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AN EXAMINATION OF THE LLIBOUTRY THEORY OF GLACIER SLIDING

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Abstract. An examination is made of the Lliboutry theory of glacier sliding. It is concluded that this theory is incompletely developed and therefore it is impossible at present to make meaningful predictions of the sliding velocity from it that might be compared with field data.

Résumé. *Examen de la théorie de Lliboutry du glissement d'un glacier.* L'auteur examine la théorie de Lliboutry du glissement d'un glacier. Il en conclut que cette théorie est incomplètement développée et que, en conséquence, il est impossible à présent de faire des prédictions ayant un sens sur la vitesse de glissement à partir de cette théorie qui puissent être comparées avec des observations faites sur le terrain.

Zusammenfassung. *Eine Überprüfung der Gletscher-Gleittheorie von Lliboutry.* Die Gleittheorie für Gletscher von Lliboutry wird einer Überprüfung unterzogen. Es zeigt sich, dass diese Theorie unvollständig entwickelt ist und daher derzeit keine zuverlässigen Vorhersagen von Gleitgeschwindigkeiten zulässt, die mit Feldbeobachtungen verglichen werden könnten.

Introduction

At the Obergurgl Symposium I presented a brief criticism of Lliboutry's theory of the sliding of glaciers (Weertman, 1962). I pointed out that Lliboutry's theory would lead to sliding velocities 10^8 to 10^{12} times larger than are normally observed. I concluded that the most likely application of Lliboutry's theory is to the problem of the avalanching of thin ice slabs where high sliding velocities are encountered. I obtained these fast velocities from an adaptation of Lliboutry's theory to the case in which the glacier bed is rough and irregular both in the direction of sliding and in the direction perpendicular to sliding. (This was the type of bed I had used in my theory of sliding.)

Lliboutry's answer to my criticism ([Union Géodésique et Géophysique Internationale], 1963, p. 62) was that he felt the bed model he used is more realistic. This model (the washboard model) is rough only in the direction of sliding and is perfectly smooth in the perpendicular direction. Moreover, he pointed out that his analysis is influenced by the pressure of the water trapped in the troughs or hollows of his type of glacier bed. (For a bed rough in two directions such as I considered, water entrapment is relatively unimportant.) My rejoinder to this reply ([Union Géodésique et Géophysique Internationale], 1963, p. 67) was:

> The washboard model of Lliboutry enables water to be trapped into the hollows of the valleys between the hills. (This water is trapped only because the bed is not considered to be rough in two directions.) The velocity that Lliboutry calculated when water is present in his hollows is completely indeterminate. That is, he found a velocity that depends on the difference between the ice overburden pressure and the pressure of water in the hollows. In his theory we are never told just what determines the water pressure in the hollow or even how it may be estimated! Moreover, we are not told what determines how much water is in the hollows.
>
> Therefore as far as the effect of water is concerned Lliboutry's theory is simply not developed to that point at which definite predictions can be made. It seems premature at this time to seriously discuss this part of his theory until it is developed beyond the very tentative state it is now in. I presume it will be so developed in the paper he has promised us for the Bari symposium.

Professor Lliboutry has not answered my criticism in any of his papers, including his recently published book (Lliboutry, 1965[a], p. 649). Moreover in a popular article (Lliboutry, 1965[b]; see however, Weertman, 1966) he implies that there are no difficulties attached to his theory and that it satisfactorily explains glacier sliding. However in the latest account of his sliding theory, which is given in his book, he again finds it necessary to assume the presence of a water pressure whose value is picked in a completely arbitrary manner (p. 651). He gives no justification whatsoever for his choice of this particular value.

It is the purpose of this paper to present my criticism of Lliboutry's theory in a more detailed form. These criticisms will be based on his theory as he has developed it in his book (Lliboutry, 1965[a], p. 649–52).

Review of Lliboutry's Theory

Lliboutry's theory is the result of adding a third mechanism of sliding to the two I had considered in my sliding theory (Weertman, 1957, 1962, 1964). We shall now review this mechanism as it is given in his book (Lliboutry, 1965[a]). The notation employed is identical to that used by Lliboutry.

Figure 1 shows Lliboutry's washboard model of a glacier bed. The surface of the bed is sinusoidal with wave-length λ and amplitude a. The glacier is moving to the right with velocity v. The average overburden pressure is $\rho g h$, where ρ is the ice density, h is the glacier thickness, and g is the gravitational acceleration. Water is trapped in the troughs, and the water pressure there is equal to p. The ice velocity is taken to be sufficiently fast so that the ice rests only on the tops of the undulations. Ice is in contact with rock over a distance X. It extends down from the tops of the crests a distance Z. Since X is assumed to be small compared to λ the distance Z is given by

$$Z = \pi^2 a X^2/\lambda^2. \tag{1}$$

The vertical pressure on the ice over the distance X is taken to be σ_1. The rock in contact with the ice pushes up with a force $\sigma_1 X$ over a distance X and the water in the hollow pushes up with a force $p(\lambda - X)$ over a distance $(\lambda - X)$. The overburden of ice produces a downward force $\rho g h \lambda$ over a distance λ. By balancing the upward and the downward forces Lliboutry obtains the equation

$$\sigma_1 = (\rho g h - p)\lambda/X. \tag{2}$$

The same type of argument is applied to the forces in the horizontal direction. Let f represent the shear stress at the bed causing ice flow to the right in Figure 1. (Lliboutry calls this shear stress the frictional stress.) The resultant equation is

$$f = (\sigma_1 - p)X^2\pi^2 a/\lambda^3. \tag{3a}$$

Lliboutry assumes that σ_1 will be much greater than p and therefore uses the approximate equation:

$$f \approx \sigma_1 X^2 \pi^2 a/\lambda^3 = (\rho g h - p)\pi^2 a X/\lambda^2. \tag{3b}$$

Ice in the regions where contact is made with rock creeps under an effective stress which is taken to be equal to $\sigma_1/3^{\frac{1}{2}}$. According to Glen's creep law, the creep rate $\dot{\gamma}$ in these regions must be:

$$\dot{\gamma} = B(\sigma_1/\sqrt{3})^n = B[(\rho g h - p)^2 \pi^2 a/\sqrt{3} f \lambda]^n \tag{4}$$

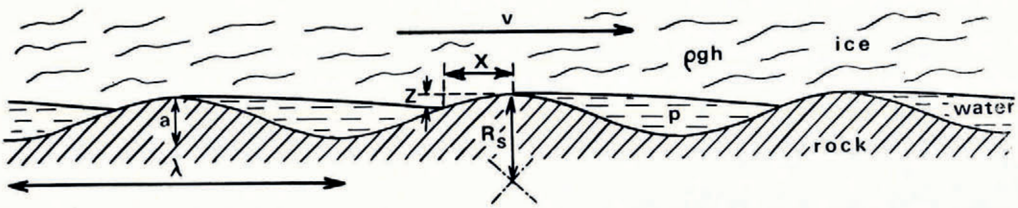

Fig. 1. The Lliboutry washboard model of a glacier bed, after figure 16.21 of Lliboutry (1965[a], p. 650)

where B is a constant and n is another constant approximately equal to 3. Since ice in contact with rock moves up a distance Z in a time interval X/v, Lliboutry considers that the creep rate $\dot{\gamma}$ thus must approximate*

$$\dot{\gamma} = 2\sqrt{3}v_c Z/X^2 = 2\sqrt{3}v_c \pi^2 a/\lambda^2. \tag{5}$$

Following Lliboutry's notation, we now let v_c instead of v represent the sliding velocity.

If Equation (4) is put equal to (5), the following value is obtained for v_c:

$$v_c = \frac{\pi^{2n-2} r^{n-2} aB(\sigma gh - p)^{2n}}{2(\sqrt{3})^{n+1} f^n} \tag{6}$$

where $r = \lambda/a$ is a measured roughness (*rugosité*) of the bed. (The smaller is r the rougher is the bed.)

This then is Lliboutry's derivation of his additional mechanism of sliding. He further develops his theory by combining this mechanism with the sliding mechanisms I considered.

In the original version of his sliding theory Lliboutry found the sliding velocity by setting the velocity v_c given by Equation (6) equal to the sliding velocity calculated from my creep rate enhancement mechanism. This latter mechanism, which he calls mechanism B, gives a velocity $v_B = C_B a f^3$, where C_B is a constant containing such factors as the roughness. (He calls the mechanism which leads to Equation (6) mechanism C.) The velocity given in Equation (6) can be rewritten as $v_c = C_c a(\rho gh - p)^6/f^3$, where C_c is another constant. Setting $v_B = v_c$ results in the equations $f = (C_c/C_B)^{1/6}(\rho gh - p)$ and $v = 2v_c = 2v_B = 2(C_B C_c)^{1/2} a(\rho gh - p)^3$. The shear stress f is determined by the term $(\rho gh - p)$. The sliding velocity depends on *both* a and $(\rho gh - p)$. Neither of these quantities can be evaluated from the theory.

Lliboutry also claims that the shear stress f is independent of sliding velocity. However an examination of these last equations does not appear to substantiate his claim since v, a, f and $(\rho gh - p)$ are all related to each other. In addition to mechanism B, Lliboutry, in the later version of his theory, discusses the regelation sliding mechanism, which he calls mechanism A. Mechanism A gives a sliding velocity $v_A = C_A f/a$, where C_A is still another constant. There is a minor mathematical error in the last equation on p. 651 of his book. A factor $\frac{1}{2}$ was omitted. If this factor is inserted in this equation and his subsequent equations are corrected, it is found that the velocity at which mechanism C first becomes more important than mechanism A is $v = 2v_c = 2v_A$. (The uncorrected equations are $v = 4v_c/3 = 4v_A$.) The sliding velocity still is given, however, by $v = 2v_B = 2v_c$.

It is seen that Lliboutry makes no attempt in this analysis to evaluate the water pressure or to decide what determines it. In comparing his theory with sliding velocities he merely picks an arbitrary value for the term $(\rho gh - p)$, and because he can pick a value of $(\rho gh - p)$ that will lead to a reasonable sliding velocity he claims his theory to be correct. It is my contention that because in the theory no reasoning or analysis is presented by which the water pressure (as well as the thickness of the water in the troughs) may be estimated *the theory is not a finished one and it cannot be used as it stands to make meaningful predictions of sliding velocities.*†

Another difficulty with the Lliboutry theory arises because no account is taken of the fact that water must be continuously removed from the bottom of a glacier. The geothermal heat alone melts each year approximately 0·5 cm.^{3} of ice per cm.^{2} of area from the bottom

* In his original paper Lliboutry (1959) assumed that the creep rate is $2v_c Z/\lambda^2$. This variation in the creep rate represents the chief difference between the two analyses.

† Of course, the theory could be considered to be a more modest phenomenological theory. A sliding velocity then could be predicted if field measurements were first obtained of *both* the water pressure *and* the thickness of the water in the cavities. Until *both* these measurements have been made, it is rather premature to claim, as has been done, that the theory does predict the correct sliding velocities.

of a glacier. This water and water produced from other sources must be removed through flow down pressure gradients.

Comments Concerning the Water Pressure

Let us investigate how the water pressure might be estimated by using Lliboutry's model. Consider again Figure 1. Suppose the water trapped in the troughs is truly isolated. If the amount of water is specified, the distance X is uniquely determined by simple geometry. Therefore, by Equation (3b) the water pressure p is equal to $\rho gh - f\lambda^2/\pi^2 aX$. Since X can have any value p can have any value from 0 to ρgh. Thus unless there is some way to predict how much water is in the hollows the theory cannot yield a sliding velocity.

There is an additional difficulty. Figure 1 represents irregularity of only one particular wavelength and amplitude. In actuality a spectrum of wavelengths and amplitudes is present. For simplicity consider a bed made up with irregularities of only two wavelengths, as shown in Figure 2. If the troughs of the smaller wavelength are filled with water the troughs of the large wavelength are essentially empty of water. The sliding past the larger obstacles thus is controlled by mechanisms I have proposed in my theory and Lliboutry's theory does not apply.

Another approach to the question of the water pressure is to assume that the pockets of water in the troughs of Figure 1 are not isolated but are connected to a source or reservoir of water which is maintained somehow at a pressure p. The amount of water in the troughs thus is not fixed. In this case it is easy to show that Figure 1 corresponds to a physical situation in unstable equilibrium. According to Equations (3b) and (6), if the sliding velocity increases the pressure p is reduced and the amount of water in the troughs is increased. The reverse holds if the sliding velocity is reduced. Suppose p is fixed and the glacier is sliding with the velocity given by Equation (6). If the sliding velocity were to be reduced slightly from this value the water pressure in the cavity would increase. Thus water would be forced from the troughs into the reservoir, which in turn would cause a further reduction in the velocity until all the water is squeezed out of the troughs. The same argument shows that an increase in velocity would cause the troughs in Figure 1 to fill completely with water. Once filled, the larger troughs such as shown in Figure 2 would commence being filled with water and eventually the sliding velocity would reach avalanche speeds *provided* the water pressure p is not itself affected by this catastrophic behavior. Obviously the water pressure will be affected and therefore a careful analysis must be made as to what factors determine p. Clearly much remains to be done with Lliboutry's theory before it can be claimed to be complete.

In particular Lliboutry (1959, p. 260) estimated that the critical thickness of the water-filled cavities is 21 cm. and the newer version of the theory appears to require an even larger thickness. Thus, according to his theory, an extremely large separation exists between the bottom of a glacier and its bed. If the cavities are interconnected so that water can flow freely between them it would be difficult to maintain a water pressure greater than atmospheric. The rate of flow of water in a water layer subjected to a pressure differential is proportional to the cube of the thickness of the water layer. A water layer only about a

Fig. 2. A washboard glacier bed containing undulations of two wavelengths

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millimeter in thickness suffices to remove the water produced by the geothermal heat (see Equation (9) of Weertman (1962)). Water flow through a layer 21 cm. in thickness would be approximately 10^9 greater! This amount of melt water is orders of magnitude larger than is normally available to any glacier.

If, as in my theory, a glacier bed is used which is rough in two dimensions rather than only one, the water pressure difficulties just discussed disappear. The water pressure is the overburden pressure. Water is free to flow along the bed. The thickness of the water film is determined by the amount of water that is flowing through a bed and the pressure gradient that is driving this flow. The pressure gradient is determined primarily by the slope of the upper ice surface and secondarily by the slope of the bed itself.*

Summary

As I see it, the difference between Lliboutry's and my theory of the sliding of glaciers reduces to the following. I maintain that Lliboutry has not derived a meaningful equation for the sliding velocity because his equation contains two unknown parameters, the water pressure, and the thickness of the water layer, whose values cannot even be estimated from the theory developed so far. Therefore as the theory now stands it is impossible (unless the theory is reduced to the status of a phenomenological theory) to compare theory with experiment. Another criticism of his theory is that he assumes a glacier bed which is rough in only one direction whereas actual beds are rough in both directions. Also no account is made in the theory for the flow of water at the bed.

Lliboutry's main critism of my theory is his claim that it leads to too slow sliding velocities. He claims it can predict velocities no larger than 1 meter per year and cannot predict velocities of the order of 100 meters per year. He arrives at this conclusion in a curious way. My sliding equation depends sensitively on the roughness of the bed. The rougher the bed the slower is the sliding velocity. In his criticism, Lliboutry says, in effect, let us take a particular value for the roughness and calculate the sliding velocity. He chooses an arbitrary value of the roughness and comes up with a sliding rate of 1 meter per year. From this calculation he concludes the theory must be wrong. At this level of logic the obvious answer to his criticism is that the theory is not wrong, he is wrong because he picked an incorrect value for the roughness. Unfortunately no one has yet made extensive measurements of the roughness of beds of glaciers. Until this is done, obviously one cannot conclude that my theory is either correct or incorrect.

The roughness of a glacier bed obviously is determined from an interaction of the erosional processes occurring at the bottom of a sliding glacier and the properties of the rock making up the bed. Both my theory and Lliboutry's could be criticized as not being completely developed since they do not lead to a prediction of the actual roughness. Because of the obvious difficulties in developing any sliding theory to this degree of refinement I hope that I have the reader's indulgence throughout this paper in regarding the roughness of the bed as a fixed parameter and not one that must be determined by theory.

The opinions expressed in this paper are those of the author and do not necessarily express the official views of the U.S. Army.

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* There are possible complications in the water flow if the bed of the glacier is not impervious to water (pointed out by W. H. Ward in private conversation). Water flow then is not confined to the space between the rock bed and the bottom of the glacier. Another complication occurs when surface melt-water streams descends to the bed of the glacier. Since the water temperature of surface melt water can exceed slightly the melting temperature of ice, streams of this water can transport heat to the bed. By melting ice, such melt-water streams can keep stream channels open and actually at times reduce their water pressure to atmospheric. These complications have not yet been analyzed although undoubtedly they will be in the future.

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Editorial note: Professor Lliboutry writes: "My actual ideas on the problem of glacier sliding are not those attributed to me by Professor Weertman. Readers are asked to refer to my paper on the subject to appear shortly in this *Journal*."