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# $xy = \cos(x + y)$ and other implicit equations that are surprisingly easy to plot

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## 1. *Introduction*

The following equations relate $y$ only *implicitly* to $x$:

$$\frac{x^2}{9} + \frac{y^2}{4} - 1 = 0; \tag{1}$$

$$2y^5 + 3y^4x + y^3x^2 + 5y^2x^3 + yx^4 + x^5 - 5 = 0. \tag{2}$$

In both equations, $y$ is a function of $x$ for a continuous range of $(x, y)$ values in the real $x$-$y$ plane. (1) represents an ellipse. (2) has been designed by the author to have a solution in the real $x$-$y$ plane at $(-1, 2)$, and because the function on the left-hand side of (2) meets certain conditions regarding continuity and partial differentiability there must be a line of points in the real $x$-$y$ plane satisfying (2) and passing continuously through $(-1, 2)$ [1, pp. 23-28].

If one desires to plot (1) or (2) graphically in the real $x$-$y$ plane, one needs to generate a set of $(x, y)$ which satisfy the respective equation, the points being close enough together to illustrate the equation.

For (1), such plotting is very easy. If one substitutes, for instance, $x = 2.44$ into (1) and rearranges, one arrives at $\frac{1}{4}y^2 = 1 - 0.6615$ so that $(x, y) = (2.44, \pm 1.1636)$. (Approximate values in this Article are given to four decimal places.) For computing an entire set of points, it is more convenient to rearrange (1) into explicit form *before* making numerical substitutions. Three possible rearrangements, all involving only elementary functions, are as follows, of which the last uses a parameter $t$:

$$y = \pm 2\sqrt{1 - \frac{x^2}{9}} \text{ substituting values of } x\text{: } -3 \leqslant x \leqslant 3; \tag{3}$$

$$x = \pm 3\sqrt{1 - \frac{y^2}{4}} \text{ substituting values of } y\text{: } -2 \leqslant y \leqslant 2; \tag{4}$$

$$x = 3\cos t,\ y = 2\sin t \text{ substituting values of } t\text{: } 0 \leqslant t < 2\pi. \tag{5}$$

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Equation (1) happens to be symmetric; if $(\alpha, \beta)$ is a solution, then so are $(\alpha, -\beta)$, $(-\alpha, \beta)$ and $(-\alpha, -\beta)$. Accordingly, one can save work by calculating points by using (3), (4) or (5) over only a quarter of the range (e.g. $x : 0 \leqslant x \leqslant 3$ for (3)), and then using this symmetry to generate the remaining points.

*Per point on the plot*, (2) *requires far more computational effort than* (1) (even if one does not take advantage the symmetry of (1)). One may identify special points on (2) such as $(-1, 2)$ and $\left(0, \sqrt[5]{\frac{5}{2}}\right)$. But general points on the plot are a different matter. If one substitutes, for instance, $x = 2.44$ into (2), one obtains

$$2y^5 + 7.32y^4 + 5.9536y^3 + 72.6339y^2 + 35.4454y + 81.4867 = 0. \qquad (6)$$

Because (6) is a general fifth-degree equation, its single real solution, $y = -4.6244$, can be calculated from (6) only by use of non-elementary functions, iterative methods, or graphical methods [2]. Likewise, because (2) is a general fifth-degree equation in both $x$ and $y$, there are no equations corresponding to (3) to (5) unless non-elementary functions are used.

The central concern of this Article is whether or not plotting of general points for an implicit equation – essential if there are to be enough points to illustrate the equation graphically – requires use of the mathematical devices mentioned in the previous paragraph. We adopt the following definitions:

*Definition, 'hard' implicit equation*: a plottable implicit equation for which the $(x, y)$ necessary for plotting can be determined only by use of non-elementary functions, iterative methods, or graphical methods.

*Definition, 'easy' implicit equation*: a plottable implicit equation that is not 'hard'.

Equation (2) is an example of a 'hard' implicit equation, and (1) of an 'easy' implicit equation.

Today, in 2022, the computational power and software to plot 'hard' implicit equations by iterative methods is readily available. One might evaluate $y$ for equations such as (6) corresponding to a sequence of $x$ values, but more likely one would use computer programs specifically written to plot implicit equations in general [3, 4]. Before electronic computers, the main computational aids available to the mathematician, scientist, or engineer were these: (a) printed tables of elementary functions, (b) hand-cranked or electromechanical desktop machines capable of performing the four basic arithmetical operations, and (c), of limited functionality and accuracy, slide rules [5, 6]. Aids (a) and (b) would have sufficed to plot 'easy' implicit equations without undue effort, but plotting a 'hard' implicit equation would have been a daunting task, not to be undertaken lightly.

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2. *Hilton on* $xy = \cos(x + y)$; *outline of the rest of this Article*

In 1960, P. J. Hilton commented that it was 'impossible to solve the equation'

$$xy = \cos(x + y) \tag{7}$$

'in any effective sense' [1, p.23]. This statement seems plausible because of the following two similarities to the 'hard' implicit equation (2). First, if one substitutes, for instance, $x = 2.44$ into (7), one obtains

$$2.44y = \cos(2.44 + y),$$

which invites solution only by use of non-elementary functions, by iterative methods, or by graphical methods (in fact, to give $y = -0.2409$). Secondly, explicit forms corresponding to (3) and (4) apparently do not exist (unless they include non-elementary functions).

But Section 3 of this Article establishes that (7) can be expressed in parametric form which, like (5), contains only elementary functions, and therefore that (7) is an 'easy' implicit equation as defined above, like (1) but unlike (2). The 'easiness' of (7) would have mattered practically even as late as 1960 when Hilton was writing, because (a) electronic computers were rare, expensive, space-consuming, and unreliable, with frequent breakdowns, (b) time on them was 'rationed', and (c) they lacked user-friendly software.

In Section 4 of the Article, we generalise the method used in Section 3. In Section 5, we apply the method, by way of example, to two further surprising 'easy' implicit equations. Our method is not of the practical significance that it might have been in 1960, but it does establish a surprising class of implicit equations that can be plotted with elegance and economy rather than by use of brute force. Section 6 comprises our conclusion and further comments.

3. *Plotting Hilton's* $xy = \cos(x + y)$

We will now show, as promised, that Hilton's equation (7) can be put into explicit parametric form and therefore is an 'easy ' implicit equation as defined. For convenience of presentation only, we use two parameters rather than one, an independent parameter $P_1$ and a second parameter $P_2$ dependent on the first:

$$x + y = P_1;$$

$$\cos P_1 = P_2.$$

Each point on the desired plot is given by the real solutions to the simultaneous equations

$$x + y = P_1; \tag{8}$$

$$xy = P_2. \tag{9}$$

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Substituting (9) into (8) gives the quadratic equation $x^2 - P_1x + P_2 = 0$. It follows that (7) has solutions –

$$x = \tfrac{1}{2}P_1 + \tfrac{1}{2}\sqrt{P_1^2 - 4P_2},\ y = \tfrac{1}{2}P_1 - \tfrac{1}{2}\sqrt{P_1^2 - 4P_2}; \tag{10}$$

$$x = \tfrac{1}{2}P_1 - \tfrac{1}{2}\sqrt{P_1^2 - 4P_2},\ y = \tfrac{1}{2}P_1 + \tfrac{1}{2}\sqrt{P_1^2 - 4P_2}, \tag{11}$$

– where in both (10) and (11) $P_2 = \cos P_1$.

Accordingly, to plot (7) we can pick plausible real numerical values for $P_1$, then evaluate $P_2$ for each one, and finally substitute each numerical pair $(P_1, P_2)$ into (10) and (11) to obtain a set of $(x, y)$ for plotting. In practice, the symmetry of (7) means that we confine our attention to solutions given by (10) for positive values of $P_1$; if (1) generates $(\alpha, \beta)$ as a point on (7) for $P_1 = \kappa$ $(\kappa > 0)$, then $P_1 = -\kappa$ in (10) or $P_1 = \pm\kappa$ in (11) will merely generate the points $(-\beta, -\alpha)$, $(\beta, \alpha)$ and $(-\alpha, -\beta)$.

By inspection, we note that (10) will have real solutions at least for all $P_1 \geqslant 0.5\pi$. Therefore we proceed, plausibly, by evaluating $x$ and $y$ according to (10) for a sequence of $P_1$ values at $0.1\pi$ intervals beginning at 0 and ending at (say) $5.5\pi$. Tabulated as they might have been in pre-computer days, the first eleven lines of calculations are set out in Table 1.

| Set $x + y$ as $P_1$ | Evaluate $\cos P_1 = P_2$ | Evaluate $P_1/2 = R_1$ | Evaluate $P_1^2 - 4P_2 = R_2$ | Evaluate $\sqrt{R_2}/2 = R_3$ | Evaluate $R_1 + R_3 = x$ | Evaluate $R_1 - R_3 = y$ |
|---|---|---|---|---|---|---|
| $0.0\pi = 0.0000$ | 1.0000 | 0.0000 | −4.0000 | not real | not real | not real |
| $0.1\pi = 0.3142$ | 0.9511 | 0.1571 | −3.7055 | not real | not real | not real |
| $0.2\pi = 0.6283$ | 0.8090 | 0.3142 | −2.8413 | not real | not real | not real |
| $0.3\pi = 0.9425$ | 0.5878 | 0.4712 | −1.4629 | not real | not real | not real |
| $0.4\pi = 1.2566$ | 0.3090 | 0.6283 | 0.3431 | 0.2929 | 0.9212 | 0.3355 |
| $0.5\pi = 1.5708$ | 0.0000 | 0.7854 | 2.4674 | 0.7854 | 1.5708 | 0.0000 |
| $0.6\pi = 1.8850$ | −0.3090 | 0.9425 | 4.7891 | 1.0942 | 2.0367 | −0.1517 |
| $0.7\pi = 2.1991$ | −0.5878 | 1.0996 | 7.1872 | 1.3405 | 2.4400 | −0.2409 |
| $0.8\pi = 2.1533$ | −0.8090 | 1.2566 | 9.5526 | 1.5454 | 2.8020 | −0.2887 |
| $0.9\pi = 2.8274$ | −0.9511 | 1.4137 | 11.7986 | 1.7175 | 3.1312 | −0.3037 |
| $1.0\pi = 3.1416$ | −1.0000 | 1.5708 | 13.8696 | 1.8621 | 3.4329 | −0.2913 |

TABLE 1

In pre-computer days, even a hand-cranked machine such as the 'FACIT' on the cover of this issue would have generated $P_1$ and $R_1$ in all 56 rows of the full table amazingly quickly; the main consumption of time would have been in entering the numbers in the table and checking that one had done so accurately. The cosine for $P_2$ would have been obtained from a printed table. The square needed for evaluating $R_2$ might have been obtained from a printed table, or else by multiplying $P_1$ by itself on a hand-cranked or electromechanical machine. The square root for $R_3$ would have been obtained from a printed table.

Plotting the full table for $P_1$ up to $5.5\pi$ results in Figure 1.

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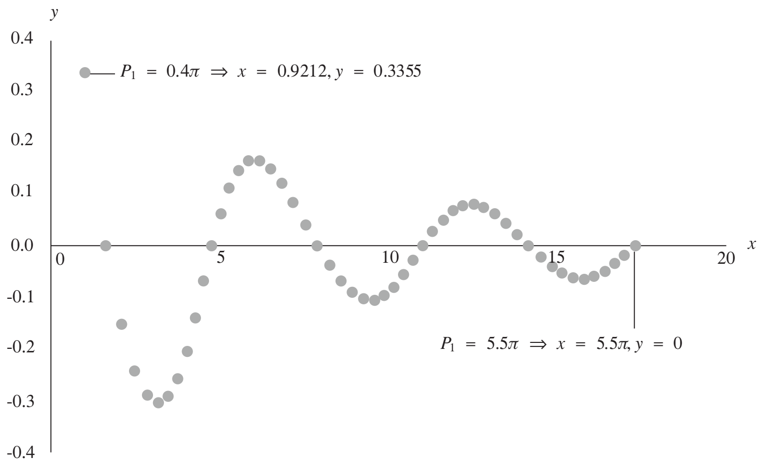


FIGURE 1

Of course, extra points can be calculated where the above points above seem rather far apart for any particular purpose.

It is worth noting that, whereas the intersections with the $x$-axis occur where $x = n\pi/2$ ($n$ odd), maxima and minima between these intersections are distinctly offset from $x = n\pi/2$ ($n$ even).

By symmetry, we arrive at the complete plot (smoothed for clarity because of the smaller scale) in Figure 2.

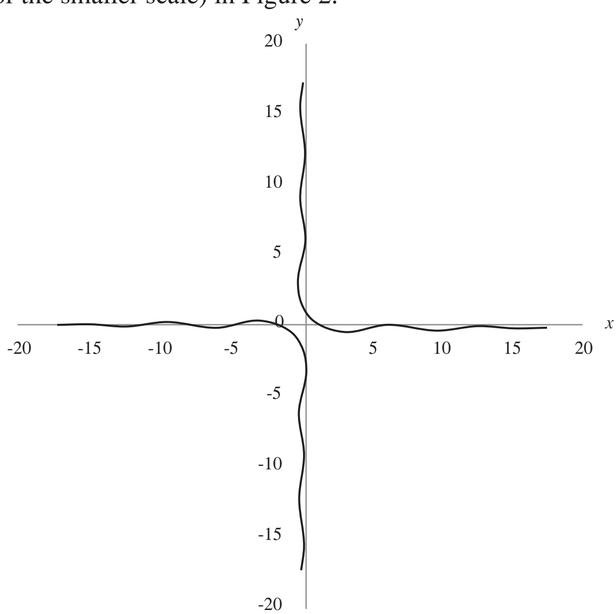


FIGURE 2

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In pre-computer days, generating Figures 1 and 2 would not have involved undue effort, even if advantage had not been taken of symmetry for Figure 2. The points plotted for Figure 2 would have been joined by hand, possibly with the aid of a flexicurve.

4. *The general case*

The method used in Section 3 can be generalised as follows (subject to the provisos in the next two paragraphs). To plot the implicit equation

$$f(x, y) = S\big(g(x, y)\big), \tag{12}$$

first we devise a sequence of numerical real values $P_1$ for $g(x, y)$, then we evaluate each $P_2 = S(P_1)$, and finally we solve the simultaneous equations $f(x, y) = P_2$ and $g(x, y) = P_1$ for each numerical pair $(P_1, P_2)$ to obtain a set of $(x, y)$ for plotting. (In (7), the three functions were $S : \gamma \to \cos\gamma$, $f : x, y \to xy$, $g : x, y \to x + y$.) Any non-real solutions of the simultaneous equations are discarded, as was exemplified in Section 3; or it may be easier, as in the two examples that will be discussed in Section 5, to avoid choosing values of $P_1$ that would lead to such solutions by identifying them in advance.

Equations of the general form (12), if they are plottable at all (see next paragraph), are ‘easy’ as defined above provided that

(i) $S$ does not include any non-elementary functions; and

(ii) $f$ and $g$ are such that the simultaneous equations $f(x, y) = P_2$ and $g(x, y) = P_1$ can be solved for $x$ and $y$ in terms of $P_1$ and $P_2$ without use of non-elementary functions, iterative numerical methods, or graphical methods. (The general question of workable $f$-$g$ combinations is discussed in Section 6, second paragraph.)

It may be of course be that a particular instance of (12) has no solutions at all for real $(x, y)$, in which case we will either be unable to devise plausible $P_1$ or else the method will generate only non-real $(x, y)$ that are discarded. However, we are unlikely to encounter equations of such a sort in scientific or engineering contexts. Even in a purely mathematical context, we may well be aware of at least one solution $(\lambda, \mu)$ in the real $x$-$y$ plane, as a consequence of which it is likely that a line of points passes continuously through $(\lambda, \mu)$ representing further solutions (see Section 1, first paragraph, final sentence).

5. *Two further surprising specific ‘easy’ implicit equations*

Our first specific further example is the equation

$$e^{x+y} = \cos(x + 2y). \tag{13}$$

We rearrange this as

$$x + y = \ln\cos(x + 2y), \tag{14}$$

i.e, in the form of (12), with $S : \gamma \to \ln\cos\gamma$, $f : x, y \to x + y$, $g : x, y \to x + 2y$.

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To solve (13) for $(x, y)$, our parameters are

$$x + 2y = P_1;$$

$$\ln \cos P_1 = P_2. \tag{15}$$

We formulate the simultaneous equations

$$x + 2y = P_1;$$

$$x + y = P_2.$$

The solutions to (13) are therefore

$$x = 2P_2 - P_1, \; y = P_1 - P_2, \text{ where } P_2 = \ln \cos P_1. \tag{16}$$

Now, $\cos(x + 2y)$ in (13), equal to parameter $P_1$, cannot be zero or negative, because, if it is, then the natural logarithm in (15) – and therefore $P_2$ – is indeterminate or non-real. Accordingly, a plausible initial approach is to evaluate $x$ and $y$ according to (16) for sequences of $P_1$ values at $0.1\pi$ intervals in the three ranges $-5\pi/2 < P_1 < -3\pi/2$, $-\pi/2 < P_1 < \pi/2$ and $3\pi/2 < P_1 < 5\pi/2$. The resulting plots, with some additional points calculated for $P_1$ values close to the range limits, are shown in Figure 3.

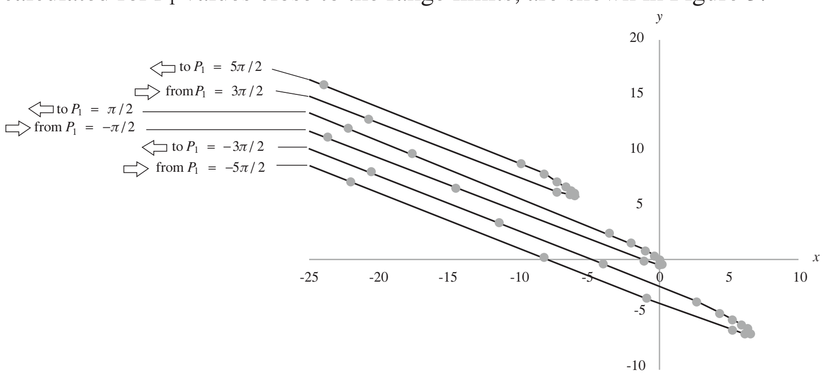


FIGURE 3

Each range of $P_1$ generates a separate two-limbed branch of the plot; further branches could readily be added above and below the three branches plotted. Each branch is tangential, from below, to the line $x + y = 0$ (not shown). This is as it must be: first, because (13) requires $e^{x+y} \leqslant 1 \Rightarrow x + y \leqslant 0$; and secondly, because (13) must have solutions corresponding to the equality, namely $(2n\pi, -2n\pi)$ where $n$ is any positive integer, any negative integer, or zero. As expected from (16), the gradient of each of the six limbs tends towards $-\frac{1}{2}$ at the left of the plot, where the limits of the ranges of $P_1$ are being approached.

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Our second further specific example is the equation

$$xy = (x + 2y)^5, \tag{17}$$

which is fifth-degree in both $x$ and $y$. With reference to (12), our functions are $S : \gamma \to \gamma^5, f : x, y \to xy, g : x, y \to x + 2y$. Our parameters are

$$x + 2y = P_1;$$

$$P_1^5 = P_2.$$

We formulate simultaneous equations and then a quadratic equation by analogy with our procedure for (7). The solutions correspond to (10) and (11) but with the numerical coefficients changed on account of the '2' on the right-hand side of (17):

$$x = \tfrac{1}{2}P_1 + \tfrac{1}{2}\sqrt{P_1^2 - 8P_2},\ y = \tfrac{1}{4}P_1 - \tfrac{1}{4}\sqrt{P_1^2 - 8P_2}; \tag{18}$$

$$x = \tfrac{1}{2}P_1 - \tfrac{1}{2}\sqrt{P_1^2 - 8P_2},\ y = \tfrac{1}{4}P_1 + \tfrac{1}{4}\sqrt{P_1^2 - 8P_2}, \tag{19}$$

where in both equations $P_2 = P_1^5$.

Now, for real solutions, $P_1^2 \geqslant 8P_2 \Rightarrow P_1^2 \geqslant 8P_1^5 \Rightarrow P_1 \leqslant \tfrac{1}{2}$. Accordingly, a plausible approach is to evaluate $x$ and $y$ according to (18) and (19) for a sequence of $P_1$ values at $-0.1$ intervals starting at 0.5 and ending at (say) $-2$. The results, with a few extra points calculated to fill out the rather attractive loop, are plotted in Figures 4 and 5.

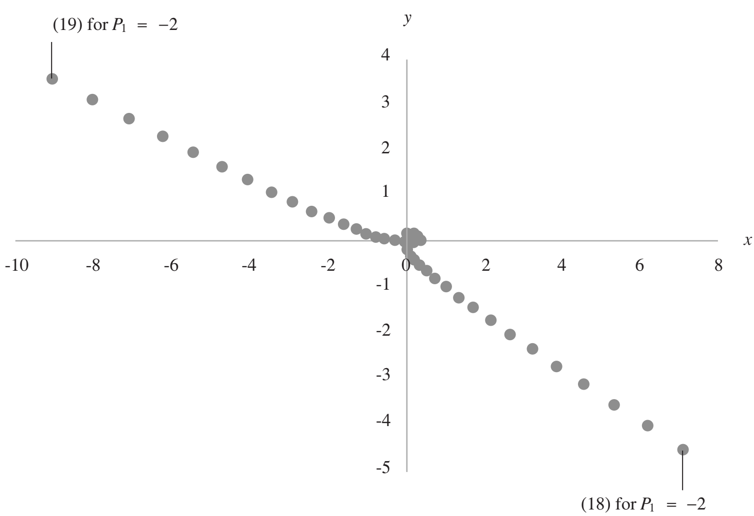


FIGURE 4

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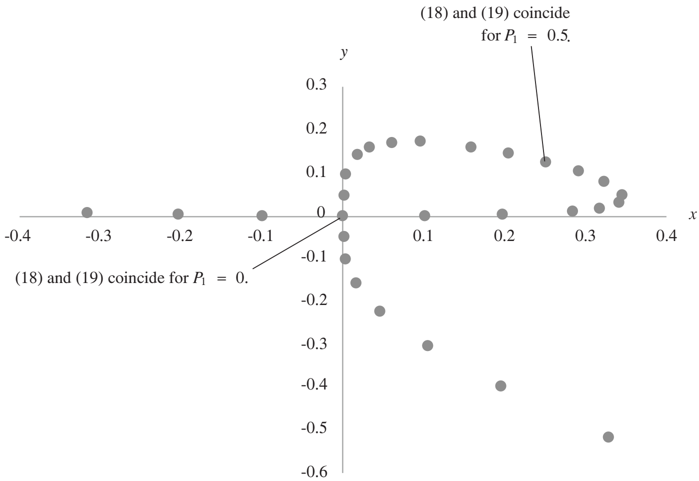


FIGURE 5

With increasingly negative $P_1$ at the right and left sides of Figure 4, the gradient of the curve is beginning to approach $-\frac{1}{2}$ as expected from (18) and (19).

## 6. *Conclusion and final comments*

Three specific implicit equations of the general form $f(x, y) = S\left(g(x, y)\right)$ (12) have been plotted. This was done without the use of non-elementary functions, iterative methods, or graphical methods; we have called such implicit equations ‘easy‘. ‘Easy’ implicit equations would have been plottable without undue effort even in pre-computer days.

The main constraint on the method used is that $f$ and $g$ must be such that the simultaneous equations $f(x, y) = P_2$ and $g(x, y) = P_1$ can be solved for $x$ and $y$ in terms of $P_1$ and $P_2$ without use of the mathematical devices just listed. This is necessarily so if both $f$ and $g$ are linear functions, as exemplified by (14). The equations can also be thus solved if either one of $f(x, y)$ and $g(x, y)$ is quadratic and the other one is a linear function, as exemplified by (7) and (17). If *both* $f(x, y)$ and $g(x, y)$ are quadratic, the equations are also thus soluble, though the algebra is a little lengthier. It would not be practical to define all workable combinations of $f$ and $g$ comprehensively, but it is worth noting that some combinations may initially look unpromising but are in fact workable, such as

(a) $f : x, y \to xy$ with $g : x, y \to x^5 + y^5$,

and

(b) $f : x, y \to x + 3 \sin y$ with $g : x, y \to 2x + 1.4 \cos y$.

The form of $S$ is unconstrained save that it must not contain non-elementary functions. Thus, when, above, we rearranged (13) as (14) to simplify $f$ so that the simultaneous equations were readily soluble, we made

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$S$ more complicated, but this was not a problem. Functions such as $S : \gamma \to \ln\cos\gamma + 2\gamma^3\sin\gamma$ and even more involved ones would present no problem of principle.

$S$ may be the identity function $S : \gamma \to \gamma$, although, if it is, there is no presentational advantage of working with two non-independent parameters $P_1$ and $P_2$ as above. An example is

$$xy = x^3 + y^3, \tag{20}$$

a *folium of Descartes*. Unsurprisingly, since (20) is of only third-degree, it is already known to be an 'easy' implicit equation, plottable parametrically with

$$x = \frac{t}{1 + t^3},\ y = \frac{t^2}{1 + t^3}, \tag{21}$$

(see [7]). According to the method of this Article, in (20) we take $S : \gamma \to \gamma$, $f : x, y \to xy$ and $g : x, y \to x^3 + y^3$. The solutions arrived using the above method are not so compact as (21), but are interestingly different:

$$x = \sqrt[3]{\tfrac{1}{2}P_1 + \sqrt{\tfrac{1}{2}\left(P_1^2 - 4P_1^3\right)}},\ y = \frac{P_1}{x}; \tag{22}$$

$$x = \sqrt[3]{\tfrac{1}{2}P_1 - \sqrt{\tfrac{1}{2}\left(P_1^2 - 4P_1^3\right)}},\ y = \frac{P_1}{x}. \tag{23}$$

In practice, we need not use (23), for if (22) generates $(\alpha, \beta)$ as a point on (20), then (23) would merely generate the point $(\beta, \alpha)$. The method could be used also for $xy = x^5 + y^5$, the cube roots in (22) and (23) being replaced by fifth roots.

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5. C. Godfrey and A. W. Siddons, *Four-figure tables* (new edn., corrected reprint), Cambridge University Press (1962). Godfrey and Siddons was a slim text dating back to 1913 which any student could afford, with at least one comparably venerable competitor. It included tables of logarithms (base 10), anti-logarithms (base 10), the six trigonometrical functions, logarithms (base 10) of those six functions, squares, square roots, reciprocals, logarithms (base $e$), $e^x$, $e^{-x}$, $\sinh x$ and $\cosh x$. No tables of powers other than of 2 and $\frac{1}{2}$ were included, but users would have raised numbers to other powers by use of the logarithm and antilogarithm tables. For instance, $20.56^{13}$ would have been calculated as $\text{antilog}_{10}(13 \times \log_{10} 20.56)$, $\sqrt[14]{20.56}$ as $\pm\ \text{antilog}_{10}\left(\frac{1}{14} \times \log_{10} 20.56\right)$ and $\sqrt[7]{-20.56}$ as $-\text{antilog}_{10}\left(\frac{1}{7} \times \log_{10} 20.56\right)$. The complex roots are not obtained, but for the purposes of plotting in the real $x$-$y$ plane, this is perfectly satisfactory.

6. Wikipedia, Mechanical calculator, accessed October 2023 at https://en.wikipedia.org/wiki/Mechanical_calculator. Some machines produced a paper record of what values and operations had been typed in together with the results obtained; this enormously facilitated checking.

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