves [Mac62]. $\square$

Remark. The assertion of Theorem A can be restated as $n_h(\mathcal{C}_\lambda) = \deg_\lambda \Lambda_h$.

Lemma 4. *[Har86]. Let* F *be a torsion free sheaf on* C*. Write* F^* *for* $\mathcal{H}om(F, \mathcal{O}_C)$*. Then* $\mathcal{E}xt^{\geqslant 1}(F, \mathcal{O}_C) = 0$ *and* $F = (F^*)^*$*. Serre duality holds in the form* $\mathrm{H}^i(F) = \mathrm{H}^{1-i}(F^* \otimes \omega_C)^*$*. For* F *with rank one and torsion free, define its degree* $d(F) := \chi(F) - \chi(\mathcal{O}_C)$*. This satisfies* $d(F) = -d(F^*)$*, and, for* L *any line bundle,* $d(F \otimes L) = d(F) + d(L)$*.*

Proposition 5. *We have* $n_h(C) = 0$ *for* $h < 0$*. Moreover,* $n_0(C)$ *is the Euler number of the compactified Jacobian of* C*. At the other extreme,* $n_{g-1}(C) = \chi(C) + 2g - 2$ *and* $n_g(C) = 1$*.*

Proof. Let $\overline{\mathrm{Pic}}^n(C)$ be the moduli of rank one, torsion free sheaves of degree n. There is a map $AJ_n : C^{[n]} \to \overline{\mathrm{Pic}}^n(C)$ taking a subscheme $Z \subset C$ to I_Z^*, the dual of the ideal sheaf cutting it out [AK80]. The inclusion $I_Z \to \mathcal{O}_C$ dualizes to a section $\mathcal{O}_C \to I_Z^*$, thus the fibre $AJ_n^{-1}(F) = \mathbb{P}(\mathrm{H}^0(F))$. Viewing $\mathrm{h}^0 : [F] \mapsto \mathrm{h}^0(F)$ as a constructible function on $\overline{\mathrm{Pic}}^n(C)$, we have $\chi(C^{[n]}) = \chi(\overline{\mathrm{Pic}}^n(C), \mathrm{h}^0)$. The involution $F \mapsto \omega_C \otimes F^*$ induces an isomorphism $\iota : \overline{\mathrm{Pic}}^n(C) \cong \overline{\mathrm{Pic}}^{2g-2-n}$. By Serre duality, $\iota \circ \mathrm{h}^0 = \mathrm{h}^1$, and, by the Riemann–Roch theorem,

$$\chi(\overline{\mathrm{Pic}}^n(C), \mathrm{h}^0) - \chi(\overline{\mathrm{Pic}}^{2g-2-n}(C), \mathrm{h}^0) = \chi(\overline{\mathrm{Pic}}^n(C), \mathrm{h}^0 - \mathrm{h}^0 \circ \iota) = (n+1-g)\chi(\overline{\mathrm{Pic}}^n(C)).$$

The choice of a degree 1 line bundle induces isomorphisms $\overline{\mathrm{Pic}}^n(C) \cong \overline{\mathrm{Pic}}^{n+1}(C)$, hence these spaces have the same Euler number. The result now follows from Proposition 1. $\square$

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Corollary 6. *Let $\mathbb{P}^1$, $\mathbb{P}^1_{\mathrm{node}}$, $\mathbb{P}^1_{\mathrm{cusp}}$ be rational curves that are smooth, have one node, and have one cusp respectively:*

- $n_0(\mathbb{P}^1) = 1$ *and all other* n_h *vanish;*
- $n_0(\mathbb{P}^1_{\mathrm{node}}) = 1$ *and* $n_1(\mathbb{P}^1_{\mathrm{node}}) = 1$ *and all other* n_h *vanish;*
- $n_0(\mathbb{P}^1_{\mathrm{cusp}}) = 2$ *and* $n_1(\mathbb{P}^1_{\mathrm{cusp}}) = 1$ *and all other* n_h *vanish.*

Proof. The proof follows from the 'in particular' of Proposition 1 and the vanishing of Proposition 5. □

More can be said by working locally at the singularities.

Definition 7. Let c be the germ of a Gorenstein curve singularity, let δ be its delta invariant, and b the number of analytic local branches. Define $n_h(c)$ by the formula

$$\sum_{h=-\infty}^{\delta} q^{\delta-h}(1-q)^{2h} n_h(c) = (1-q)^b \sum_{n=0}^{\infty} q^n \,\chi(c^{[n]}).$$

Remark. Theorem A′ asserts that when c is planar, $n_h(c) = \deg_{[c]} \mathbb{V}_h$.

Proposition 8. *If C has singularities $c_1, \ldots c_k$ and geometric genus $\tilde{g}$, then*

$$n_h(C) = \sum_{i_1+\cdots+i_k+\tilde{g}=h} n_{i_1}(c_1)\cdots n_{i_k}(c_k).$$

Proof. Stratifying the Hilbert scheme of C by the number of points at each of the c_i, we see

$$\begin{aligned}\sum_{n=0}^{\infty} q^n \chi(C^{[n]}) &= \left(\sum_{n=0}^{\infty} q^n \chi\left(\left(C \Big\backslash \coprod c_i\right)^{[n]}\right)\right) \prod_i \left(\sum_{n=0}^{\infty} q^n \chi(c_i^{[n]})\right)\\ &= (1-q)^{2\tilde{g}-2+\sum b(c_i)} \prod_i \sum_{n=0}^{\infty} q^n \chi(c_i^{[n]}).\end{aligned}$$

Substituting in the definitions of the n_h, we obtain

$$\sum_{h=0}^{g} q^{g-h}(1-q)^{2h-2} n_h(C) = (1-q)^{2\tilde{g}-2} \prod_i \sum_{h=-\infty}^{\delta} q^{\delta-h}(1-q)^{2h} n_h(c).$$

Collecting terms and writing $z^2 = q^{-1}(1-q)^2$, we obtain

$$\sum_{h=0}^{g} z^{2h} n_h(C) = z^{2\tilde{g}} \prod_i \sum_{h=-\infty}^{\delta} z^{2h} n_h(c).$$

Comparison of the coefficients of z yields the result. □

Corollary 9. *If C is a rational curve with a single singularity c, then $n_h(C) = n_h(c)$.*

Corollary 10. *For c the germ of a plane curve singularity, $n_h(c)$ vanishes for $h < 0$.*

Corollary 11. *For C a complete, locally planar curve, $n_h(C)$ vanishes for $h < \tilde{g}(C)$.*

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COROLLARY 12. *If C is nodal of geometric genus $\tilde{g}$ and arithmetic genus g, then $n_h(C) = \binom{g-\tilde{g}}{g-h}$.*

COROLLARY 13. *The following are equivalent.*

Theorem A: for a family $\mathcal{C} \to \Lambda$ of integral locally planar curves locally versal at λ, $n_h(C_\lambda) = \deg_\lambda \Lambda_h$.

Theorem A′: for c a plane curve singularity, $n_h(c) = \deg_{[c]} \mathbb{V}_h$.

Proof. The '$A' \implies A$' direction follows from comparing the relation between the multiplicities asserted in (2) with the relation between the n_h established in Proposition 8. To see '$A \implies A'$', consider locally versal deformations of curves with unique singular points. □

We now remark on the relation between smoothness of relative Hilbert schemes and relative compactified Jacobians. The result and its proof are closely analogous to [PT10, Theorem 4].

PROPOSITION 14. *Let $\mathcal{C} \to S$ be a family over a smooth base of complete integral Gorenstein curves of arithmetic genus g. Then the following are equivalent.*

(i) *The total space of the relative Hilbert scheme $\mathcal{C}_S^{[n]}$ is smooth for some $n \geqslant 2g - 1$.*

(ii) *The total space of the relative compactified Jacobian $\overline{\mathrm{Pic}}^0(\mathcal{C}/S)$ is smooth.*

(iii) *The total space of the relative Hilbert scheme $\mathcal{C}_S^{[n]}$ is smooth for all $n \geqslant 2g - 1$.*

(iv) *The total space of the relative Hilbert scheme $\mathcal{C}_S^{[n]}$ is smooth for all n.*

Proof. It suffices to take S to be a small polydisc. As in Proposition 5, the Riemann–Roch theorem for Gorenstein curves ensures that the Abel–Jacobi map $\mathcal{C}^{[n]} \to \overline{\mathrm{Pic}}^n(\mathcal{C}/S)$ is a bundle with fibres $\mathbb{P}^{n-g}$ once $n \geqslant 2g - 1$. Choose a section of $S \to \mathcal{C}$ with image in the smooth locus of each fibre gives a line bundle of relative degree 1 over S, to induce identifications between $\overline{\mathrm{Pic}}^n(\mathcal{C}/S)$ for varying n. Thus (i) implies (ii) and (ii) implies (iii). The section also induces an embedding $\mathcal{C}_S^{[n]} \subset \mathcal{C}_S^{[n+1]}$. For $p \in \mathcal{C}_S^{[n]}$ corresponding to a subscheme supported away from the section, some analytic neighborhood $p \in U \subset \mathcal{C}_S^{[n+1]}$ is analytically a product of $\overline{U} = U \cap \mathcal{C}_S^{[n]}$ with a disc. Thus if $\mathcal{C}_S^{[n+1]}$ is smooth, so is $\overline{U}$. By choosing different sections, we may cover $\mathcal{C}_S^{[n]}$ with such neighborhoods. Thus (iii) implies (iv). It is clear that (iv) implies (i). □

COROLLARY 15. *Let $\mathcal{C} \to \Lambda$ be a family of integral, locally planar curves, locally versal at $\lambda \in \Lambda$. Let δ be the difference between the arithmetic and geometric genera of the curve $\mathcal{C}_\lambda$. Then for any h, any $k \geqslant \delta$, and any generic, sufficiently small $\lambda \in \mathbb{D}^k \subset \Lambda$, the relative Hilbert scheme $\mathcal{C}_{\mathbb{D}^k}^{[h]}$ is smooth.*

Proof. It is shown in [FGS99] that the relative compactified Jacobian over $\mathbb{D}^k$ is smooth in this situation; more precisely, smoothness holds once $T_\lambda \mathbb{D}^k$ is transverse to the reduced tangent cone of the equigeneric stratum. The result now follows from Proposition 1. □

3. The proof of Theorem A

THEOREM 16. *Let $\mathcal{C} \to \Lambda$ be a family of integral, locally planar curves. Assume Λ is locally versal at λ. Let Λ_h be the locus of curves of geometric genus less than or equal to h. Then $n_h(\mathcal{C}_\lambda) = \deg_\lambda \Lambda_h$.*

Proof. The right-hand side of this equality vanishes unless $\tilde{g} \leqslant h \leqslant g$ since in this case $\lambda \notin \Lambda_h$; the left-hand side vanishes as well by Proposition 5 and Corollary 11. So assume $\tilde{g} \leqslant h \leqslant g$. Then Λ_h is of pure codimension $g-h$, and is the closure of the locus Λ_h^+ of curves with $g-h$ nodes [DH88, Tes80].

Choose a small polydisc $\lambda \in D = \mathbb{D}^{g-h} \times \mathbb{D} \subset \Lambda$ subject to the following conditions:

(i) $D_0 := \mathbb{D}^{g-h} \times \{0\}$ intersects Λ_h only at λ;

(ii) $D_\epsilon := \mathbb{D}^{g-h} \times \{\epsilon\}$ intersects Λ_h generically, i.e., at $\deg_\lambda \Lambda_h$ points of Λ_h^+. That is, the points of intersection correspond to nodal curves of genus h;

(iii) the relative Hilbert schemes $\mathcal{C}_D^{[i]}$, $\mathcal{C}_{D_0}^{[i]}$, $\mathcal{C}_{D_\epsilon}^{[i]}$ are smooth for $i \leqslant g-h$.

Each of these conditions is generically true; the third by Theorem B, which we prove as Corollary 20 in § 4; so we may satisfy them all simultaneously. By condition (iii) above, $\mathcal{C}_{D_0}^{[i]}$ and $\mathcal{C}_{D_\epsilon}^{[i]}$ are deformation equivalent smooth varieties for $i \leqslant g-h$. In particular, they are diffeomorphic, and hence have the same Euler numbers.

We define constructible functions n_i, $\chi_i : \Lambda \to \mathbb{Z}$ by their values on the fibres: for $p \in \Lambda$,

$$n_i : p \mapsto n_i(\mathcal{C}_p),$$
$$\chi_i : p \mapsto \chi(\mathcal{C}_p^{[i]}).$$

Observe that $\chi(D_0, \chi_i) = \chi(\mathcal{C}_{D_0}^{[i]}) = \chi(\mathcal{C}_{D_\epsilon}^{[i]}) = \chi(D_\epsilon, \chi_i)$ for $i \leqslant g-h$. However, by Proposition 1, there is a linear change of variables between the $\chi_0, \ldots, \chi_{g-h}$ and the $n_g, \ldots, n_h$. Therefore,

$$\chi(D_0, n_j) = \chi(D_\epsilon, n_j) \quad \text{for } g \geqslant j \geqslant h.$$

As we know from Corollary 11 that n_h is supported on Λ_h, we have

$$n_h(\mathcal{C}_\lambda) = \chi(D_0, n_h) = \chi(D_\epsilon, n_j) = \sum_{p \in D_\epsilon \cap \Lambda_h} n_h(\mathcal{C}_p) = \# D_\epsilon \cap \Lambda_h = \deg_\lambda \Lambda_h.$$

We have already explained the first two equalities. The third holds again because n_h is supported on Λ_h. The fourth because each $\mathcal{C}_p$ is nodal of geometric genus h so $n_h(\mathcal{C}_p) = 1$ by Corollary 12. The final equality holds by definition of the multiplicity. □

4. Smoothness of relative Hilbert schemes

Let $V \subset \mathbb{C}[x, y]$ be a finite-dimensional, smooth family of polynomials, and consider the family of curves

$$\mathcal{C}_V := \{(f \in V, p \in \mathbb{C}^2) \mid f(p) = 0\} \subset V \times \mathbb{C}^2.$$

We have $\mathcal{C}_V^{[k]} \subset V \times (\mathbb{C}^2)^{[k]}$. The Hilbert scheme of points on a surface is smooth [Fog68], and for $I \subset \mathbb{C}[x, y]$ the tangent space is $T_I(\mathbb{C}^2)^{[k]} = \mathrm{Hom}_{\mathbb{C}[x,y]}(I, \mathbb{C}[x, y]/I)$, where a map η corresponds to the tangent vector $I(\eta) := \{\phi + \epsilon\phi' \mid \phi \in I,\ \eta(\phi) = \phi' \bmod I\} \subset \mathbb{C}[x, y, \epsilon]/\epsilon^2$. Writing $\tilde{I}$ for the image of I in $\mathbb{C}[x, y]/f$, we have an exact sequence:

$$0 \to T_{(f,\tilde{I})}\mathcal{C}_V^{[k]} \to T_f V \times T_I(\mathbb{C}^2)^{[k]} \xrightarrow{(f+\epsilon g, \eta) \mapsto \eta(f) - g \bmod I} \mathbb{C}[x, y]/I. \tag{6}$$

If f is squarefree, then all fibres in a neighborhood of $f \in U \subset V$ will be reduced, and the relative Hilbert schemes $\mathcal{C}_U^{[k]}$ are reduced, of pure dimension $k + \dim V$, and locally complete intersections [BGS81]. Thus for squarefree f, the space $\mathcal{C}_V^{[k]}$ is smooth at $(f, \tilde{I})$ if and only if $\dim T_{(f,\tilde{I})}\mathcal{C}_V^{[k]} = k + \dim V$. By counting dimensions, this occurs if and only if the final map of

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the sequence in (6) is surjective. The easiest way to ensure this is to ask for surjectivity already at $\eta = 0$, i.e., that $T_f V \twoheadrightarrow \mathbb{C}[x, y]/I$.

We now recall basic notions from the deformation theory of singularities; for details we refer to [GLS07]. Let (X, x) be the germ of a complex analytic space. A deformation of (X, x) is a flat morphism of germs of complex analytic spaces, $(\mathscr{X}, x) \to (B, b)$, together with an isomorphism from (X, x) to the fibre over b. A deformation $(\mathscr{X}, x) \to (\mathbb{V}, v)$ is said to be *versal* if given a flat morphism $(\mathscr{Y}, y) \to (A, a)$, a closed subgerm $(A', a) \subset (A, a)$, a map $\phi' : (A', a) \to (\mathbb{V}, v)$ and an isomorphism of deformations $(\mathscr{Y}|_{A'}, y) \cong_{A'} (\mathscr{X}|_{A'}, x)$, there is a (non-unique) extension $\phi : (A, a) \to (\mathbb{V}, v)$ of ϕ' which admits a compatible isomorphism $(\mathscr{Y}, y) \cong_A (\mathscr{X}|_A, x)$. If the Zariski tangent map to ϕ *is* always uniquely determined by the given data, then $(\mathscr{X}, x) \to (\mathbb{V}, v)$ is said to be *miniversal*. The existence of a versal deformation $(\mathscr{X}, x) \to (\mathbb{V}, v)$ guarantees the existence of a miniversal $(\overline{\mathscr{X}}, \overline{x}) \to (\overline{\mathbb{V}}, \overline{v})$, and, moreover, there are compatible isomorphisms $(\mathbb{V}, v) \cong (\overline{\mathbb{V}}, \overline{v}) \times (\mathbb{C}^k, 0)$ and $(\mathscr{X}, x) \cong (\overline{\mathscr{X}}, \overline{x}) \times (\mathbb{C}^k, 0)$.

The miniversal deformation of an isolated plane curve singularity has an explicit description. Let $(C, 0)$ be the germ at the origin of the zero locus of some $f \in (x, y)\mathbb{C}[x, y]$. Fix $g_1 \cdots g_\tau \in \mathbb{C}[x, y]$ whose images form a basis of the vector space $\mathcal{T}^1 = \mathbb{C}[x, y]/(f, \partial_x f, \partial_y f)$. Then consider $F : \mathbb{C}^\tau \times \mathbb{C}^2 \to \mathbb{C}^\tau \times \mathbb{C}$ given by $F(t_1, \ldots, t_\tau, x, y) = (t_1, \ldots, t_\tau, (f + \sum g_i t_i)(x, y))$. Taking the fibre over $\mathbb{C}^\tau \times 0$ gives a family of curves over $\mathbb{C}^\tau$; taking germs at the origin gives the miniversal deformation $(\mathcal{C}, 0) \to (\mathbb{C}^\tau, 0)$ of $(C, 0)$. Moreover, if $g'_1, \ldots, g'_s \in \mathbb{C}[x, y]$ are any functions and $(\mathcal{C}', 0) \to (\mathbb{C}^s, 0)$ the analogously formed deformation of $(C, 0)$, then the tangent map $\mathbb{C}^s \to \mathcal{T}^1$ is just induced by the quotient $\mathbb{C}[x, y] \to \mathbb{C}[x, y]/(f, \partial_x f, \partial_y f)$. As soon as $\mathbb{C}^s \twoheadrightarrow \mathcal{T}^1$, the family $\mathcal{C}' \to (\mathbb{C}^s, 0)$ is itself versal.

Proposition 17. *Let $(C, 0)$ be the analytic germ of a plane curve singularity and let $(\mathcal{C}, 0) \to (\mathbb{V}, 0)$ be an analytically versal deformation of $(C, 0)$. For sufficiently small representatives $\mathcal{C} \to \mathbb{V}$, the relative Hilbert scheme $\mathcal{C}_{\mathbb{V}}^{[k]}$ is smooth.*

Proof. The relative compactified Jacobian over such a family is known to be smooth [FGS99, Corollary B.2], so the result follows from Corollary 15.

The following direct argument was suggested to us by Rahul Pandharipande. Choose $\mathbb{V} \subset \mathbb{C}[x, y]$ containing an equation f determining $(C, 0)$, such that $(\mathcal{C}_{\mathbb{V}}, 0) \to (\mathbb{V}, f)$ determines a versal family for this singularity, and such that $T_f\mathbb{V}$ contains all polynomials of degree less than or equal to k. Then $T_f\mathbb{V}$ projects surjectively onto $\mathbb{C}[x, y]/I$ for any I of colength k, hence by (6) the space $\mathcal{C}_{\mathbb{V}}^{[k]}$ is smooth. Now let $\overline{\mathcal{C}} \to \overline{\mathbb{V}}$ be the miniversal deformation. By versality there are compatible isomorphisms $\mathbb{V} \cong \overline{\mathbb{V}} \times (\mathbb{C}^t, 0)$ and $\mathcal{C} \cong \overline{\mathcal{C}} \times (\mathbb{C}^t, 0)$ [GLS07, p. 237], and hence also $\mathcal{C}_{\mathbb{V}}^{[k]} \cong \overline{\mathcal{C}}_{\overline{\mathbb{V}}}^{[k]} \times (\mathbb{C}^t, 0)$. Thus smoothness of the relative Hilbert schemes over any versal deformation is equivalent to smoothness of relative Hilbert schemes over the miniversal deformation. □

For fixed I of colength k, a generic choice of k-dimensional V ensures surjectivity of the final map in (6). We must now show that some fixed V works for all I containing the equation of the curve.

Lemma 18. *Let $\mathcal{O}$ be the complete local ring at a point on a reduced curve, and let $\bar{\mathcal{O}}$ be a finite length quotient of $\mathcal{O}$. Let $W \subset \bar{\mathcal{O}}$ be a generic k-dimensional vector subspace. Then for $\bar{I}$ the image in $\bar{\mathcal{O}}$ of any ideal of colength less than or equal to k in $\tilde{\mathcal{O}}$, we have $W + \bar{I} = \bar{\mathcal{O}}$.*

Proof. We employ the semigroup of the curve. Fix a normalization $\mathcal{O} \subset \mathbb{C}[[t]]^{\oplus r}$. Define

$$\mathrm{ord} : \mathbb{C}[[t]]^{\oplus r} \backslash \{\text{zero divisors}\} \to \mathbb{N}^{\oplus r}$$

which takes an r-tuple of power series to the r-tuple of degrees of leading elements. Removing the zero divisors ensures this is well defined.

All colength k ideals will contain the kth power of the maximal ideal M. Let $\bar{\mathcal{O}} = \mathcal{O}/M^k$, and $\Sigma = \mathrm{ord}(\mathcal{O})\backslash\mathrm{ord}(M^k)$. If $\mathrm{ord}(f) = \mathrm{ord}(g)$, then some linear combination of f and g has higher order. Therefore we may choose a vector space basis of $\bar{\mathcal{O}}$ of the form $\{f_s, s \in \Sigma |\ \mathrm{ord}(f_s) = s\}$. For any $\Delta \subset \Sigma$, we define the projection $\pi_\Delta : \bar{\mathcal{O}} \to \mathrm{Span}\{f_s\}_{s\in\Delta}$ by

$$\pi_\Delta : \sum_{s\in\Sigma} c_s f_s \mapsto \sum_{s\in\Delta} c_s f_s.$$

Fix an ideal $\tilde{I} \subset \mathcal{O}$ and write $\overline{I} = I/M^k$. Let $\iota = \mathrm{ord}(\tilde{I})$. Then $\pi_{\iota\cap\Sigma}|_{\overline{I}}$ is a vector space isomorphism. Thus for a sub vector space $W \subset \bar{\mathcal{O}}$, we have $W + \bar{I} = \bar{\mathcal{O}}$ if and only if $\pi_{\Sigma\backslash\iota}|_W$ is surjective. This determines a Zariski open locus in the Grassmannian of $(\dim W)$-dimensional subspaces of $\bar{\mathcal{O}}$, which is non-empty if $\dim W \geqslant \#\Sigma\backslash\iota = \dim \bar{\mathcal{O}}/\bar{I} = \dim \mathcal{O}/(\tilde{I} + M^k)$. It suffices for $\dim W \geqslant \dim \mathcal{O}/\tilde{I}$.

Thus requiring that $\pi_{\{s_1,\ldots,s_k\}}|_W$ is surjective for all $\{s_1, \ldots, s_k\} \subset \Sigma$ ensures that W is transverse to all ideals of colength bounded by k. The intersection of these finitely many non-empty Zariski open sets remains a non-empty Zariski open set. □

Theorem 19. *Let $(C, 0)$ be the analytic germ of a plane curve singularity and let $(\mathcal{C}, 0) \to (\mathbb{V}, 0)$ be an analytically versal deformation of $(C, 0)$. Then, for sufficiently small representatives $\mathcal{C} \to \mathbb{V}$, and generic discs $0 \in \mathbb{D}^k \subset \mathbb{V}$, the space $\mathcal{C}^{[h]}_{\mathbb{D}^k}$ is smooth for $h \leqslant k$.*

Proof. As in Proposition 17, it suffices to show this for any versal deformation $\mathbb{V}$. Let $(C, 0)$ be given by the germ at the origin of the zero locus of $f \in \mathbb{C}[x, y]$, and choose $g_1, \ldots, g_\tau$ whose images in $\mathbb{C}[[x, y]]/(f, \partial_x f, \partial_y f)$ form a basis; as discussed above the miniversal deformation $\mathcal{C} \to \mathbb{V} = \mathbb{C}^\tau$ has as fibres the curves $f + \sum t_i g_i = 0$. Let $0 \in \mathbb{D}^k \subset \mathbb{V}$ be a generic, k-dimensional disc. Lemma 18 ensures that the image of its tangent space in $\mathbb{C}[[x, y]]/(f, \partial_x f, \partial_y f)$ is complementary to any ideal of colength $h \leqslant k$. Thus the final map of (6) is surjective, and $\mathcal{C}^{[h]}_{\mathbb{D}^k}$ is smooth at points over $0 \in \mathbb{D}^k$ which correspond to subschemes supported at the singularity. Finally, let $z \subset \mathcal{C}_0$ be any subscheme of length h; and let z' be its component supported at the singularity, say of length h'. Then an analytic neighborhood of z in $\mathcal{C}^{[h]}_{\mathbb{D}^k}$ differs from an analytic neighborhood of z' in $\mathcal{C}^{[h']}_{\mathbb{D}^k}$ by a smooth factor. □

Corollary 20. *Let $\mathcal{C} \to \Lambda$ be a family of integral, locally planar curves, locally versal at $\lambda \in \Lambda$. Then for any generic, sufficiently small $\lambda \in \mathbb{D}^k \subset \Lambda$, the relative Hilbert scheme $\mathcal{C}^{[h]}_{\mathbb{D}^k}$ is smooth for $h \leqslant k$.*

Proof. This situation is analytically locally smooth over that in the theorem; and a compactness argument yields smoothness uniformly over an open neighborhood on the base. □

5. ADE singularities

A singularity is said to be simple if it has no non-trivial equisingular deformations. Simple singularities of hypersurfaces famously fall into an ADE classification [AGV88]:

- $A_n : y^2 + x^{n+1}$;
- $D_n : xy^2 + x^{n-1}$;

– $E_6 : y^3 + x^4$;
– $E_7 : y^3 + yx^3$;
– $E_8 : y^3 + x^5$.

We calculate the Euler numbers of some related Hilbert schemes. Consider the non-reduced germs at the origin $A_\infty : y^2 = 0$, $D_\infty : xy^2 = 0$, and $E_\infty : y^3 = 0$. In each case the curve is preserved by the full $\mathbb{C}^* \times \mathbb{C}^*$ action on $\mathbb{C}^2$. The action lifts to the Hilbert schemes, and the fixed points are monomial ideals in $\mathbb{C}[[x, y]]$ containing the equation. Counting fixed points gives the following formulas:

$$\sum q^n \chi(A_\infty^{[n]}) = \frac{1}{(1-q)(1-q^2)}$$
$$\sum q^n \chi(D_\infty^{[n]}) = \frac{1-q+q^3}{(1-q)^2(1-q^2)}$$
$$\sum q^n \chi(E_\infty^{[n]}) = \frac{1}{(1-q)(1-q^2)(1-q^3)}.$$

We now observe that the equation for any simple singularity is equal to its '∞' version modulo $(x, y)^\delta$, where δ is the delta invariant of the singularity. In particular, the first δ punctual Hilbert schemes are equal as subvarieties of the punctual Hilbert scheme of $\mathbb{C}^2$ at the origin. By Corollary 10, their Euler characteristics suffice to determine the whole series. Explicitly, we have:

– $A_{2\delta-1}$: $n_h = \binom{\delta+h}{\delta-h}$;
– $A_{2\delta}$: $n_h = \binom{\delta+h+1}{\delta-h}$;
– $D_{2\delta-2}$: $n_h = \binom{\delta+h-3}{\delta-h} + 2\binom{\delta+h-3}{\delta-h-1} + \binom{\delta+h-2}{\delta-h-2}$;
– $D_{2\delta-1}$: $n_h = \binom{\delta+h-2}{\delta-h} + 2\binom{\delta+h-2}{\delta-h-1} + \binom{\delta+h-1}{\delta-h-2}$;
– E_6: $(n_0, \ldots, n_3) = (5, 10, 6, 1)$;
– E_7: $(n_0, \ldots, n_4) = (2, 11, 15, 7, 1)$;
– E_8: $(n_0, \ldots, n_4) = (7, 21, 21, 8, 1)$.

Theorem A′ asserts these numbers are the multiplicities of the Severi strata. We present a heuristic argument computing these multiplicities directly. To an ADE singularity c is associated the Dynkin diagram with of the same name. Its points form a natural basis of vanishing cycles [AGV88]. Generic points in $\mathbb{V}_h$ correspond to curves with $\delta - h$ nodes. As these singular curves are deformations of c, only vanishing cycles of c can collapse at the nodes. Moreover, simultaneously contracting intersecting cycles yields singularities worse than nodes. Thus the multiplicity of $\mathbb{V}_h$ is the number of different ways to pick $\delta - h$ disjoint vanishing cycles, or, equivalently, $\delta - h$ vertices of the Dynkin diagram so that no two are connected. The resulting numbers are precisely the ones given.

We expect this argument can be made rigorous by using either the description of the discriminant of the versal deformation of an ADE singularity in terms of the associated Weyl group and root lattice [AGV88], or Grothendieck's classification of the degenerations of ADE singularities in terms of the Dynkin diagrams [Dem73]. Such results seem to be completely out of reach for general singularities. On the other hand, there are so-called D-diagrams attached to all curve singularities [AGV88]; we would be extremely interested to learn of a procedure to compute the Severi degrees from the D-diagrams.

6. BPS numbers

Our original motivation for considering the series on the left-hand side of Theorem A comes from certain curve counting theories on three-folds. We now briefly sketch this connection; further details may be found in the papers of Pandharipande and Thomas [PT09, PT10].

For Y a Calabi–Yau three-fold, a parameter count suggests that only finitely many genus g curves will represent any given homology class $\beta \in \mathrm{H}_2(Y)$. In fact, the curves may come in positive-dimensional families; nonetheless, the *Gromov–Witten invariants* are defined to be the degree of the virtual fundamental class of the Kontsevich moduli space of stable maps [Beh97, BF97, LT98]. These invariants suffer from two major failings: first, they are fractional due to the stack structure on the moduli space; second, maps from genus g curves will give rise to undesirable maps from genus $h > g$ curves due to ramified covers and collapsing of components. Conjecturally, both problems may be simultaneously eliminated by the Faber–Pandharipande [FP00] multiple cover formula, which repackages the Gromov–Witten numbers into conjecturally integral invariants $n^{GW}_{h,\beta}(Y)$:

$$\sum_{\beta\neq 0}\sum_{h=0}^{\infty} \deg\,[\overline{\mathcal{M}}_g(Y,\beta)]^{\mathrm{vir}} u^{2h-2} v^{\beta} = \sum_{\beta\neq 0}\sum_{h=0}^{\infty} n^{GW}_{h,\beta}(Y)\, u^{2h-2} \sum_{k\geqslant 1} \frac{v^{k\beta}}{k}\left(\frac{\sin(ku/2)}{u/2}\right)^{2h-2}.$$

Gopakumar and Vafa [GV98] explained the physical meaning of the $n^{GW}_{h,\beta}(Y)$. They consider M2-branes in the M-theory in the space $\mathbb{R}^{4,1}\times Y$, i.e., real three-dimensional manifolds whose projection to Y is a complex curve in the class β and whose projection to $\mathbb{R}^{4,1}$ is the world-line of a particle. Integrating out the Calabi–Yau degrees of freedom suggests that at low energy, the state space of the particle is the cohomology of the relative compactified Jacobian of the family of embedded curves in class β. We write this as $\mathrm{H}^*(\mathcal{M}_{\mathrm{GV}})$. The theory transforms under $SO(4,\mathbb{R}) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$; in particular, this group should act $\mathrm{H}^*(\mathcal{M}_{\mathrm{GV}})$. The $\mathrm{SU}(2)_R$ induces a weight grading. Forgetting the action and collapsing the grading, so that the odd graded pieces become negative virtual $\mathrm{SU}(2)_L$ representations, yields $\mathrm{H}^*(\mathcal{M}_{\mathrm{GV}})|_L \in \mathrm{Rep}(\mathrm{SU}(2)_L)$. Enumerative invariants may be extracted via the prescription

$$\mathrm{H}^*(\mathcal{M}_{\mathrm{GV}})|_L = \sum_h n^{GV}_{h,\beta}(Y)(\mathbb{C}\oplus V_{\mathrm{std}}\oplus\mathbb{C})^{\otimes h}.$$

In a certain limit, the M2-branes become strings, and the $n^{GV}_{h,\beta}(Y)$ are related to the Gromov–Witten invariants by precisely the multiple cover formula. That is, $n^{GV}_{h,\beta}(Y) = n^{GW}_{h,\beta}(Y)$. The $n^{GV}_{h,\beta}(Y)$ may be calculated by computing the kernels of powers of the $SU(2)_L$ raising operator, which, at least in simple cases, is the cup product with the class of the relative theta divisor. In [KKV99], it is shown how the Abel–Jacobi map expresses these traces in terms of the Euler numbers of relative Hilbert schemes of points. According to Kawai [Kaw03], the Hilbert schemes should be interpreted as moduli of D2–D0 branes.

The moduli of D2–D0 branes is made mathematically precise in the work of Pandharipande and Thomas [PT09, PT10]. They define

$$P_n(Y,\beta) = \{[\phi:\mathcal{O}_Y\to F]\,|F \text{ pure},\ \chi(F)=n,\ [\mathrm{supp}(F)]=\beta,\ \dim_{\mathbb{C}} F/\phi(\mathcal{O}_Y)<\infty\}.$$

This space carries a virtual class $[P_n(Y,\beta)]^{\mathrm{vir}}$ of dimension zero. Integers $n^{PT}_{h,\beta}$ are defined by

$$\log\biggl(1+\sum_{\beta\neq 0}\sum_n(-q)^n v^\beta \deg[P_n(Y,\beta)]^{\mathrm{vir}}\biggr)=\sum_{h>-\infty}\sum_{\beta\neq 0} n^{PT}_{h,\beta}(Y)\sum_{k\geqslant 1}\frac{v^{k\beta}}{k}(q^{-k/2}-q^{k/2})^{2h-2}.$$

It is conjectured [MNOP06, PT09] that $n^{PT}_{h,\beta}(Y)=n^{GW}_{h,\beta}(Y)$.

We consider only irreducible β; for these, the expression simplifies to

$$\sum_n(-q)^n\deg[P_n(Y,\beta)]^{\mathrm{vir}}=\sum_{h>-\infty} n^{PT}_{h,\beta}(Y)(q^{-1/2}-q^{1/2})^{2h-2}.$$

In [PT09, PT10], it is observed that the P_n carry symmetric perfect obstruction theories; Behrend [Beh09] has shown the resulting virtual degrees can be computed as $\deg[P_n(Y,\beta)]^{\mathrm{vir}}=\chi(P_n(Y,\beta),\nu^b)$. Here ν^b is a constructible function depending only on the scheme structure in an analytic local neighborhood, and not on the obstruction theory. This makes it possible to discuss the contribution of a single curve. That is, if $\mathcal{C}\to\Lambda$ is the family of curves in class β, then for $\lambda\in\Lambda$ we define $n^{PT}_{h,\beta}(\mathcal{C}_\lambda)$ by

$$\sum_n(-q)^n\,\chi\left(P_n(\mathcal{C}_\lambda),\nu^b|_{P_n(\mathcal{C}_\lambda)}\right)=\sum_{h>-\infty} n^{PT}_{h,\beta}(\mathcal{C}_\lambda)(q^{-1/2}-q^{1/2})^{2h-2}.$$

Here, the space $P_n(\mathcal{C}_\lambda)\subset P_n(Y,\beta)$ is the locus where the sheaf F is (scheme-theoretically) supported on the curve $\mathcal{C}_\lambda$. The function $\nu^b|_{P_n(\mathcal{C}_\lambda)}$ is restricted from $P_n(Y,\beta)$ and *is not* intrinsic to $P_n(\mathcal{C}_\lambda)$. If $n^{PT}_h:\Lambda\to\mathbb{Z}$ is the function $\lambda\mapsto n^{PT}_h(\mathcal{C}_\lambda)$, then $n^{PT}_{h,\beta}(Y)=\chi(\Lambda,n_h)$.

Assume $\mathcal{C}_\lambda$ is integral and locally planar. Then [PT10, Appendix B], since $\mathcal{C}_\lambda$ is Gorenstein, we can identify $P_{n+1-g}(\mathcal{C}_\lambda)=\mathcal{C}^{[n]}_\lambda$. It follows from Corollary 15 that if the total space of the relative compactified Jacobian of the family $\mathcal{C}\to\Lambda$ is smooth at points over λ, then $\nu^b|_{P_n(\mathcal{C}_\lambda)}=(-1)^{n-1+g+\Lambda}$. This certainly holds at points where Λ is smooth and $\mathcal{C}\to\Lambda$ is locally versal; in fact [FGS99], it suffices for its image in the product of the versal deformations of the singularities to be transverse to the tangent cone of the equigeneric stratum. In this case, $(-1)^{\dim\Lambda}n^{PT}_h(\mathcal{C}_\lambda)=n_h(\mathcal{C}_\lambda)$, the left-hand side being the invariants discussed in this article.

The $n^{PT}_h(\mathcal{C}_\lambda)$ should count the 'number of curves of geometric genus h occurring at λ'. In the situation we have been discussing, Theorem A gives a sense in which this is true.

7. The HOMFLY polynomial of the link

A knot is a smooth embedding $S^1\to S^3$, considered up to isotopy; more generally, a link is an smooth embedding of possibly several circles. Singularities naturally give rise to links. If $p\in C\subset S$ is a point on a curve on a surface, and $B_\epsilon(p)$ is a small ball containing p, then we write $\mathrm{Link}(C,p)$ for $C\cap\partial B_\epsilon(p)\subset\partial B_\epsilon(p)$. Data about the singularity is reflected in the topology of the link; for instance, the link is trivial if and only if the p is a smooth point, and the number of components of the link is equal to the number of analytic local branches at p. In fact [Zar71], the link determines the equisingularity class of the germ of C at p. For discussions of the interplay between singularities and knots, see [AGV88, Mil68, Wal04].

A central project of knot theory is the classification of knots and links by means of invariants. Given the close relationship between a singularity and its link, one may ask what various topological invariants of the link capture about the geometry the singularity, and, conversely, what algebro-geometric invariants say about the topology. For example, Campillo, Delgado, and Gusein and Zade have proven that the multivariate Alexander polynomial of the link is a certain

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graded Euler number of the ring of functions at the singular point [CDG03]. It is known that the link type in turn can be recovered from the multivariate Alexander polynomial [Yam84].

There is a generalization of the (usual univariate) Alexander polynomial, variously called the skein, Jones–Conway, HOMFLY, or HOMFLY-PT polynomial [FYHLMO85]. We denote it by $\mathbf{P}$. It associates an element of $\mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ to any oriented link, and is characterized by its behavior when strands of the link pass through one another:

$$\begin{aligned} a^{-1}\,\mathbf{P}(\times) - a\,\mathbf{P}(\times) &= z\,\mathbf{P}()(), \\ a^{-1} - a &= z\,\mathbf{P}(\bigcirc). \end{aligned}$$

We write $P_\infty \in \mathbb{Z}[z^{\pm 1}]$ for the coefficient of the lowest power of a.

Suppose C is rational with a unique singularity at p. Oblomkov and the present author [OS10] have conjectured a relation between the Hilbert schemes of points on C and the HOMFLY polynomial of the link of C at p. Here we state only its specialization to P_∞:

Conjecture 21 [OS10]. *Let $p \in C$ be a point on a locally planar curve; let c denote the analytic germ at this point. Let c have b branches and contribute δ to the arithmetic genus. Then,*

$$P_\infty(\mathrm{Link}(C,p)) = \sum_{h=0}^{\delta} n_h(c) z^{2h-b}.$$

Theorem A′ gives an enumerative interpretation of the coefficients on the right-hand side. One may ask whether any such meaning exists for the analogous coefficients on the left-hand side. We will find one in the work of Jaeger [Ja91].

Recall that the braid group is $\pi_1(\mathrm{Conf}_n(\mathbb{C}), \star)$, where

$$\mathrm{Conf}_n(\mathbb{C}) = \{n \text{ unlabelled, distinct points in } \mathbb{C}\}.$$

At the basepoint $\star \in \mathbb{C}^{(n)}$, we label the n points as $p_1, \dots p_n$. The braid group is generated by $\tau_1, \dots, \tau_{n-1}$, where τ_i is the counter-clockwise half-twist interchanging p_i and p_{i+1} while leaving all other points fixed. Their inverses are the analogous clockwise half-twists. The relations are generated by $\tau_i\tau_j = \tau_j\tau_i$ if $|i-j| \neq 1$ and $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$. A braid may be 'closed' to form an oriented link; this is done by associating to a loop $S^1 \to \mathbb{C}^{(n)}$ its evaluation graph in the solid torus $S^1 \times \mathbb{C}$, and then embedding the solid torus in the usual way into S^3. The orientation lifts from the orientation of S^1. That any link may be obtained in this manner is a classical theorem of Alexander [Alx23].

We now describe Jaeger's formula. Fix some sequence $\tau_{i_1}^{\pm 1} \dots \tau_{i_N}^{\pm 1}$. We denote both the sequence and its product braid by β. Consider now the set of all sequences formed from β by replacing some of the τ_i with symbols $\not\tau_i$ and likewise some τ_i^{-1} with $\not\tau_i^{-1}$. Jaeger calls these 'circuit partitions'. Such a sequence determines an element of the braid group by viewing all $\not\tau_i^{\pm 1}$ as identity elements.

Consider tracing through the braid closure of the new sequence in the following manner. Start at the point $(1, p_1) \in S^1 \times \mathbb{C}$, and move according to the orientation lifted from the circle. While travelling, keep track of the *strand number*, which begins at 1 and when passing $\tau_i^{\pm 1}$ is changed by the transposition $i \leftrightarrow i+1$. Continue until returning to $(1, p_1)$. If there are multiple link components, now jump to the first point $(1, p_k)$ which has not yet been encountered, set the strand number to k, and continue. Along this path, each of the half-twists or removed half-twists is encountered twice. The sequence is *admissible* if the first encounter of a given $\not\tau$ (respectively $\not\tau^{-1}$) has lower (respectively higher) strand number than the second.

We write $A(\beta)$ for the set of admissible sequences. Denote by $w(\beta)$ the *writhe*, i.e., the number of τ minus the number of τ^{-1}; it does not depend on the presentation of the braid. Given $\pi \in A(\beta)$, let $b(\pi)$ denote the number of components of the braid closure of the braid associated to π. Jaeger proves [Ja91]:[1]

$$\mathbf{P}(\overline{\beta}) = a^{w(\beta)} \sum_{\pi \in A(\beta)} (-1)^{\#\not\tau^{-1}} z^{\#\not\tau} a^{n-b(\pi)} \mathbf{P}(\bigcirc)^{b(\pi)}.$$

Example. Consider the braid on $n=2$ strands given by τ_1^3; the corresponding knot is the right handed trefoil which is the link of an ordinary cusp. Then the admissible sequences are $\not\tau_1\not\tau_1\not\tau_1$, $\not\tau_1\not\tau_1\tau_1$, $\tau_1\tau_1\not\tau_1$, $\not\tau_1\tau_1\tau_1$, $\tau_1\tau_1\tau_1$. The resulting formula for the HOMFLY polynomial is

$$\mathbf{P}(\text{trefoil}) = a^3(z^3\mathbf{P}(\bigcirc)^2 + z^2 a\mathbf{P}(\bigcirc) + 2z\mathbf{P}(\bigcirc)^2 + a\mathbf{P}(\bigcirc)) = a^2(2 - a^2 + z^2)\mathbf{P}(\bigcirc).$$

Thus $P_\infty(\text{trefoil}) = 2z^{-1} + z$, matching the $n_0(\text{cusp}) = 2$ and $n_1(\text{cusp}) = 1$ observed in Corollary 6.

Henceforth we discuss only *positive* braids, i.e., those which are products of counter-clockwise half twists. The description of links of singularities as iterated torus knots (see e.g. [Wal04]) yields positive braid presentations. For such braids the writhe w is just the number of twists appearing, and it can be seen from Jaeger's formula that the number $w - n$ is an invariant of the closed braid. Since $\tau^{\pm 1}$ changes the number of link components by 1, any circuit partition π with $b(\pi) = n$ must have an even number of $\not\tau^{\pm 1}$. Denote the set of admissible circuit partitions with $2r$ half twists by $A_{n,r}(\beta)$. Counting link components, we see that $n - b(\beta) \leqslant w - 2r$. Restricting Jaeger's formula to the lowest degree term in a,

$$P_\infty(\overline{\beta}) = \sum_{r=0}^{(w+b-n)/2} \#A_{n,r}(\beta)\, z^{w-n-2r}.$$

Remark. The polynomial $P_\infty(L)$ has non-negative coefficients if L admits a positive braid presentation. Thus Conjecture 21 predicts that $n_h(C) \geqslant 0$. The identification of the $n_h(C)$ as multiplicities establishes this positivity.

Example. Consider the braid on $n=3$ strands given by $(\tau_1\tau_2)^4$; the corresponding knot is the right handed (3, 4) torus knot, which is the link of the E6 singularity. Let us abbreviate $\tau_1 = \tau$ and $\tau_2 = \sigma$. The admissible sequences π with $b(\pi) = 3$ are as follows:

$T_{3,4}$	Admissible sequences
$A_{3,0}$	$(\not\tau\not\sigma)^4$
$A_{3,1}$	$(\tau\not\sigma)^2(\not\tau\not\sigma)^2, (\not\tau\not\sigma)(\tau\not\sigma)^2(\not\tau\not\sigma), (\not\tau\not\sigma)^2(\tau\not\sigma)^2$ $(\not\tau\sigma)^2(\not\tau\not\sigma)^2, (\not\tau\not\sigma)(\not\tau\sigma)^2(\not\tau\not\sigma), (\not\tau\not\sigma)^2(\not\tau\sigma)^2$
$A_{3,2}$	$(\tau\not\sigma)^4, (\not\tau\sigma)^4$ $\tau\not\sigma\tau\not\sigma\not\tau\sigma\not\tau\sigma, \tau\not\sigma\tau\sigma\not\tau\sigma\not\tau\not\sigma, \not\tau\not\sigma\tau\not\sigma\tau\sigma\not\tau\sigma, \not\tau\sigma\not\tau\sigma\tau\not\sigma\tau\not\sigma$ $\tau\sigma\not\tau\sigma\tau\not\sigma\not\tau\not\sigma, \not\tau\not\sigma\tau\sigma\not\tau\sigma\tau\not\sigma, \not\tau\sigma\tau\not\sigma\tau\sigma\not\tau\not\sigma, \not\tau\not\sigma\not\tau\sigma\tau\not\sigma\tau\sigma$
$A_{3,3}$	$(\tau\not\sigma\tau\sigma)^2, (\tau\sigma\not\tau\sigma)^2, (\tau\sigma)^3\not\tau\not\sigma, \not\tau\not\sigma(\tau\sigma)^3, \not\tau(\sigma\tau)^3\not\sigma$

Jaeger's formula gives $P_\infty(T_{3,4}) = z^{-1} + 6z + 10z^3 + 5z^5$, matching the values for $n_h(E_6)$ in § 5.

[1] Our expression is slightly different due to a different convention for the HOMFLY polynomial.

LEMMA 22. *Consider a singularity with b analytic local branches, delta invariant δ, and Milnor number μ. Let β be a positive braid presentation of the link of the singularity, with n strands and w crossings. Then $\mu = w - n + 1$, or, equivalently, $2\delta = w - n + b$.*

Proof. Specialize Jaeger's formula to the Alexander polynomial, and use known properties relating its degree with the Milnor number of the singularity [Mil68]. □

Fix a singularity, c, and a positive braid presentation β of its link. Let $\mathbb{V}^{+}_{\delta-r} \subset \mathbb{V}(c)$ denote, as usual, the locus of deformations of c with r nodes and no other singularities. Let $\mathbb{D}^r$ be a generic disc in $\mathbb{V}(c)$. In light of Theorem A, Conjecture 21 is equivalent to the assertion that $\# A_{n,r} = \mathbb{V}_{\delta-r} \cap \mathbb{D}^r$. For $r = 0, 1$, this is straightforward. There is evidently a unique element of $A_{n,0}$, and, as $\mathbb{V}_\delta = \mathbb{V}(c)$, a generic space of complementary dimension is a single point. An element of $A_{n,1}$ must have two remaining τ_i, for some fixed i. The admissibility condition ensures that the second τ_i must be the first one occurring in the original β after the first τ_i. Thus $\# A_{n,1} = w - (n - 1)$. This is equal to the Milnor number of the singularity, which is in turn equal to the multiplicity of the discriminant locus.

We now speculate about how a bijection may be established between $A_{n,r}$ and $\mathbb{V}_{\delta-r} \cap \mathbb{D}^r$. That is, how to match deformations with r nodes to circuit partitions with $2r$ remaining half-twists. View c as the germ at the origin of some curve C in $\mathbb{C}^2$. Choose a projection $\mathbb{C}^2 \to \mathbb{C}$; it induces a finite map $C \to \mathbb{C}$. In a small punctured disc $\mathbb{D}^* \subset \mathbb{C}$ the map is unramified; say it has degree n. Thus the boundary of the disc gives a S^1 family of n points moving in $\mathbb{C}$; the closure of the corresponding braid β is the link of the singularity. Now deform c very slightly to a smooth curve c_0 whose projection to $\mathbb{D}$ has only simple ramification, say at points in $\mathcal{R} \subset \mathbb{D}$. Comparing Euler numbers reveals that $\#\mathcal{R} = 2\delta - b + n = w$.

Above each point in $\mathbb{D} \backslash \mathcal{R}$ is a collection of n points in the fibre, which is $\mathbb{C}$. Fix a point d on the boundary of the disk and let $\star \in \mathrm{Conf}_n(\mathbb{C})$ be the points in c_0 lying over it. We can thus form the braid monodromy [Moi81]:

$$BM : \pi_1(\mathbb{D} \backslash \mathcal{R}, d) \to \pi_1(\mathrm{Conf}_n(\mathbb{C}), \star).$$

The image of a loop containing no ramification points is the trivial braid; the image of the loop $\partial\mathbb{D}$ containing all the ramification points is a braid whose closure is the link of the singularity. The image of a loop containing exactly one ramification point is a braid which interchanges two points in the fibre by a positive half-twist. The description of β as an iterated torus link gives a positive braid presentation on n strands; this presentation must have exactly $w = \mu + n - 1$ half-twists. As $w = \#\mathcal{R}$, we find it plausible that there exists a decomposition $\partial\mathbb{D} = \ell_1 \cdots \ell_w$ into loops ℓ_j containing one ramification point each, such that $BM(\ell_j) \in \{\tau_1, \ldots, \tau_{n-1}\}$. Fix such a decomposition.

Consider an intersection point of a generic hyperplane with $\mathbb{V}_{\delta-h}$. This corresponds to a curve c_h with exactly h nodes. By genericity, c_h projects to $\mathbb{D}$ with only simple ramification, and with no nodes over the ramification points. We denote the ramification points by $\mathcal{R}(c_h) \subset \mathbb{D}$, and the images of the nodes by $\mathcal{N}(c_h)$. Evidently $\#\mathcal{R} - \#\mathcal{R}(c_h) = 2h$. We again have the braid monodromy,

$$BM : \pi_1(\mathbb{D} \backslash (\mathcal{R}(c_h) \cup \mathcal{N}(c_h))) \to \pi_1(\mathrm{Conf}_n(\mathbb{C}), \star).$$

Choose a path in $\mathbb{V}(c)$ from c_0 to c_h. Traversing this path, some ramification points will remain ramification points and others will collide to form nodes; thus we have a map $\phi : \mathcal{R} \to \mathcal{R}(c_h) \cup \mathcal{N}(c_h)$. We define a circuit partition $\pi(c_h)$ by taking the sequence $BM(\ell_1) \cdots BM(\ell_w)$ and replacing the $BM(\ell_i)$ for $i \notin \phi^{-1}(\mathcal{N}(c_h))$ with $\not\tau$ terms. The chosen path induces an inclusion $\pi_1(\mathbb{D}\backslash(\mathcal{R}(c_h)\cup\mathcal{N}(c_h)))\hookrightarrow\pi_1(\mathbb{D}\backslash\mathcal{R})$ compatible with the braid monodromy; by construction, the braid associated to $\pi(c_h)$ comes from a loop in $\pi_1(\mathbb{D}\backslash(\mathcal{R}(c_h)\cup\mathcal{N}(c_h)))$ which goes around all the nodes and none of the ramification points. Thus the braid closure has n components.

The admissibility of $\pi(c_h)$ presumably depends on the path chosen from c_0 to c_h; and we do not know how to choose paths in a systematic way. Even having done this, one must somehow show every admissible circuit partition occurs exactly once. We leave the further study of these matters to future work.

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