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Distance from Idempotents to Nilpotents

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Abstract. We give bounds on the distance from a non-zero idempotent to the set of nilpotents in the set of $n \times n$ matrices in terms of the norm of the idempotent. We construct explicit idempotents and nilpotents which achieve these distances, and determine exact distances in some special cases.

1 Introduction

Let H be a Hilbert space and let $\mathcal{B}(H)$ denote the space of all bounded linear operators on H with the usual operator norm: for A in $\mathcal{B}(H)$, the norm of A is

$$\|A\| = \sup\{\|A\mathbf{x}\| : \mathbf{x} \in H, \|\mathbf{x}\| = 1\}.$$

An operator E in $\mathcal{B}(H)$ is called idempotent if $E^2 = E$ and an operator N in $\mathcal{B}(H)$ is called nilpotent $N^k = 0$ for some $k \in \mathbb{N}$. If $\mathbf{x}$ and $\mathbf{y}$ are in H, $\mathbf{x} \otimes \mathbf{y}$ denotes the rank one operator in $\mathcal{B}(H)$ defined on $\mathbf{z}$ in H by

$$(\mathbf{x} \otimes \mathbf{y})(\mathbf{z}) = \langle \mathbf{z}, \mathbf{y} \rangle \mathbf{x},$$

where $\langle \cdot, \cdot \rangle$ is the inner product on H.

We shall consider the problem of finding the shortest distance from a non-zero idempotent to the set of nilpotents (all distances in this paper are with respect to the operator norm) in $\mathcal{B}(H)$. If we denote the set of all nilpotents in $\mathcal{B}(H)$ by

$$Nil = \{N \in \mathcal{B}(H) : N^k = 0 \text{ for some } k \in \mathbb{N}\},$$

then the distance from an operator A to the set of nilpotents is defined to be

$$\operatorname{dist}(A, Nil) = \inf\{\|A - N\| : N \in Nil\}.$$

We shall mainly consider the problem of computing distance to nilpotents when the underlying Hilbert space is finite-dimensional, and denote the n-dimensional complex Hilbert space by $\mathbb{C}^n$. Related to this problem are two sequences:

$$\delta_n = \inf\{\|P - N\| : P = P^2 = P^*, P \neq 0, N \in Nil \text{ and } P, N \in \mathcal{B}(\mathbb{C}^n)\}$$

and

$$\delta'_n = \inf\{\|E - N\| : E = E^2, E \neq 0, N \in Nil \text{ and } E, N \in \mathcal{B}(\mathbb{C}^n)\}.$$

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So δ_n (resp., δ_n') is the distance from the set of non-zero projections (resp., non-zero idempotents) to the set of nilpotents in $\mathcal{B}(\mathbb{C}^n)$. The analysis of δ' is staightforward. As shown in [5], if we define

$$E = \frac{1}{n}\begin{bmatrix} 1 \\ \varepsilon \\ \varepsilon^2 \\ \vdots \\ \varepsilon^{n-1} \end{bmatrix} \begin{bmatrix} 1 & \varepsilon^{-1} & \varepsilon^{-2} & \cdots & \varepsilon^{-n+1} \end{bmatrix}$$

then E is idempotent and if we let N be the strictly upper triangular part of E (so N is nilpotent) and take ε arbitrarily small, we have that $\delta_n' \leq \frac{1}{n}$. Note also that for any non-zero idempotent E and nilpotent N in $\mathcal{B}(\mathbb{C}^n)$

$$1 \leq \operatorname{tr}(E) = \operatorname{tr}(E - N) \leq \|\operatorname{tr}\|\|E - N\| = n\|E - N\|,$$

so it must be that $\delta_n' \geq \frac{1}{n}$. Hence, $\delta_n' = \frac{1}{n}$ so it would seem that there is nothing more to say about the idempotent case. However, in order to get the idempotent E above close to within δ_n' of a nilpotent, you must let its norm increase without bound. Is this necessary? This leads to the following definition:

Definition 1.1 For $n \in \mathbb{N}$ and β a real number greater than or equal to one, define

$$\delta_n(\beta) = \inf\{\|E - N\| : E = E^2, 0 < \|E\| \leq \beta, N \in \mathit{Nil} \text{ and } E, N \in \mathcal{B}(\mathbb{C}^n)\}.$$

Since the zero operator is nilpotent, clearly, $\delta(\beta) \leq \beta$. Also, if a nilpotent N had norm greater than 2β and an idempotent E had norm less than or equal to β, then

$$\|N - E\| \geq \|N\| - \|E\| \geq \beta.$$

So when determining $\delta_n(\beta)$ we can restrict our attention to nilpotents of norm 2β or less. This is a compact set, as is the set of idempotents with norm less than or equal to some β, so the shortest distance $\delta_n(\beta)$ is achieved, so the above infimum is actually a minimum.

Determination of the value of $\delta_n(\beta)$ would tell you how large you must choose the norm of an idempotent to be within a given distance from a nilpotent. Some properties of $\delta_n(\beta)$ are easily deduced.

Lemma 1.2 *For $\delta_n(\beta)$ defined as above we have that*

(i) $\delta_n(1) = \delta_n$;
(ii) *if $n_1 \leq n_2$ are two natural numbers, then $\delta_{n_1}(\beta) \geq \delta_{n_2}(\beta)$;*
(iii) *if $1 \leq \beta_1 \leq \beta_2 < \infty$, then $\delta_n(\beta_1) \geq \delta_n(\beta_2)$;*
(iv) *as $\beta \to \infty$, $\delta_n(\beta) \to \delta_n' = \frac{1}{n}$.*

Proof Since idempotents of norm 1 are projections, (i) follows, and since when $n_1 \leq n_2$, the embedding of $\mathcal{B}(\mathbb{C}^{n_1})$ into $\mathcal{B}(\mathbb{C}^{n_2})$ defined by mapping A to $A \oplus 0$ is an isometry which preserves idempotence and nilpotence, (ii) follows. Conditions (iii) and (iv) follow directly from the definition. ∎

The evaluation of of δ_n (or $\delta_n(1)$) is more difficult than that of δ'_n. The evolution of information went somewhat as follows: Herrero [2] determined that δ_n converges to $\frac{1}{2}$; Salinas [8] showed that

$$\delta_n \leq \frac{1}{2} + \frac{1}{2\sqrt{n}};$$

Herrero [3, 4] showed that

$$\frac{1}{2} \leq \delta_n \leq \frac{1}{2} + \sin\left(\frac{\pi}{\left\lfloor \frac{n-1}{2} \right\rfloor + 1}\right),$$

where $\lfloor x \rfloor$ denotes the greatest integer which is less than or equal to x. Finally, this author [6] improved the upper bound on δ_n by showing that ν_n, the distance from the set of *rank-one* projections to the set of nilpotents in $\mathcal{B}(\mathbb{C}^n)$, is exactly

$$\frac{1}{2}\sec\left(\frac{\pi}{n+2}\right).$$

We also generalize the definition of ν_n in the same direction as δ_n.

Definition 1.3 For $n \in \mathbb{N}$ and β a real number greater than or equal to one, define

$$\nu_n(\beta) = \inf\Big\{ \|E - N\| : E = E^2, \|E\| \leq \beta, \\ \operatorname{rank}(E) = 1, N \in Nil \text{ and } E, N \in \mathcal{B}(\mathbb{C}^n)\Big\}.$$

The quantities $\nu_n(\beta)$ have properties analogous to those of $\delta_n(\beta)$ described in Lemma 1.2. In addition, it follows from trace considerations (see [6]) that $\delta_n(1) = \nu_n(1) = \frac{1}{2}\sec\left(\frac{\pi}{n+2}\right)$ for $n = 2$ or 3 and it was conjectured that $\delta_n(1) = \frac{1}{2}\sec\left(\frac{\pi}{n+2}\right)$ for all n. Also, if the rank of an idempotent E is 2 or greater, trace considerations give that the distance from E to the nilpotents is greater than or equal to $\frac{2}{n}$. From this we obtain that for each n there exists a β_n, such that $\delta_n(\beta) = \nu_n(\beta)$ for $\beta > \beta_n$.

The core technique used to obtain the formula for ν_n was the Arveson distance formula (see [7]) which gives an exact formula for the distance from a matrix to the strictly upper triangular matrices in terms of certain compressions of the matrix. The key idea in this paper is a refinement of the Arveson distance formula, which we develop in Section 2. In many cases this refinement, in addition to giving the distance to upper triangular matrices, also gives information on the "closest pairs" which achieve that distance. We then show this extension applies, at least in the case where the matrix under consideration is a rank-one projection or idempotent. In Section 3, we revisit the problem of determining ν_n and use different methods to rediscover the above bounds. With these methods, we are now able to give a complete concise description of the closest pairs. We also give some slight advances towards proving that $\delta_n = \frac{1}{2}\sec\left(\frac{\pi}{n+2}\right)$ for all $n = 1, 2, 3, \ldots$.

Finally in Section 4 we determine upper bounds for $\delta_n(\beta)$ (which we conjecture are the actual values of $\delta_n(\beta)$). These bounds are obtained by applying the new methods developed in Section 2. We conclude by using the information obtained about $\delta_n(\beta)$ to obtain bounds for

$$\inf\left\{ \|E - N\| : E = E^2, \|E\| \leq \beta, N \in Nil \text{ and } E, N \in \mathcal{B}(H)\right\},$$

the distance from non-zero idempotents of norm less than or equal to β to the nilpotents in the case where H is an infinite dimensional Hilbert space.

2 Extending the Arveson Distance Formula

The Arveson distance formula ([7]) in its full generality gives the distance from an operator T in $\mathcal{B}(H)$ to a given nest algebra. We are interested only in the special case where the nest algebra is $\mathcal{T}_n(\mathbb{C})$, the set of all strictly upper triangular $n \times n$ matrices over $\mathbb{C}$. In that case the distance formula is as follows.

Theorem 2.1 (Arveson Distance Formula) *Let $\{\mathbf{e}_i\}_{i=1}^k$ denote the standard basis for $\mathbb{C}^n$ and let $P_0 = 0$ and $P_i = \sum_{j=1}^i \mathbf{e}_j \otimes \mathbf{e}_j$ for $i = 1, \ldots, n$ denote the orthogonal projection onto the i-dimensional subspace spanned by $\{\mathbf{e}_j\}_{j=1}^i$. Then, for an arbitrary operator A in $\mathcal{B}(\mathbb{C}^n)$, the distance from A to the set of operators whose matrix with respect to the standard basis is strictly upper triangular is*

$$\operatorname{dist}(A, \mathcal{T}_n(\mathbb{C})) = \inf\{\|A - T\| : T \in \mathcal{T}_n(\mathbb{C})\} = \max_{1 \leq i \leq k} \|P_{i-1}^{\perp} A P_i\|.$$

When, for a given operator A, all the compressions cited in the Arveson distance formula have equal norm, we would like to show the difference of A and its closest upper-triangular approximation is a multiple of a unitary. We will need the following lemma.

Lemma 2.2 *Suppose B is an $n \times n$ matrix, A is a $k \times n$ matrix, $\mathbf{y}$ is in $\mathbb{C}^k$ and that*

(i) *the $(n+k) \times n$ matrix*

$$\begin{bmatrix} B \\ A \end{bmatrix}$$

has orthonormal columns;

(ii) *the $k \times (n+1)$ matrix*

$$\begin{bmatrix} A & \mathbf{y} \end{bmatrix}$$

has norm one.

Then there exists $\mathbf{x}$ in $\mathbb{C}^n$ such that the matrix

$$\begin{bmatrix} B & \mathbf{x} \\ A & \mathbf{y} \end{bmatrix}$$

has orthonormal columns.

Proof Since $\left[\begin{smallmatrix} B \\ A \end{smallmatrix}\right]$ has orthonormal columns, $B^*B + A^*A = I_n$ so $B^*B = I_n - A^*A$. Hence, for any vector $\mathbf{v}$ in $\mathbb{C}^n$,

$$\|B\mathbf{v}\|^2 = \langle B^*B\mathbf{v}, \mathbf{v}\rangle = \langle (I_n - A^*A)\mathbf{v}, \mathbf{v}\rangle = \|(I_n - A^*A)^{\frac{1}{2}}\mathbf{v}\|^2.$$

Thus, we can define a linear map $L\colon \mathbb{C}^n \to \mathbb{C}^n$ by

$$L(B\mathbf{v}) = (I_n - A^*A)^{\frac{1}{2}}\mathbf{v}$$

and L is an isometry on the range of B and so can be extended to a unitary on $\mathbb{C}^n$ satisfying $LB = (I_n - A^*A)^{\frac{1}{2}}$. Now

$$\begin{bmatrix} L & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} B & \mathbf{x} \\ A & \mathbf{y} \end{bmatrix} = \begin{bmatrix} (I_n - A^*A)^{\frac{1}{2}} & \mathbf{x}' \\ A & \mathbf{y} \end{bmatrix},$$

where $L\mathbf{x} = \mathbf{x}'$, so with no loss of generality we may assume $B = (I_n - A^*A)^{\frac{1}{2}}$.

Now we also have that

$$\begin{bmatrix} A & \mathbf{y} \end{bmatrix}$$

has norm one, so $\mathbf{y}\mathbf{y}^* \leq I_k - AA^*$ and the associated quadratic forms are equal at some unit vector. Thus, similarly to above, the functional $\varphi\colon \operatorname{Ran}((I_k - AA^*)^{\frac{1}{2}}) \to \mathbb{C}$ defined by

$$\varphi((I_k - AA^*)^{\frac{1}{2}}\mathbf{v}) = \langle \mathbf{v}, \mathbf{y}\rangle$$

has norm one and so, by defining φ to be zero on the orthogonal complement of the range of $(I_k - AA^*)^{\frac{1}{2}}$, we have a norm one functional φ defined on $\mathbb{C}^k$. By the Riesz representation theorem, there exists a vector $\mathbf{z}$ of norm one such that $\varphi(\mathbf{v}) = \langle \mathbf{v}, \mathbf{z}\rangle$ and so $(I_k - AA^*)^{\frac{1}{2}}\mathbf{z} = \mathbf{y}$. Let $\mathbf{x}' = -A^*\mathbf{z}$. Then we claim

$$Z = \begin{bmatrix} (I_n - A^*A)^{\frac{1}{2}} & -A^*\mathbf{z} \\ A & \mathbf{y} \end{bmatrix}$$

has orthonormal columns. This is verified by confirming that $Z^*Z = I$. Note that

$$\begin{aligned} Z^*Z &= \begin{bmatrix} (I_n - A^*A)^{\frac{1}{2}} & A^* \\ -\mathbf{z}^*A & \mathbf{y}^* \end{bmatrix} \begin{bmatrix} (I_n - A^*A)^{\frac{1}{2}} & -A^*\mathbf{z} \\ A & \mathbf{y} \end{bmatrix} \\ &= \begin{bmatrix} (I_n - A^*A) + A^*A & -(I_n - A^*A)^{\frac{1}{2}}A^*\mathbf{z} + A^*\mathbf{y} \\ -\mathbf{z}^*A(I_n - A^*A)^{\frac{1}{2}} + \mathbf{y}^*A & \mathbf{z}^*AA^*\mathbf{z} + \mathbf{y}^*\mathbf{y} \end{bmatrix}. \end{aligned}$$

The $(1,1)$ entry above clearly equals I_n and using that $\mathbf{z}$ is of norm one and $(I_k - AA^*)^{\frac{1}{2}}\mathbf{z} = \mathbf{y}$, the $(2,2)$ entry simplifies to

$$\mathbf{z}^*AA^*\mathbf{z} + \mathbf{z}^*(I_k - AA^*)^{\frac{1}{2}}(I_k - AA^*)^{\frac{1}{2}}\mathbf{z} = \mathbf{z}^*\mathbf{z} = I_k.$$

Using that $A^*(I_n - AA^*)^{\frac{1}{2}} = (I_n - A^*A)^{\frac{1}{2}}A^*$, the $(1,2)$ entry (and hence, similarly for its adjoint in the $(2,1)$ entry) simplifies to

$$-A^*(I_n - AA^*)^{\frac{1}{2}}\mathbf{z} + A^*\mathbf{y} = -A^*\mathbf{y} + A^*\mathbf{y} = 0,$$

so the lemma is established. ■

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We are now ready to prove our key structural theorem.

Theorem 2.3 *Let $\{\mathbf{e}_i\}_{i=1}^{k}$ denote the standard basis for $\mathbb{C}^n$ and let $P_0 = 0$ and $P_i = \sum_{j=1}^{i} \mathbf{e}_j \otimes \mathbf{e}_j$ for $i = 1, \ldots, j$ denote the orthogonal projection onto the i-dimensional subspace spanned by $\{\mathbf{e}_j\}_{j=1}^{i}$. If, for an arbitrary operator $A \in \mathcal{B}(\mathbb{C}^n)$, $\gamma = \|P_{i-1}^{\perp} A P_i\|$ is a constant for $i = 1, 2, \ldots, n$, then there is a unitary operator U and a strictly upper triangular operator T such that $A - T = \gamma U$.*

Proof By scaling A, there is no loss of generality in assuming that $\gamma = 1$. The proof proceeds as follows: Consider the submatrix of A consisting of the first two columns of A. By Lemma 2.2 we can modify the $(1, 2)$ entry so that the resulting matrix has two orthonormal columns. Considering the matrix consisting of this matrix and the third column of A, we can again apply Lemma 2.2 to adjust the $(3, 1)$ and $(3, 2)$ entries so that resulting matrix has three orthonormal columns. We continue in this manner, modifying entries above the main diagonal, until we have finished all the columns and have a unitary matrix which is of the form $A - T$ where T is upper triangular. ■

Remark 2.4 Theorem 2.3 may be of independent interest, however it may not be applicable to many chains of subspaces beyond maximal chains in $\mathbb{C}^n$ as in Theorem 2.3. The theorem does not hold for non-maximal chains, as can be seen by considering the chain $\{\{0\}, \mathbb{C}^2 \oplus \{0\}, \mathbb{C}^3\}$ and choosing a 3×3 matrix A whose first two columns are not orthogonal but are a norm one operator and whose last row has norm one (for example $A = \mathbf{e}_1 \otimes \mathbf{e}_3$). The defect to orthogonality is in the second column and cannot be corrected by adjusting the last column. It also does not hold for infinite chains as there is an index obstruction, as can be seen on $\ell^2(\mathbb{N})$ by taking $P_i = \sum_{j=1}^{i} \mathbf{e}_j \otimes \mathbf{e}_j$ for $i = 1, 2, \ldots$ and taking A to be the forward unilateral shift.

As shown in [6, Lemma 3], in the case of an orthogonal projection P of rank one, there is a choice of basis for which the compressions in Theorem 2.3 are equal and so we can assume Theorem 2.3 applies. Hence, [6, Theorem 6] gives all closest pairs (up to unitary equivalence). It is also a consequence of Theorem 6 that there is only one closest pair up to unitary equivalence, and that any closest pair $\{P, N\}$ satisfies $P - N = \nu_n U$ for some unitary. In the next section we shall provide an alternate description of these unitaries based on spectral information and use this to give a new construction of closest pairs, one that can more easily be generalized to other cases.

First, however we would like a result similar to [6, Lemma 3] which applies to general idempotents. As with projections, the Arveson distance formula and the fact that all rank-one idempotents of a given norm are unitarily equivalent gives that

$$\nu_n(\beta) = \inf\big\{ \max_{1 \le i \le n} \|P_{i-1}^{\perp}\mathbf{e}\| \|P_i \mathbf{f}\| : \mathbf{e}, \mathbf{f} \in \mathbb{C}^n, \|\mathbf{e}\| = \sqrt{\beta}, \|\mathbf{f}\| = \sqrt{\beta}, \langle \mathbf{e}, \mathbf{f} \rangle = 1 \big\}$$

where P_i is the projection of a vector onto its first i coordinates. Using this we can prove the following.

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Lemma 2.5 *The above infimum is achieved when* $\mathbf{e}$ *and* $\mathbf{f}$ *are chosen so that* $\|P_{i-1}^{\perp}\mathbf{e}\|\|P_i\mathbf{f}\|$ *are equal for all* $i = 1, 2, \dots, n$.

Proof Let

$$\mathbf{e} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then $\nu_n(\beta)^2$ is the infimum of

$$\max_{1\le i\le n}\Big(\sum_{j=i}^{n}|a_j|^2\Big)\Big(\sum_{j=1}^{i}|b_j|^2\Big)$$

taken over all $\{a_i\}_{i=1}^{n}$ and $\{b_j\}_{j=1}^{n}$ where

$$\Big(\sum_{j=1}^{n}|a_j|^2\Big)\Big(\sum_{j=1}^{n}|b_j|^2\Big) = \beta^2 \text{ and } \sum_{j=1}^{n} a_j\overline{b_j} = 1.$$

Now suppose that we have a sequence

$$X_i = \Big(\sum_{j=i}^{n}|a_j|^2\Big)\Big(\sum_{j=1}^{i}|b_j|^2\Big) \text{ for } i = 1, 2, \dots, n$$

for some

$$\Big(\sum_{j=1}^{n}|a_j|^2\Big)\Big(\sum_{j=1}^{n}|b_j|^2\Big) = \beta^2 \quad \text{and} \quad \sum_{j=1}^{n} a_j\overline{b_j} = 1$$

and where not all terms are equal. Step (1) is to let i^* be an index where: (1) the maximum is achieved; and (2) the maximum is not achieved at index $i^* + 1$. So

$$\Big(\sum_{j=i^*}^{n}|a_j|^2\Big)\Big(\sum_{j=1}^{i^*}|b_j|^2\Big) > \Big(\sum_{j=i^*+1}^{n}|a_j|^2\Big)\Big(\sum_{j=1}^{i^*+1}|b_j|^2\Big).$$

(If no such i^* exists, this means that the maximum is achieved at some i^* and at all indices greater than i^* and so we go immediately to step 2 below.) First suppose that b_{i^*} does not equal zero. Then we shall adjust $a_{i^*}, a_{i^*+1}, b_{i^*}$ and b_{i^*+1} to $a'_{i^*}, a'_{i^*+1}, b'_{i^*}$ and b'_{i^*+1} so that

$$|a_{i^*}|^2 + |a_{i^*+1}|^2 = |a'_{i^*}|^2 + |a'_{i^*+1}|^2,$$
$$|b_{i^*}|^2 + |b_{i^*+1}|^2 = |b'_{i^*}|^2 + |b'_{i^*+1}|^2,$$
$$a_{i^*}\overline{b_{i^*}} + a_{i^*+1}\overline{b_{i^*+1}} = a'_{i^*}\overline{b'_{i^*}} + a'_{i^*+1}\overline{b'_{i^*+1}},$$

so that the new vectors $\mathbf{e}'$ and $\mathbf{f}'$ that we create satisfy the above conditions which guarantee that $\mathbf{e}' \otimes \mathbf{f}'$ is idempotent and so that the maximum of the X_i either decreases, or at least is achieved at one less index (namely i^*). Note that the above adjustments will only change the value of X_{i^*} and X_{i^*+1} and no other X_j. The adjustments we make are best described geometrically in $\mathbb{C}^2$. Let

$$\mathbf{a} = \begin{bmatrix} a_{i^*} \\ a_{i^*+1} \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_{i^*} \\ b_{i^*+1} \end{bmatrix}.$$

Since b_{i^*} is not zero, we can rotate $\mathbf{a}$ and $\mathbf{b}$ by a small fixed angle to

$$\mathbf{a}' = \begin{bmatrix} a'_{i^*} \\ a'_{i^*+1} \end{bmatrix} \quad \text{and} \quad \mathbf{b}' = \begin{bmatrix} lb'_{i^*} \\ b'_{i^*+1} \end{bmatrix}$$

so that b'_{i^*} is decreased and so the new X'_{i^*} is decreased but still greater than X'_{i^*+1} (which may be slightly increased). If b_{i^*} does equal zero, then it is clear that we must have that $X_{i^*-1} = X_{i^*}$ and $a_{i^*-1} = 0$. Applying a similar rotation to the vectors

$$\mathbf{a} = \begin{bmatrix} a_{i^*-1} \\ a_{i^*} \end{bmatrix} = \begin{bmatrix} 0 \\ a_{i^*} \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_{i^*-1} \\ b_{i^*} \end{bmatrix} = \begin{bmatrix} b_{i^*-1} \\ 0 \end{bmatrix}$$

we can lower the value of both X_{i^*-1} and X_{i^*} and keep them equal and leave the remaining X_i unchanged.

Step (2) is to let i^* be an index where: (1) the maximum is achieved; and (2) the maximum is not achieved at index $i^* - 1$, and do a similar analysis. The argument is slightly changed in the case $i^* = 1$ or n, but in those cases it is easier to decrease the value of X_{i^*}, since the degenerate cases where a entry is zero cannot occur.

Thus we have either decreased the maximum or decreased the index at which the maximum occurs. We may repeat this process until all terms are equal, and so the lemma follows. ■

From this Lemma and Theorem 2.3, it follows that in the calculation of $\nu_n(\beta)$ as well, we may assume that closest pairs $\{E, N\}$ satisfy $E - N = \nu_n(\beta)U$ for some unitary U. In Section 4, we work from unitaries to find bounds for $\nu_n(\beta)$, which we conjecture are sharp, and describe all closest pairs $\{E, N\}$ which achieve this distance. It is not at all clear what unitaries should be chosen, but the projection case gives us some hints.

3 Special Case: Projections to Nilpotents

As mentioned in the introduction, it was shown in [6] that the shortest distance from the set of rank-one projections to nilpotents in $M_n(\mathbb{C})$ is $\nu_n = \frac{1}{2}\sec\left(\frac{\pi}{n+2}\right)$. Explicit descriptions of some closest pairs which achieve that distance were also given. In this section we use new methods, based on the analysis of the unitaries arising from Theorem 2.3, to obtain the values of ν_n and completely characterize the closest pairs, *i.e.*, $\{P, N\}$ where P is a projection and N is a nilpotent in $M_n(\mathbb{C})$, and $\|P - N\| = \nu_n$.

Let $\alpha = \frac{1}{2}\sec\left(\frac{\pi}{n+2}\right)$, let $\{\mathbf{e}_i\}_{i=1}^n$ denote the standard basis for $\mathbb{C}^n$, and define unit vectors $\{\mathbf{x}_i\}_{i=1}^n$ in $\mathbb{C}^n$ as follows

$$\mathbf{x}_1 = \mathbf{e}_1, \mathbf{x}_i = \cos\theta_i \mathbf{e}_{i-1} + \sin\theta_i \mathbf{e}_i \text{ for } i = 2, \dots, n+1,$$

where $\cos\theta_1 = \alpha$ and $\cos\theta_{i+1}\sin\theta_i = \alpha$. (So if $f(t) = \frac{\alpha^2}{1-t^2}$, then $\cos(\theta_i) = \sqrt{f^{(i)}(0)}$.) As shown in [6], $f^{(n)}(0) = 1$, so $\mathbf{x}_{n+1}$ is well defined in $\mathbb{C}^n$ by the above formula, with $\mathbf{x}_{n+1} = \mathbf{e}_n$. It is clear from the above construction that

$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ \alpha & \text{if } |i-j| = 1, \\ 0 & \text{if } |i-j| > 1, \end{cases}$$

for $i, j = 1, 2, \dots, n+1$. We shall use these vectors to construct some special unitary matrices.

Theorem 3.1 *Define a linear operator U in $\mathcal{B}(\mathbb{C}^n)$ by setting $U\mathbf{x}_i = \mathbf{x}_{i+1}$ for $i = 1, 2, \dots, n$ and extending by linearity. Then U has the following properties:*

(i) *U is unitary;*
(ii) *$U^{n+2} = (-1)^{n+1} I$;*
(iii) *the spectrum of U consists of all $(n+2)$-th roots of $(-1)^{n+1}$ except the two closest to -1.*

Proof (i) Given any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{C}^n$, we can write them as a linear combinations of $\{\mathbf{x}_i\}_{i=1}^n$ (since this is a basis for $\mathbb{C}^n$) and if

$$\mathbf{u} = \sum_{i=1}^n u_i \mathbf{x}_i \quad \text{and} \quad \mathbf{v} = \sum_{j=1}^n v_j \mathbf{x}_j,$$

then

$$\begin{aligned} \langle U\mathbf{u}, U\mathbf{v} \rangle &= \sum_{i=1}^n \sum_{j=1}^n u_i \overline{v_j} \langle U\mathbf{x}_i, U\mathbf{x}_j \rangle = \sum_{i=1}^n \sum_{j=1}^n u_i \overline{v_j} \langle \mathbf{x}_{i+1}, \mathbf{x}_{j+1} \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n u_i \overline{v_j} \langle \mathbf{x}_i, \mathbf{x}_j \rangle = \langle \mathbf{u}, \mathbf{v} \rangle \end{aligned}$$

and so U is unitary.

(ii) We have that $U^j \mathbf{e}_1 = \mathbf{x}_{j+1}$ for $j = 1, 2, \dots, n$. Let us define $\mathbf{x}_{n+2}$ to be $U^{n+1}\mathbf{e}_1$. If

$$\mathbf{x}_{n+2} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

then for $j = 1, 2, \dots, n$,

$$\langle \mathbf{x}_{n+2}, \mathbf{x}_{j+1} \rangle = x_j \cos\theta_j + x_{j+1} \sin\theta_j,$$

and using the fact that U is unitary we have that

$$\begin{aligned}\langle \mathbf{x}_{n+2}, \mathbf{x}_{j+1} \rangle &= \langle U^{n+1}\mathbf{e}_1, U^j\mathbf{e}_1 \rangle = \langle U^n\mathbf{e}_1, U^{j-1}\mathbf{e}_1 \rangle \\ &= \langle \mathbf{x}_{n+1}, \mathbf{x}_j \rangle = \begin{cases} \alpha & \text{if } j = n, \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

for $j = 1, 2, \dots, n$. So we get an upper triangular linear system in the coefficients of $\mathbf{x}_{n+2}$,

$$\begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 & \cdots & 0 & 0 \\ 0 & \cos\theta_2 & \sin\theta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \cos\theta_1 \end{bmatrix},$$

which can solved by back-substitution to obtain that

$$x_k = (-1)^{n-k}\alpha \prod_{i=k}^{n-1} \tan\theta_i.$$

Then we define $\mathbf{x}_{n+3}$ to be $U^{n+2}\mathbf{e}_1$. If we let its coordinates be

$$\mathbf{x}_{n+3} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},$$

then, similarly to above, using the fact that U is unitary, and considering the inner products $\langle \mathbf{x}_{n+3}, U^j\mathbf{e}_1 \rangle$ for $j = 2, 3 \dots, n+2$, we obtain the linear system in the entries of $\mathbf{x}_{n+3}$

$$\begin{bmatrix} 0 & \cos\theta_2 & \sin\theta_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cos\theta_3 & \sin\theta_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ (-1)^{n+1}\cos\theta_1 & \cdot & \cdot & \cdot & \cdots & \cdot & \cos\theta_1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \cos\theta_1 \end{bmatrix}.$$

It is clear by inspection that $(-1)^{n+1}\mathbf{e}_1$ is the solution, so $U^{n+2}\mathbf{e}_1 = (-1)^{n+1}\mathbf{e}_1$ and hence $U^{n+2}U^j\mathbf{e}_1 = (-1)^{n+1}U^j\mathbf{e}_1$ for all $j = 0, 1, 2, \dots, n-1$, and so $U^{n+2} = (-1)^{n+1}I$.

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(iii) By (ii), the spectrum of U is contained in the set of $(n+2)$-th roots of $(-1)^{n+1}$. Let ρ be such that $\rho^{n+2} = (-1)^{n+1}$ and let

$$\mathbf{v}_\rho = \sum_{j=0}^{n+1} \rho^j U^j \mathbf{e_1}.$$

Then

$$U\mathbf{v}_\rho = \sum_{j=0}^{n+1} \rho^j U^{j+1}\mathbf{e_1} = \sum_{j=1}^{n+1} \rho^{j-1} U^j \mathbf{e_1} + \rho^{n+1} U^{n+2}\mathbf{e_1}$$
$$= \overline{\rho} \sum_{j=1}^{n+1} \rho^j U^j \mathbf{e_1} + \rho^{n+1}(-1)^{n+1}\mathbf{e_1} = \overline{\rho}\mathbf{v}_\rho.$$

We must consider the possibility that $\mathbf{v}_\rho = 0$. If this were the case, then

$$0 = \langle \mathbf{v}_\rho, \mathbf{v}_\rho \rangle = \sum_{i,j=0}^{n+2} \rho^i \rho^{-j} \langle U^i \mathbf{e_1}, U^j \mathbf{e_1} \rangle$$
$$= \left\langle W \begin{bmatrix} 1 \\ \rho \\ \vdots \\ \rho^{n+1} \end{bmatrix}, \begin{bmatrix} 1 \\ \rho \\ \vdots \\ \rho^{n+1} \end{bmatrix} \right\rangle,$$

where

$$W = \begin{bmatrix} 1 & \alpha & 0 & 0 & \cdots & (-1)^{n+1}\alpha \\ \alpha & 1 & \alpha & 0 & \cdots & 0 \\ 0 & \alpha & 1 & \alpha & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ (-1)^{n+1}\alpha & 0 & \cdots & 0 & \alpha & 1 \end{bmatrix}$$

So $\mathbf{v}_\rho$ will be zero when

$$W \begin{bmatrix} 1 \\ \rho \\ \vdots \\ \rho^{n+1} \end{bmatrix} = \mathbf{0}.$$

Solving the homogeneous system, we obtain that $\alpha\overline{\rho} + 1 + \alpha\rho = 0$, or equivalently $Re(\rho) = -\cos(\frac{\pi}{n+2})$. These correspond to the two $(n+2)$-th roots of $(-1)^{n+1}$ which are closest to -1. ■

With this theorem we can recapture the results on the shortest distance from rank-one projections to nilpotents, and can classify closest pairs.

Theorem 3.2 *In $M_n(\mathbb{C})$, the shortest distance from a rank-one projection $P = \mathbf{e} \otimes \mathbf{e}$, where $\mathbf{e}$ is a unit vector in $\mathbb{C}^n$, to the set of nilpotents is $\nu_n = \frac{1}{2}\sec\left(\frac{\pi}{n+2}\right)$ and the closest nilpotents to P are of the form $P - \nu_n U$ where U is constructed as in Theorem* 3.1*, using any orthonormal basis for $\mathbb{C}^n$ whose first element is $\mathbf{e}$.*

Proof The first part of the theorem is Theorem 1 of [6]. Also, as shown in [6], under the assumption that closest pairs $\{P, N\}$ satisfy $P - N = \nu_n U$ for some unitary U (which by Theorem 2.3 we now know to be true), all closest pairs are unitarily equivalent.

If $P = \mathbf{e}_1 \otimes \mathbf{e}_1$, then take U to be the unitary in Theorem 3.1. To verify that $\mathbf{e}_1 \otimes \mathbf{e}_1 - \alpha U$ is nilpotent, apply it to the basis $\{U^{-j}\mathbf{e}_1\}_{j=1}^{n}$.

$$(\mathbf{e}_1 \otimes \mathbf{e}_1 - \alpha U)\, U^{-1}\mathbf{e}_1 = \langle \mathbf{e}_1, U\mathbf{e}_1\rangle \mathbf{e}_1 - \alpha \mathbf{e}_1 = \mathbf{0},$$

while for $j = -2, -3, \dots, -n$,

$$(\mathbf{e}_1 \otimes \mathbf{e}_1 - \alpha U)U^{-j}\mathbf{e}_1 = \langle \mathbf{e}_1, U^{j}\mathbf{e}_1\rangle \mathbf{e}_1 - \alpha U^{-j+1}\mathbf{e}_1 = -\alpha U^{-j+1}\mathbf{e}_1,$$

so with respect to this basis $\mathbf{e}_1 \otimes \mathbf{e}_1 - \alpha U$ is $-\alpha$ times a single Jordan cell and hence is nilpotent.

For general $P = \mathbf{e} \otimes \mathbf{e}$, conjugate by any unitary which maps $\mathbf{e}$ to $\mathbf{e}_1$ to obtain the result. ∎

As mentioned above, in [6] it is shown that, at least in dimensions 1, 2 and 3, the closest projection to the set of nilpotents is of rank one, and it is conjectured that this is always the case. The following estimation allows us to obtain the result for dimension 4.

Lemma 3.3 *In $M_n(\mathbb{C})$, if P is a projection of rank k and N is nilpotent, then*

$$\|P - N\| \geq \sqrt{\frac{k}{2n}\left(1 + \frac{k}{n}\right)}.$$

Proof We shall use $\|\cdot\|_2$ to denote the Hilbert–Schmidt norm and $\|\cdot\|_1$ to denote the trace norm. If the rank of a projection P is k and N is any nilpotent, upper triangularize N and with respect to this triangularization let $\Delta(X)$, $\mathcal{L}(X)$ and $\mathcal{U}(X)$ denote the diagonal, lower triangular and upper triangular parts of any operator X, respectively. Then

$$\begin{aligned}
\|P - N\|^2 &\geq \frac{1}{n}\|P - N\|_2^2 = \frac{1}{n}\left(\|\mathcal{L}(P)\|_2^2 + \|\Delta(P)\|_2^2 + \|\mathcal{U}(P - N)\|_2^2\right)\\
&\geq \frac{1}{n}\left(\|\mathcal{L}(P)\|_2^2 + \|\Delta(P)\|_2^2\right)\\
&\geq \frac{1}{2n}\left(\|\mathcal{L}(P)\|_2^2 + \|\Delta(P)\|_2^2 + \|\mathcal{U}(P)\|_2^2\right) + \frac{1}{2n}\|\Delta(P)\|_2^2\\
&= \frac{1}{2n}\left(\|P\|_2^2 + \|\Delta(P)\|_2^2\right)
\end{aligned}$$

Now a rank k projection has Hilbert–Schmidt norm $\sqrt{k}$, and applying the Holder inequality we have $k = \operatorname{tr}(\Delta(P)) \leq \|\Delta(P)\|_1 \leq \|\Delta(P)\|_2\|I\|_2$ so $\|\Delta(P)\|_2^2 \geq \frac{k^2}{n}$. Substituting these and taking square roots we have that

$$\|P - N\| \geq \sqrt{\frac{k}{2n}\left(1 + \frac{k}{n}\right)} \qquad \blacksquare$$

Theorem 3.4 *In $M_4(\mathbb{C})$, the shortest distance from the non-zero projections to the nilpotents is achieved by a rank-one projection and is $\delta_4 = \nu_4 = \frac{1}{\sqrt{3}}$.*

Proof Apply Lemma 3.3 with $n = 4$ and $k > 1$. The computation shows that a projection of rank greater than 1 must be at least .612 away from the nilpotents, while Theorem 1 of [6] allows us to get a closest rank-one projection within $\frac{1}{\sqrt{3}}$. $\blacksquare$

Closer analysis of the estimate from Lemma 3.3 shows that the rank of the closest projection to nilpotents can be no more than one-third of the dimension of the underlying space.

4 Idempotents to Nilpotents

Starting from a unitary, we would like to mimic the construction from the projection case in Section 3 for the case of a general idempotent. The fact that closest pairs $\{E, N\}$, (E idempotent of rank one and N nilpotent) have the property that $E - N$ is a multiple of a unitary imposes spatial conditions.

Lemma 4.1 *For $E = \mathbf{x} \otimes \mathbf{y}$ a rank-one operator, N a nilpotent operator and U a unitary operator in $M_n(\mathbb{C})$, and $\gamma \in \mathbb{C}$, if $E - N = \gamma U$, then*

(i) *on the $n - 1$ dimensional subspace $\mathbf{y}^{\perp}$, $-N = \gamma U$;*
(ii) *N has degree of nilpotency n;*
(iii) *if $\mathbf{z} \in \ker(N)$ then $U\mathbf{z} \in \operatorname{Ran}(E)$.*

Proof (i) is immediate. If $\mathbf{z}$ is in $\ker(N)$, then $E\mathbf{z} = \gamma U\mathbf{z}$, so (iii) is established, and if there were two linearly independent vectors in $\ker(N)$, then applying U would give two linearly independent vectors in $Ran(E)$. Hence the nullity of N is 1, and so from Jordan canonical form, N must consist of a single Jordan cell and have degree of nilpotency n. $\blacksquare$

This lemma tells us that the following construction is canonical.

We begin with a unitary matrix U in $M_n(\mathbb{C})$ with a cyclic vector $\mathbf{e}$ of norm one. Define a nilpotent matrix N as the Jordan cell (the matrix of zeroes and ones with the only nonzero entries on the first superdiagonal) with respect to the basis $\{U^{-j}\mathbf{e}\}_{j=1}^{n}$. So the restriction of $U - N$ to the $n - 1$ dimensional subspace $\bigvee_{j=2}^{n} U^{-j}\mathbf{e}$ is 0 and $(U - N)U^{-1}\mathbf{e} = \mathbf{e}$.

Let $\mathbf{y}$ be a vector in $(\bigvee_{j=2}^{n} U^{-j}\mathbf{e}t)^{\perp}$ with $\langle \mathbf{e}, \mathbf{y}\rangle = 1$ and define $E = \mathbf{e} \otimes \mathbf{y}$.

Lemma 4.2 *E has the following properties:*

(i) *E is idempotent of norm $\|\mathbf{y}\|$.*
(ii) *The distance from E to the set of nilpotents is $\frac{1}{|\operatorname{tr}(U)|}$.*

Proof Part (i) is obvious. Taking $N_1 = -\langle U^{-1}\mathbf{e}, \mathbf{y}\rangle N$, we see that

$$E - N_1 = \mathbf{e} \otimes \mathbf{y} + \langle U^{-1}\mathbf{e}, \mathbf{y}\rangle N.$$

If $j = 2, 3, \dots, n$, then $U^{-j}\mathbf{e} \perp \mathbf{y}$ and U and N act identically on $U^{-j}\mathbf{e}$, so

$$(E - N_1)U^{-j}\mathbf{e} = \langle U^{-1}\mathbf{e}, \mathbf{y}\rangle NU^{-j}\mathbf{e} = \langle U^{-1}\mathbf{e}, \mathbf{y}\rangle UU^{-j}\mathbf{e};$$

while $U^{-1}\mathbf{e}$ is in the kernel of N_1, so

$$(E - N_1)U^{-1}\mathbf{e} = (\mathbf{e} \otimes \mathbf{y})U^{-1}\mathbf{e} = \langle U^{-1}\mathbf{e}, \mathbf{y}\rangle\mathbf{e},$$

so $E - N_1$ and $\langle U^{-1}\mathbf{e}, \mathbf{y}\rangle U$ agree on a basis and hence are equal.

To complete the proof, take trace of both sides of $E - N_1 = \langle U^{-1}\mathbf{e}, \mathbf{y}\rangle U$. Since E is a rank-one idempotent, its trace is 1 and N_1 is nilpotent and so has trace zero. Hence $\operatorname{tr}(U) = \frac{1}{\langle U^{-1}\mathbf{e}, \mathbf{y}\rangle}$. ∎

We will want to compute the norm of the vector $\|\mathbf{y}\|$. We have that $\mathbf{y}$ satisfies the n independent linear equations: $\langle U^{-j}\mathbf{e}, \mathbf{y}\rangle = 0$ for $j = 2, 3, \dots, n$ and $\langle U^{-1}\mathbf{e}, \mathbf{y}\rangle = \frac{1}{\operatorname{tr}(U)}$, so we can find $\mathbf{y}$ and hence $\|\mathbf{y}\|$ by solving this system.

So we want to choose unitaries so that the above construction will give good bounds. In the projection case the eigenvalues of the unitary had good separation properties, that is, the angle between adjacent eigenvalues on the unit circle was constant. We shall use similar eigenvalue properties in the general idempotent case, but first we need to interpret the above formulas in terms of the eigenvalues of U.

Set

$$U = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \text{and} \quad \mathbf{e} = \begin{bmatrix} lx_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where $\{\lambda_i\}_{i=1}^n$ lie on the unit circle in the complex plane, and all x_i are distinct (so $\mathbf{e}$ is cyclic for U) and $\sum_{i=1}^n |x_i|^2 = 1$. Then

$$U^k\mathbf{e} = \begin{bmatrix} \lambda_1^k x_1 \\ \lambda_2^k x_2 \\ \vdots \\ \lambda_n^k x_n \end{bmatrix}.$$

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Let D denote the $n \times n$ diagonal matrix with $\lambda_i x_i$ in the (i,i) entry and $\mathbf{z} = D\mathbf{y}$. Then the above equations for $\mathbf{y}$ translate into the following linear system for $\mathbf{z}$:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-2} & \lambda_2^{n-2} & \lambda_3^{n-2} & \cdots & \lambda_n^{n-2} \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{\sum_{i=1}^n \lambda_i} \end{bmatrix}.$$

The coefficient matrix above is a Vandermonde matrix, and using the well-known formula for the determinant of a Vandermonde matrix V and Cramer's rule, we obtain that

$$|z_j|^2 = \frac{1}{|\sum_{i=1}^n \lambda_i|^2}\Big(\frac{1}{\prod_{i\neq j}|\lambda_j - \lambda_i|^2}\Big),$$

so

$$|y_j|^2 = \frac{1}{|x_j|^2 |\sum_{i=1}^n \lambda_i|^2}\Big(\frac{1}{\prod_{i\neq j}|\lambda_j - \lambda_i|^2}\Big)$$

and hence

$$\|E\| = \|\mathbf{y}\| = \frac{1}{|\sum_{i=1}^n \lambda_i|}\sum_{j=1}^n \sqrt{\frac{1}{|x_j|^2 \prod_{i\neq j}|\lambda_j - \lambda_i|^2}}.$$

We want to achieve a given distance with an idempotent of minimal norm, so we choose the cyclic vector $\mathbf{e}$ in such a way as to minimize the norm of E. This can be done via a standard Lagrange multiplier method with variables $t_j = |x_j|^2$ for $j = 1, 2, \ldots, n$, where, if

$$A_j = \frac{1}{|\sum_{i=1}^n \lambda_i|^2}\Big(\frac{1}{\prod_{i\neq j}|\lambda_j - \lambda_i|^2}\Big),$$

the objective function is

$$f(t_1, t_2, \ldots, t_n, \lambda) = \sum_{i=1}^n \frac{A_i}{t_i} - \lambda\Big(\sum_{i=1}^n t_i\Big).$$

This gives us our choice of

$$t_i = \frac{\sqrt{A_i}}{\Big(\sum_{j=1}^n \sqrt{A_j}\Big)^2}$$

which, when substituted into the formula for the norm of E, gives us

$$\|E\| = \frac{1}{|\sum_{i=1}^n \lambda_i|}\sum_{j=1}^n \frac{1}{\prod_{i\neq j}|\lambda_j - \lambda_i|}.$$

We want $|\sum_{i=1}^{n} \lambda_i|$ to be as large as possible, to get close to nilpotents, But if they are too close the norm of E will blow up. It seems reasonable that, as in the projection case, equally spaced eigenvalues on some arc of the unit circle are desirable for U. Since the formulas for both $\|E\|$ and $\mathrm{dist}(E, Nil)$ are invariant under rotation of the eigenvalues $\{\lambda_i\}_{i=1}^{n}$, with no loss of generality we may assume that the eigenvalues $\{\lambda_i\}$ of the unitary matrix U sum to a positive real number. This leads to the following geometric conjecture.

Conjecture 4.3 *Fix $\beta \geq 1$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n points ($n \geq 2$) on the unit circle which sum to β. For each $j = 1, 2, \ldots, n$ consider D_j which is the reciprocal of the product of the distances from λ_j to each of the other λ_i. We conjecture that $D_1 + D_2 + \cdots + D_n$ is minimized when $\lambda_1, \lambda_2, \ldots, \lambda_n$ are equally spaced on some arc of the unit circle.*

This conjecture is vacuously true for $n = 2$ and can be verified for $n = 3$ by setting $\lambda_1 = e^{2ix}$, $\lambda_2 = 1$ and $\lambda_3 = e^{-2iy}$. (Of course, a suitable rotation can be applied so that $\lambda_1 + \lambda_2 + \lambda_3$ is a positive real number). Then it suffices to show that

$$D_1 + D_2 + D_3 = \frac{1}{\sin(x)\sin(y)} + \frac{1}{\sin(x)\sin(x+y)} + \frac{1}{\sin(y)\sin(x+y)} = f(x, y)$$

subject to the constraint $|e^{2ix} + 1 + e^{-2iy}| = 1$, which simplifies to

$$\cos(x)\cos(y)\cos(x+y) = \frac{\beta^2 - 1}{8},$$

has its minimum where $x = y$. In this case, the computations are unpleasant but manageable. In higher dimensions, experimental evidence points to the truth of the conjecture but a computational proof seems intractable.

Assuming the conjecture, we want to take the eigenvalues of U to be $e^{2ij\theta}$ for $j = 0, 1, \ldots, n-1$ for some fixed angle 2θ. (The notational choice of 2θ rather than θ is just to slightly simplify formulas to follow, and of course these eigenvalues could be rotated so their sum is a positive real number without changing the calculation of $\|E\|$ or $\mathrm{dist}(E, Nil)$.)

Substituting these choices for the eigenvalues, and using standard trigonometric identites, we obtain the following bounds:

$$\|E\| = \frac{\sin\theta}{\sin n\theta}\frac{1}{2^{n-1}}\left(\sum_{k=1}^{n} \frac{1}{\left(\prod_{j=1}^{k-1}\sin j\theta\right)\left(\prod_{j=1}^{n-k}\sin j\theta\right)}\right)$$

(where empty products are interpreted as 1) and

$$\mathrm{dist}(E, Nil) \leq \frac{\sin\theta}{\sin n\theta}.$$

Proving the above conjecture is the only impediment to confirming that the above bounds actually give $\nu_n(\beta)$.

For $n = 2$, there is only one choice of eigenvalues up to rotation for each fixed β, and this gives that

$$\|E\| = \frac{1}{\sin 2\theta}$$

and

$$\operatorname{dist}(E, Nil) \leq \frac{1}{2\cos\theta}.$$

So setting $\|E\| = \beta$ and solving we obtain that

$$\operatorname{dist}(E, Nil) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1 + \sqrt{1 - \beta^{-2}}}}$$

and so

$$\delta_2(\beta) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1 + \sqrt{1 - \beta^{-2}}}}.$$

Note that this gives the known values $\delta_2(1) = \frac{1}{\sqrt{2}}$ and $\lim_{\beta\to\infty} \delta_2(\beta) = \frac{1}{2}$. Expanding at the point $\beta = 1$ we have that

$$\delta_2(\beta) = \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}}(\beta - 1) + o(\beta - 1)^2.$$

The above formulas can be used to numerically determine any value $\nu_n(\beta)$ (assuming Conjecture 4.3), or at least give bounds on the value of $\nu_n(\beta)$. They also give parametric descriptions of the curves $y = \nu_n(x)$ as

$$x = \frac{\sin\theta}{\sin n\theta} \frac{1}{2^{n-1}} \left(\sum_{k=1}^{n} \frac{1}{\left(\prod_{j=1}^{k-1} \sin j\theta\right)\left(\prod_{j=1}^{n-k} \sin j\theta\right)} \right),$$

$$y = \frac{\sin\theta}{\sin n\theta},$$

where the parameter θ ranges in the interval $0 < \theta \leq \frac{\pi}{n+2}$. As θ varies in this interval, x, which is the norm of the idempotent, ranges in the interval $1 \leq x < \infty$ and y, which is the distance to nilpotents, ranges in the interval $\frac{1}{n} < y \leq \frac{1}{2}\sec(\frac{\pi}{n+2})$.

Figure 1 is a plot of these curves for $n = 2, 3, \dots, 100$, which gives upper bounds on the corresponding $y = \nu_n(\beta)$, and assuming Conjecture 4.3 are actually plots of $y = \nu_n(\beta)$. The limiting curve gives a bound (which we conjecture is exact) on the distance from rank-one idempotents of norm β to the nilpotents in infinite dimensional Hilbert space.

A better view of the functions may be obtained by applying the function $t = \frac{1}{x}$ to the x coordinate. Plots of these functions (which bound $y = \nu_n\left(\frac{1}{\beta}\right)$) are given in Figure 2 for $n = 2, 3, \dots, 100$.

Both these plots were generated by plotting the parametric curves described above using Maple. It is perhaps surprising that, as can be seen from the above plots, the

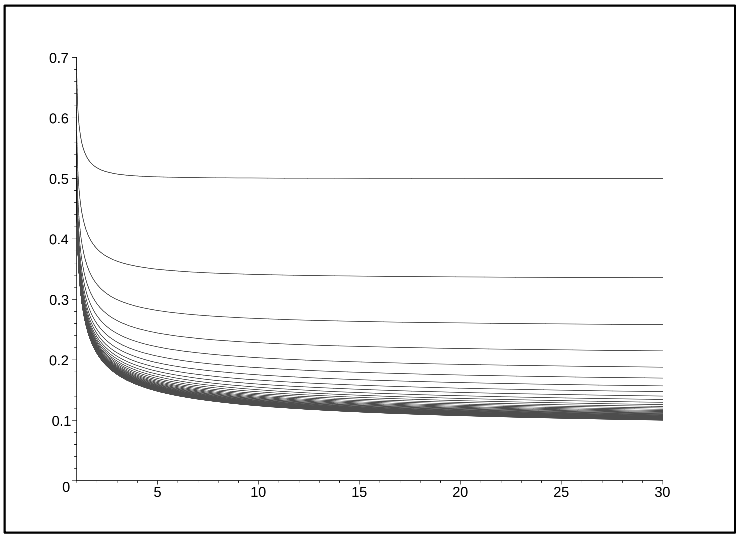

Figure 1

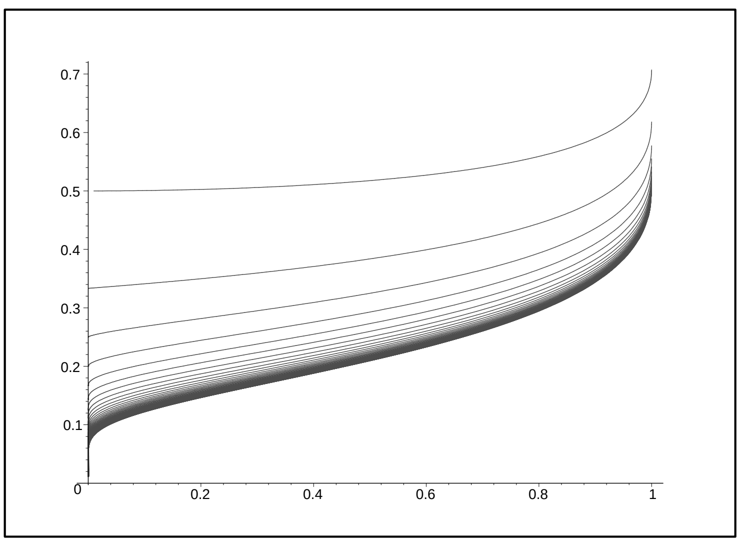

Figure 2

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norm of an idempotent does not have to be very large (on the order of 2 to 3 depending on n) to be within .2 of a nilpotent, but must be very large (on the order of 25 to 30 depending on n) to be within .1 of an idempotent.

Based on numerical evidence derived from the above formulas, we make the following conjecture regarding the behavior of

$$\nu_\infty(\beta) = \inf\Big\{ \|E - N\| : E = E^2, \|E\| \leq \beta, \\ \operatorname{rank}(E) = 1, N \in \mathit{Nil} \text{ and } E, N \in \mathcal{B}(H)\Big\}$$

where H is a separable, infinite-dimensional Hilbert space.

Conjecture 4.4 *For β large,*

$$\nu_\infty(\beta) \approx \frac{1}{2\ln(\beta)}$$

while for β near 1*,*

$$\nu_\infty(\beta) \approx \frac{1}{2}\Big(1 - \sqrt{\beta - 1}\Big).$$

Except for low dimensional cases and large β cases, it is still open as to whether $\delta_n(\beta) = \nu_n(\beta)$, that is: "Is the closest idempotent (of a given norm) to the set of nilpotents in $B(H)$ always rank-one?" It is virtually certain that this is the case, but the proof is elusive. Perhaps the fact that in $M_n(\mathbb{C})$, the closest pairs $\{E_\beta, N_\beta\}_{\beta\geq 1}$ (where E_β is a rank-one idempotent of norm β and N_β is the closest nilpotent) form a continuous path, and are closest pairs among idempotents of all ranks when β is large, can be exploited to verify that $\delta_n(\beta) = \nu_n(\beta)$.

Since every rank-one operator is either nilpotent or a multiple of an idempotent, once Conjecture 4.3 is established, the distance to the set of nilpotents of any rank-one operator will be confirmed. Determination of the distances of higher-rank operators to nilpotents is more difficult. However it seems reasonable that for any non-nilpotent operator A, if N is the nilpotent closest to A, then $A - N$ is a multiple of a unitary. If so, then at least in the case where A is normal, there is perhaps hope that methods similar to those in this paper could yield a general distance-to-nilpotents formula, in terms of the spectral data of A. It is known [1] that this distance should have something to do with the width of the gaps between eigenvalues.

We close by noting that since the eigenvalues of the unitary U in our construction can, with no loss of generality, be taken to be symmetric about the real line, all the constructed matrices U, E and N which satisfy $E - N = \nu_n(\beta)U$ can be realized in $M_n(\mathbb{R})$. Thus it seems that, as in the projection case, the distance from an idempotent of a given norm to the nilpotents is independent of whether we are over the real or complex field.

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References

[1] D. Herrero, *Normal limits of nilpotent operators.* Indiana Univ. Math. J. **23**(1973/74), 1097–1108.

[2] ———, *Toward a spectral characterization of the set of norm limits of nilpotent operators.* Indiana Univ. Math. J. **24**(1974/75), 847–864.

[3] ———, *Quasidiagonality, similarity and approximation by nilpotent operators.* Indiana Univ. Math. J. **30**(1981), no. 2, 199–233.

[4] ———, *Unitary orbits of power partial isometries and approximation by block-diagonal nilpotents.* In: Topics in Modern Operator Theory, Operator Theory: Adv. Appl. 2, Birkhuser, Basel, 1981, pp. 171–210.

[5] ———, *Approximation of Hilbert space operators. I,* Research Notes in Mathematics 72, Pitman, Boston, MA, 1982.

[6] G. MacDonald, *Distance from projections to nilpotents.* Canad. J. Math. **47**(1995), no. 4, 841–851.

[7] S. Power, *The distance to upper triangular operators.* Math. Proc. Camb. Philos. Soc. **88**(1980), no. 2, 327–329.

[8] N. Salinas, *On the distance to the set of compact perturbations of nilpotent operators.* J. Operator Theory **3**(1980), no. 2, 179–194.

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