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ON FINITE ESSENTIAL EXTENSIONS OF TORSION FREE ABELIAN GROUPS

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ABSTRACT. Let A be a torsion free abelian group. We say that a group K is a finite essential extension of A if K contains an essential subgroup of finite index which is isomorphic to A. Such K admits a representation as $(A \oplus \mathbb{Z}_k x)/\mathbb{Z}_k y$ where $y = Nx + a$ for some $k \times k$ matrix N over $\mathbb{Z}$ and $a \in A^k$ satisfying certain conditions of relative primeness and $\mathbb{Z}_k = \{(\alpha_1, \ldots, \alpha_k) : \alpha_i \in \mathbb{Z}\}$. The concept of absolute width of an $f.e.e.$ K of A is defined and it is shown to be strictly smaller than the rank of A. A kind of basis substitution with respect to Smith diagonal matrices is shown to hold for homogeneous completely decomposable groups. This result together with general properties of our representations are used to provide a self contained proof that acd groups with two critical types are direct sum of groups of rank one and two.

1. **Introduction.** Let A be a torsion free group. We say that a group K is a *finite essential extension* of A if K contains an essential subgroup of finite index isomorphic to A. We write K is an $f.e.e.$ of A. Note that K is also torsion free. We denote by $\mathcal{F}(A)$ the class of all $f.e.e.$'s of A.

Some general results about the class $\mathcal{F}(A)$ were given in [10] via the study of $\operatorname{Ext}(C, A)$ exploiting the isomorphism $\operatorname{Ext}(C, A) \cong \operatorname{Hom}(C, \bar{A})$ where C is a finite group and $\bar{A} = A/eA$ where $eC = 0$. In this article we propose another approach obtaining the class $\mathcal{F}(A)$ as special homomorphic images of the direct sum of A with a finite rank free group L. This leads to a simple representation of members of $\mathcal{F}(A)$ which can be handled with an appropriate and suggestive linear calculus. We develop general properties of this representation and show its usefulness by giving a new proof of the structure theorem of almost completely decomposable groups with two critical types. We begin with an observation which is at the root of our development.

THEOREM 1.1. *Let A be a torsion free group. A group K is an $f.e.e.$ of A if and only if there exists a free group L of finite rank and an epimorphism $\phi: A \oplus L \longrightarrow K$ such that $\ker\phi$ is $\tilde{A}$-high in $A \oplus L$, where $\tilde{A} = \{(a, 0) : a \in A\}$.*

PROOF. Suppose that K is an $f.e.e.$ of A, without loss of generality we may assume that K contains A and K/A is finite. Write K/A in its canonical direct sum of cyclic groups namely, $K/A = \bigoplus_{i=1}^{k} \langle x_i + A \rangle$ where $1 < o(x_i + A) \mid o(x_{i+1} + A)$, $i = 1, \ldots, k-1$. It is easy to see that $\{x_i\}_1^k$ is an independent set in K. Thus $K = A + L$ where $L = \bigoplus_{i=1}^{k} \langle x_i \rangle$ is a free group of rank k. Put $G = A \oplus L$ and $\phi: G \longrightarrow K$ defined by $\phi(a, l) = a + l$, where

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$a \in A, l \in L$. Now $\ker\phi = \{(a,-a) : a \in A \cap L\}$ is a pure subgroup of G of rank k disjoint from $\tilde{A} = \{(a,0) : a \in A\}$. Clearly $\ker\phi$ is maximal disjoint from $\tilde{A}$. Conversely, if $\phi: A \oplus L \longrightarrow K$ is an epimorphism where L is free of finite rank and $\ker\phi$ is A-high in $A \oplus L$ then $\phi(A)$ is an essential subgroup of K of finite index and $\phi(A) \cong A$. ∎

We see from Theorem 1.1. that K is an $f.e.e.$ of A if and only if $K \cong (A \oplus L)/H$ where L is free of finite rank and H is an A-high subgroup of $A \oplus L$. Note that H also is free and has the same rank as L.

2. **Linear calculus of finite essential extensions.** In order to give a useful analytical description of A-high subgroups of $G = A \oplus L$ where L is free of finite rank, we have found it convenient to introduce some notations and to state explicitly some linear algebra inherent to this subject.

Let G be a torsion free group and k a positive integer. We put

$$G^k = \{(g_1, \dots, g_k) : g_i \in G, i = 1, \dots, k\}$$

the element $(g_1, \dots, g_k)$ of G^k we denote by $\mathbf{g}$. Thus if we write $\mathbf{h} \in G^k$ it means $\mathbf{h} = (h_1, \dots, h_k)$, $h_i \in G$, $i = 1, \dots, k$. G^k is a $\mathbb{Z}$-module with the usual addition and scalar multiplication. G^k is also naturally a left and right module over the ring $M_k(\mathbb{Z})$ of $k \times k$ matrices over the ring of integers $\mathbb{Z}$. If $\mathbf{g} \in G^k$ and $N \in M_k(\mathbb{Z})$, $\mathbf{g}N$ is the formal line-column matrix product of $\mathbf{g}$ considered as $1 \times k$ matrix by N, whereas $N\mathbf{g}$ is $\mathbf{g}N^T$ or the transposed of the line-column product of N by $\mathbf{g}$ considered as a $k \times 1$ matrix. In the following, for a general group G, we use only the left $M_k(\mathbb{Z})$-module structure of G^k. However in the case of $\mathbb{Z}$ we use the right $M_k(\mathbb{Z})$-module structure and we write $\mathbb{Z}_k$ instead of $\mathbb{Z}^k$. The elements of $\mathbb{Z}_k$ will generally be denoted by Greek letters such as $\boldsymbol{\alpha}, \boldsymbol{\beta}, \dots$. If we have to consider a sequence $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_r \in \mathbb{Z}_k$ we adopt the convention that the coefficients of $\boldsymbol{\alpha}_i$ will bear a double index the first of which is i. Thus $\boldsymbol{\alpha}_i = (\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{ik})$.

Finally, in handling linear combinations of elements of G we find it useful to consider the bilinear map f from $\mathbb{Z}_k \times G^k$ to G defined by

$$f(\boldsymbol{\alpha}, \mathbf{g}) = \sum_{i=1}^{k} \alpha_i g_i$$

we denote $f(\boldsymbol{\alpha}, \mathbf{g})$ simply by $\boldsymbol{\alpha}\mathbf{g}$.

We gather in the next proposition some of the rules of calculation involving a mixture of these various operations.

PROPOSITION 2.1. *Let* $m, n \in \mathbb{Z}$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Z}_k$, $N, D \in M_k(\mathbb{Z})$, $\mathbf{g}, \mathbf{h} \in G^k$. *Then*

(1) $(m\boldsymbol{\alpha})\mathbf{g} = \boldsymbol{\alpha}(m\mathbf{g}) \in G$

(2) $\boldsymbol{\alpha}(m\mathbf{g} + n\mathbf{h}) = m\boldsymbol{\alpha}\mathbf{g} + n\boldsymbol{\alpha}\mathbf{h} \in G$

(3) $(m\boldsymbol{\alpha} + n\boldsymbol{\beta})\mathbf{g} = m\boldsymbol{\alpha}\mathbf{g} + n\boldsymbol{\beta}\mathbf{g} \in G$

(4) $\boldsymbol{\alpha}(N\mathbf{g}) = (\boldsymbol{\alpha}N)\mathbf{g} \in G$

(5) $N(D\mathbf{g}) = (ND)\mathbf{g} \in G^k$

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(6) $m(N\mathbf{g}) = (mN)\mathbf{g} = N(m\mathbf{g}) \in G^k$
(7) $N(\mathbf{g}+\mathbf{h}) = N\mathbf{g} + N\mathbf{h} \in G^k$
(8) $(\boldsymbol{\alpha}+\boldsymbol{\beta})N = \boldsymbol{\alpha}N + \boldsymbol{\beta}N \in \mathbb{Z}_k$.

NOTATION. Let $g_1,\ldots,g_k$ be a sequence of elements of G, let $\mathbf{g}$ be the corresponding element of G^k then the subgroup generated by $g_1,\ldots,g_k$ can be represented as:

$$\mathbb{Z}_k\mathbf{g} = \{\boldsymbol{\alpha}\mathbf{g} : \boldsymbol{\alpha} \in \mathbb{Z}_k\} = \langle g_1,\ldots,g_k\rangle < G.$$

Also for $N \in M_k(\mathbb{Z})$,

$$NG^k = \{N\mathbf{g} : \mathbf{g} \in G^k\} < G^k.$$

DEFINITION 2.2. Let $\mathbf{g} \in G^k$, $N \in M_k(\mathbb{Z})$, and p a prime number. We define the *p-annihilator* of $\mathbf{g}$ by

$$\mathrm{Ann}_p(\mathbf{g}) = \{\boldsymbol{\alpha} \in \mathbb{Z}_k : \boldsymbol{\alpha}\mathbf{g} \in pG\}$$

and the *p-kernel* of N by

$$K_p(N) = \{\boldsymbol{\alpha} \in \mathbb{Z}_k : \boldsymbol{\alpha}N \in p\mathbb{Z}_k\}.$$

These are subgroups of $\mathbb{Z}_k$ containing $p\mathbb{Z}_k$.

DEFINITION 2.3. Let $\mathbf{g} \in G^k$, we say that $\mathbf{g}$ is *distinguished* if $g_1,\ldots,g_k$ are independent elements in G. Thus $\mathbf{g}$ is distinguished if and only if $\mathbb{Z}_k\mathbf{g}$ is a free group of rank k. In particular we will denote a free group L having $x_1,\ldots,x_k$ as a basis by the symbol $L = \mathbb{Z}_k\mathbf{x}$.

PROPOSITION 2.4. *Let G be a torsion free group, $\mathbf{g} \in G^k$ a distinguished element of G^k, $N \in M_k(\mathbb{Z})$ and p a prime number. Then*
(a) $\mathrm{Ann}_p(\mathbf{g}) = p\mathbb{Z}_k$ *iff* $\mathbb{Z}_k\mathbf{g}$ *is a p-pure subgroup of* G
(b) $\mathrm{Ann}_p(\mathbf{g}) = \mathbb{Z}_k$ *iff* $\mathbb{Z}_k\mathbf{g} \subset pG$
(c) $K_p(N) = p\mathbb{Z}_k$ *iff* $(p, \det N) = 1$, *i.e. p does not divide* $\det N$
(d) $K_p(N) = \mathbb{Z}_k$ *iff* $N = pM$ *for some* $M \in M_k(\mathbb{Z})$.
(e) $\det N = 0$ *iff* $\exists \boldsymbol{\alpha} \neq 0$, $\boldsymbol{\alpha} \in \mathbb{Z}_k$ *such that* $\boldsymbol{\alpha}N = 0$.

Before we proceed to the characterization of A-high subgroups of $A \oplus L$ where $L = \mathbb{Z}_k\mathbf{x}$ is a free group of rank k, we need a generalization of the notion of relatively prime integers.

DEFINITION 2.5. Let $N \in M_k(\mathbb{Z})$ and $\mathbf{g} \in G^k$ where G is a torsion free group. We say that N and $\mathbf{g}$ are *relatively prime* and we write $(N,\mathbf{g}) = 1$, if for every prime number p we have

$$\text{(I)} \qquad \mathrm{Ann}_p(\mathbf{g}) \cap K_p(N) = p\mathbb{Z}_k.$$

It can be seen that if $k = 1$ and $G = \mathbb{Z}$ that it is the usual relative primeness of integers. This notion is useful when the determinant of N is a non-zero integer, and in this case it is only necessary to verify (I) for those prime numbers that divide $\det N$.

3. **Characterization of A-high subgroups of $A \oplus \mathbb{Z}_k\mathbf{x}$.** Let $G = A \oplus \mathbb{Z}_k\mathbf{x}$ and H an A-high subgroup of G, then H is isomorphic to a subgroup of the free group $\mathbb{Z}_k\mathbf{x}$, and as such H is also free. In fact H has a basis $\{y_1, y_2, \ldots, y_k\}$ and for each y_i there exist a unique $a_i \in A$ and a unique $\boldsymbol{\alpha}_i \in \mathbb{Z}_k$ such that

$$y_i = \boldsymbol{\alpha}_i\mathbf{x} + a_i.$$

Let $\mathbf{y} = (y_1, \ldots, y_k)$, N the matrix whose rows are $\boldsymbol{\alpha}_i = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ik})$ and $\mathbf{a} = (a_1, \ldots, a_k)$, we have:

$$\mathbf{y} = N\mathbf{x} + \mathbf{a} \quad \text{and} \quad H = \mathbb{Z}_k\mathbf{y}.$$

Thus every A-high subgroup H of $A \oplus \mathbb{Z}_k\mathbf{x}$ determines a matrix $N \in M_k(\mathbb{Z})$ and an element $\mathbf{a} \in A^k$ for each choice of basis of H. More precisely we have:

THEOREM 3.1. *Let A be a torsion free group and $G = A \oplus \mathbb{Z}_k\mathbf{x}$, where $\mathbb{Z}_k\mathbf{x}$ is a free group of rank k. Then every A-high subgroup of G is of the form $\mathbb{Z}_k\mathbf{y}$ where $\mathbf{y} = N\mathbf{x} + \mathbf{a}$ for some $N \in M_k(\mathbb{Z})$ and $\mathbf{a} \in A^k$. Furthermore, $\mathbb{Z}_k\mathbf{y}$ is an A-high subgroup of G if and only if* $\det N \neq 0$ *and* $(N, \mathbf{a}) = 1$.

PROOF. In view of the preceding discussion we need only to show the second part of this theorem. Suppose $\mathbb{Z}_k\mathbf{y}$ is A-high in G. Then $\mathbf{y}$ is distinguished since $\mathbb{Z}_k\mathbf{y}$ is of rank k. Let $\boldsymbol{\alpha} \in \mathrm{Ann}_p(\mathbf{a}) \cap K_p(N)$ for a prime p, this means that $\boldsymbol{\alpha}\mathbf{a} \in pG$, and $\boldsymbol{\alpha}N = p\boldsymbol{\beta}$, for some $\boldsymbol{\beta} \in \mathbb{Z}_k$. Therefore, $\boldsymbol{\alpha}\mathbf{y} = \boldsymbol{\alpha}N\mathbf{x} + \boldsymbol{\alpha}\mathbf{a} = p\boldsymbol{\beta}\mathbf{x} + \boldsymbol{\alpha}\mathbf{a} \in pG$, so that $\boldsymbol{\alpha} \in \mathrm{Ann}_p(\mathbf{y})$ and since $\mathbb{Z}_k\mathbf{y}$ is pure (p-pure for all primes p), $\mathrm{Ann}_p(\mathbf{y}) = p\mathbb{Z}_k$, by Proposition 2.4a. It follows that $\boldsymbol{\alpha} \in p\mathbb{Z}_k$ and $\mathrm{Ann}_p(\mathbf{a}) \cap K_p(N) = p\mathbb{Z}_k$, for every prime p. This is precisely what we mean by $(N, \mathbf{a}) = 1$. Now, let $\boldsymbol{\gamma} \in \mathbb{Z}_k$ such that $\boldsymbol{\gamma}N = \mathbf{0}$ then $\boldsymbol{\gamma}\mathbf{y} = \boldsymbol{\gamma}\mathbf{a} \in A \cap \mathbb{Z}_k\mathbf{y} = 0$. Thus $\boldsymbol{\gamma}\mathbf{y} = 0$ and since $\mathbf{y}$ is distinguished, $\boldsymbol{\gamma} = \mathbf{0}$. Therefore, by Proposition 2.4e, $\det N \neq 0$. Conversely, suppose $\det N \neq 0$ and $(N, \mathbf{a}) = 1$, we need to show that $\mathbf{y}$ is distinguished, $\mathbb{Z}_k\mathbf{y} \cap A = 0$ and $\mathbb{Z}_k\mathbf{y}$ is p-pure for all primes p. Suppose $\boldsymbol{\alpha}\mathbf{y} = 0$, then $\boldsymbol{\alpha}N\mathbf{x} = -\boldsymbol{\alpha}\mathbf{a} \in A \cap \mathbb{Z}_k\mathbf{x} = 0$ *i.e.*, $\boldsymbol{\alpha}N = \mathbf{0}$, but $\det N \neq 0$, therefore $\boldsymbol{\alpha} = \mathbf{0}$, and $\mathbf{y}$ is distinguished. Similarly, one can show that $\mathbb{Z}_k\mathbf{y} \cap A = 0$. Now, let $\boldsymbol{\beta} \in \mathrm{Ann}_p(\mathbf{y})$, then $\boldsymbol{\beta}\mathbf{y} \in pG$ and since $G = A \oplus \mathbb{Z}_k\mathbf{x}$ we have $\boldsymbol{\beta}N\mathbf{x} \in p\mathbb{Z}_k\mathbf{x}$ and $\boldsymbol{\beta}\mathbf{a} \in pA$. But $\mathbf{x}$ is distinguished so that $\boldsymbol{\beta}N = p\boldsymbol{\gamma}$ for some $\boldsymbol{\gamma} \in \mathbb{Z}_k$. It follows that $\boldsymbol{\beta} \in \mathrm{Ann}_p(\mathbf{a}) \cap K_p(N) = p\mathbb{Z}_k$, hence $\mathrm{Ann}_p(\mathbf{y}) = p\mathbb{Z}_k$. Therefore, by Proposition 2.4a, $\mathbb{Z}_k\mathbf{y}$ is p-pure for all primes p. ∎

The next step is to show that the matrix N can be replaced by a special diagonal matrix through a suitable change of bases. This is an application of the well-known stacked bases theorem.

PROPOSITION 3.2. *Let $G = A \oplus \mathbb{Z}_k\mathbf{x}$, where $\mathbb{Z}_k\mathbf{x}$ is a free group of rank k, and $\mathbb{Z}_k\mathbf{y}$ an A-high subgroup of G. Then there exist $\mathbf{z}$, $\mathbf{w} \in G^k$ and $\mathbf{b} \in A^k$ and D a diagonal $k \times k$ matrix over $\mathbb{Z}$, $D = \mathrm{diag}(d_1, \ldots, d_k)$ and $d_1 | d_2 | \cdots | d_k$, such that:*

$$\mathbb{Z}_k\mathbf{z} = \mathbb{Z}_k\mathbf{x}, \quad \mathbb{Z}_k\mathbf{w} = \mathbb{Z}_k\mathbf{y}, \quad \textit{and} \quad \mathbf{w} = D\mathbf{z} + \mathbf{b}.$$

PROOF. Let $\mathbf{y} = N\mathbf{x} + \mathbf{a}$, where $N \in M_k(\mathbb{Z})$, $\det N \neq 0$, $\mathbf{a} \in A^k$ and where $(N, \mathbf{a}) = 1$. Since $\det N \neq 0$, $\mathbb{Z}_k(N\mathbf{x})$ is a free group of rank k. By the stacked bases theorem for finite rank free abelian groups, there exist a basis $z_1, \ldots, z_k$ of $\mathbb{Z}_k(N\mathbf{x})$ and $d_i \in \mathbb{Z}$, $d_i | d_{i+1}$ such that $d_1 z_1, \ldots, d_k z_k$ is a basis of $\mathbb{Z}_k(N\mathbf{x})$. Thus we have $\mathbb{Z}_k \mathbf{x} = \mathbb{Z}_k \mathbf{z}$ and $\mathbb{Z}_k(N\mathbf{x}) = \mathbb{Z}_k D\mathbf{z}$ where $\mathbf{z} = (z_1, \ldots, z_k)$ and $D = \operatorname{diag}(d_1, \ldots, d_k)$. Let M be the matrix corresponding to this change of basis in $\mathbb{Z}_k(N\mathbf{x})$, M is an invertible matrix in $M_k(Z)$ and $N\mathbf{x} = MD\mathbf{z}$. Let $\mathbf{b} = M^{-1}\mathbf{a}$ and $\mathbf{w} = M^{-1}\mathbf{y}$ then $\mathbf{w} = D\mathbf{z} + \mathbf{b}$ and $\mathbb{Z}_k \mathbf{w} = \mathbb{Z}_k \mathbf{y}$. ∎

From the preceding exposition it follows that if K is an *f. e. e.* of A then K is isomorphic to $(A \oplus \mathbb{Z}_k \mathbf{x}) / \mathbb{Z}_k \mathbf{y}$ where $\mathbb{Z}_k \mathbf{x}$ is a free group of rank k and $\mathbf{y} = D\mathbf{x} + \mathbf{a}$ where $D = \operatorname{diag}(d_1, \ldots, d_k)$, $\det D \neq 0$, $d_1 | d_2 | \cdots | d_k$ and $\mathbf{a} \in A^k$ such that $(D, \mathbf{a}) = 1$. We will say that $(A \oplus \mathbb{Z}_k \mathbf{x}) / \mathbb{Z}_k \mathbf{y}$ is in *canonical form* when D is such a diagonal matrix.

We have the exact sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \frac{A \oplus \mathbb{Z}_k \mathbf{y}}{\mathbb{Z}_k \mathbf{y}} & \longrightarrow & \frac{A \oplus \mathbb{Z}_k \mathbf{x}}{\mathbb{Z}_k \mathbf{y}} & \longrightarrow & \frac{A \oplus \mathbb{Z}_k \mathbf{x}}{A \oplus \mathbb{Z}_k \mathbf{y}} \simeq \frac{\mathbb{Z}_k \mathbf{x}}{\mathbb{Z}_k (D\mathbf{x})} & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & K & \longrightarrow & \frac{K}{A} = \oplus_{i=1}^{k} \langle x_i + A \rangle & \longrightarrow & 0
\end{array}$$

where $o(x_i + A) = d_i$ and $d_i | d_{i+1}, i = 1, \ldots, k-1$.

We say that $(A \oplus \mathbb{Z}_k \mathbf{x}) / \mathbb{Z}_k \mathbf{y}$ is a *reduced representation* of an *f. e. e.* of A if in addition, $1 < d_1 \leq d_2 \leq \cdots \leq d_k$. In other words that $\mathbb{Z}_k \mathbf{x} / \mathbb{Z}_k (D\mathbf{x})$ is the direct sum of k non-zero cyclic groups $\langle x_i + \mathbb{Z}_k D\mathbf{x} \rangle$ of order d_i respectively and $d_i | d_{i+1}$. In this case we say that $(A \oplus \mathbb{Z}_k \mathbf{x}) / \mathbb{Z}_k \mathbf{y}$ is a reduced *f. e. e.* of A of *relative width k*. In general we define the relative width of an extension in canonical form $(A \oplus \mathbb{Z}_k \mathbf{x}) / \mathbb{Z}_k \mathbf{y}$ to be the number of diagonal entries in D larger than 1. It can be shown that the relative width is equal to the maximum of $\dim(\mathbb{Z}_k \mathbf{x} / \mathbb{Z}_k N\mathbf{x})[p]$ over the primes p that divide $\det N$.

In general if A is a fixed torsion free group and K is an *f. e. e.* of A, K may be isomorphic to different canonical reduced representations. It is easy to see that the relative width of these representations is at most equal to the rank of A. We can define the *absolute width* of K over A to be the minimum of the relative widths of all canonical reduced representations isomorphic to K. In other words given a K which contains an essential subgroup B of finite index isomorphic to A, we define the absolute width of K over A to be the smallest width of K/B where B ranges over all subgroups of finite index in K that are isomorphic to A. (The width of a finite group G being the number of cyclic summands in any canonical decomposition of G.) The absolute width of an *f. e. e.* K of A of rank k is always strictly smaller than k. This is shown in the next theorem. But first a definition and some technical lemmas.

DEFINITION 3.3. Let $M \in M_k(\mathbb{Z})$ we say that M is a *Smith diagonal* matrix if
i) $\det M \neq 0$ and
ii) $M = \operatorname{diag}(m_1, \ldots, m_k)$ where $1 < m_1 | m_2 | \cdots | m_k$.

LEMMA 3.4. *Let $D \in M_k(\mathbb{Z})$ be a Smith diagonal matrix, and $\mathbf{a} \in A^k$ such that $(D, \mathbf{a}) = 1$ where A is a torsion free group, then $\mathbf{a}$, is distinguished and $k \leq \operatorname{rank}(A)$.*

PROOF. Let $\boldsymbol{\alpha} \in \mathbb{Z}_k$, such that $\boldsymbol{\alpha}\mathbf{a} = 0$, then $\boldsymbol{\alpha} \in \operatorname{Ann}_p(\mathbf{a})$, for all primes p. If $\boldsymbol{\alpha} \neq 0$, we may assume that $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$, and $\gcd(\alpha_1, \ldots, \alpha_k) = 1$. Let p be a prime divisor of d_1, then $K_p(D) = \mathbb{Z}_k$, since $D = pM$ where $M = \operatorname{diag}(m_1, \dots, m_k)$ and $d_i = m_i p$. Therefore $\boldsymbol{\alpha} \in K_p(D) \cap \operatorname{Ann}_p(\mathbf{a}) = p\mathbb{Z}_k$ and $\boldsymbol{\alpha} = p\boldsymbol{\beta}$ for some $\boldsymbol{\beta} \in \mathbb{Z}_k$. This contradicts the fact that $\gcd(\alpha_1, \dots, \alpha_k) = 1$ therefore $\boldsymbol{\alpha} = \mathbf{0}$. It follows that $k \leq \operatorname{rank}(A)$. ∎

LEMMA 3.5. *Let $G = A \oplus \mathbb{Z}_k\mathbf{x}$ and $\mathbf{y} = D\mathbf{x} + \mathbf{a}$ where $D, A, \mathbf{a}$ are as in Lemma 3.4 and* $\operatorname{rank}(A) = k$. *Then $A \subset (d_1 A \oplus \mathbb{Z}_k d_1 \mathbf{x}) + \mathbb{Z}_k\mathbf{y}$.*

PROOF. Let $b \in A$, since by Lemma 3.4, $\mathbf{a}$ is distinguished we have $k = \operatorname{rank}(\mathbb{Z}_k\mathbf{a})$, thus there exists $n \in \mathbb{Z}$ such that $nb = \boldsymbol{\alpha}\mathbf{a}$ for some $\boldsymbol{\alpha} \in \mathbb{Z}_k$. If p is a prime divisor of n and d_1, we have $D = pM$ where $M = \operatorname{diag}(m_1, \dots, m_k)$ and $m_i p = d_i$. Therefore $K_p(D) = \mathbb{Z}_k$ and consequently $\operatorname{Ann}_p(\mathbf{a}) = p\mathbb{Z}_k$. But $\boldsymbol{\alpha} \in \operatorname{Ann}_p(\mathbf{a})$, since $\boldsymbol{\alpha}\mathbf{a} \in pA$, thus $\boldsymbol{\alpha} = p\boldsymbol{\beta}$. This gives $p(mb - \boldsymbol{\beta}\mathbf{a}) = 0$, where $n = pm$ and since A is torsion free, we have $mb = \boldsymbol{\beta}\mathbf{a}$. Therefore we may suppose that $(n, d_1) = 1$. Let $s, t \in \mathbb{Z}$ such that $tn + sd_1 = 1$, then $b = tnb + sd_1 b$ but $tnb = t\boldsymbol{\alpha}\mathbf{a} = t\boldsymbol{\alpha}(D\mathbf{x} - \mathbf{y}) \in \mathbb{Z}_k d_1\mathbf{x} + \mathbb{Z}_k\mathbf{y}$ since $D = d_1 D'$ where $D' = \operatorname{diag}(1, d_2', \dots, d_k')$ and $d_i = d_i' d_1$. It follows that $b \in (d_1 A \oplus \mathbb{Z}_k d_1\mathbf{x}) + \mathbb{Z}_k\mathbf{y}$ and $A \subset (d_1 A \oplus \mathbb{Z}_k d_1\mathbf{x}) + \mathbb{Z}_k\mathbf{y}$. ∎

Now, we are ready to establish our theorem.

THEOREM 3.6. *Let A be a torsion free group of rank n and K an $f.e.e.$ of A. Then the absolute width of K over A is strictly less than n.*

PROOF. Let $K = (A \oplus \mathbb{Z}_k\mathbf{x})/\mathbb{Z}_k\mathbf{y}$ such that $\mathbf{y} = D\mathbf{x} + \mathbf{a}, D$ a Smith diagonal matrix, $\mathbf{a} \in A^k$ and $(D, \mathbf{a}) = 1$. From Lemma 3.4 k is less or equal to n. Now $\mathbb{Z}_k\mathbf{y} \subset A \oplus \mathbb{Z}_k d_1\mathbf{x}$. Indeed let $d_i = d_1 d_i'$ and set $M = \operatorname{diag}(1, d_2', \dots, d_k')$ then for every $\boldsymbol{\alpha} \in \mathbb{Z}_k$,

$$\boldsymbol{\alpha}\mathbf{y} = \boldsymbol{\alpha}D\mathbf{x} + \boldsymbol{\alpha}\mathbf{a} = \boldsymbol{\alpha}Md_1\mathbf{x} + \boldsymbol{\alpha}\mathbf{a} \in A \oplus \mathbb{Z}_k d_1\mathbf{x}.$$

Now, $\mathbb{Z}_k\mathbf{y}$ is also A-high in $A \oplus \mathbb{Z}_k d_1\mathbf{x}$, thus $H = (A \oplus \mathbb{Z}_k d_1\mathbf{x})/\mathbb{Z}_k\mathbf{y}$ is an $f.e.e.$ of A. Furthermore $H \subset K$, and

$$K/H \cong \mathbb{Z}_k\mathbf{x}/\mathbb{Z}_k d_1\mathbf{x} = \bigoplus_{i=1}^{k} \langle x_i + \mathbb{Z}_k d_1\mathbf{x}\rangle o(x_i + \mathbb{Z}_k d_1\mathbf{x}) = d_1, i = 1, \dots, k.$$

It follows that $d_1 K \subset H$. Now if $k < n$, there is nothing to prove, but if $k = n$, from Lemma 3.5 we have

$$\begin{aligned}(A \oplus \mathbb{Z}_k d_1\mathbf{x})/\mathbb{Z}_k\mathbf{y} \subset \big((d_1 A \oplus \mathbb{Z}_k d_1\mathbf{x}) + \mathbb{Z}_k\mathbf{y}\big)/\mathbb{Z}_k\mathbf{y} &= \big(d_1(A + \mathbb{Z}_k\mathbf{x}) + \mathbb{Z}_k\mathbf{y}\big)/\mathbb{Z}_k\mathbf{y} \\ &= d_1\big((A + \mathbb{Z}_k\mathbf{x})/\mathbb{Z}_k\mathbf{y}\big) \\ &= d_1 K.\end{aligned}$$

Therefore $H = d_1 K$ but K is torsion free thus $K \cong H$. Clearly, H is an $f.e.e.$ of A of relative width $\leq k - 1$. Therefore K contains an essential subgroup B of finite index isomorphic to A such that width of $K/B < k$, and the absolute width of K over A is strictly less than n. ∎

This theorem has many consequences. In particular we see immediately that for a group A of rank 1 all $f.e.e.$'s of A have absolute width 0. That is to say that all $f.e.e.$'s are isomorphic to A. In fact it is well-known that all $f.e.e.$'s of a homogeneous completely decomposable group A of finite rank are isomorphic to A. But it is interesting to see how we can obtain this from Theorem 3.6.

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COROLLARY 3.7. *Let A be a homogeneous completely decomposable group of finite rank and K an f.e.e. of A. Then $K \cong A$. In other words the absolute width of K over A is* 0.

PROOF. We induct on the rank of A. If $\operatorname{rank}(A) = 1$ it follows from the remark above. Let A be of rank $k > 1$. Then by Theorem 3.6 if K is an $f.e.e.$ of A it can be represented as $(A \oplus \mathbb{Z}_m\mathbf{x})/\mathbb{Z}_m\mathbf{y}$ where $m < k$ and $\mathbf{y} = D\mathbf{x} + \mathbf{a}$ where $\mathbf{a} \in A^m$, $D \in M_m(\mathbb{Z})$, D is a Smith diagonal matrix and $(D, \mathbf{a}) = 1$. By Lemma 3.4 $\mathbf{a}$ is distinguished so that $\mathbb{Z}_m\mathbf{a}$ is a subgroup of rank m of A. Since A is homogeneous completely decomposable of finite rank every pure subgroup of A is a summand of A (see [5, vol. II, p. 115]), thus $A = (\mathbb{Z}_m\mathbf{a})_* \oplus H$, then

$$K = \left(\frac{(\mathbb{Z}_m\mathbf{a})_* \oplus \mathbb{Z}_m\mathbf{x}}{\mathbb{Z}_m\mathbf{y}}\right) \oplus \left(\frac{H \oplus \mathbb{Z}_m\mathbf{y}}{\mathbb{Z}_m\mathbf{y}}\right).$$

The right hand summand is isomorphic to H. The left hand summand is an $f.e.e.$ of the homogeneous completely decomposable group $(\mathbb{Z}_m\mathbf{a})_*$ of rank $m < k$, hence by induction it is isomorphic to $(\mathbb{Z}_m\mathbf{a})_*$. Therefore $K \cong H \oplus (\mathbb{Z}_m\mathbf{a})_* = A$. ∎

To conclude this section we observe that our representation of $f.e.e.$'s remains invariant under certain substitutions. Let $K = (A \oplus \mathbb{Z}_k\mathbf{x})/\mathbb{Z}_k\mathbf{y}$, where $\mathbf{y} = N\mathbf{x} + \mathbf{a}$, $\det N \neq 0$, $\mathbf{a} \in A^k$, $(N, \mathbf{a}) = 1$. Then K does not change if we replace $\mathbf{y}$ by $M\mathbf{y}$ where M is an invertible $k \times k$ matrix over $\mathbb{Z}$. It does not change either if we replace $\mathbf{x}$ by $\mathbf{x} + \mathbf{b}$ where $\mathbf{a} = N\mathbf{b} + \mathbf{c}$, $\mathbf{b}, \mathbf{c} \in A^k$. Thus $\mathbf{a} \in A^k$ can be replaced by $\mathbf{c} \in A^k$ provided $\mathbf{a} \equiv \mathbf{c} \bmod N$, *i.e.* $\mathbf{a} - \mathbf{c} \in NA^k$. These transformations play an important role in the sequel.

4. **Homogeneous completely decomposable groups.**

Starting here we consider $f.e.e.$'s of completely decomposable groups. We need to establish a crucial property of homogeneous completely decomposable groups (hcd groups) of finite rank. We call this the *substitution property* with respect to Smith diagonal matrices, and it is explicitly stated in the following

THEOREM 4.1. *Let $D \in M_k(\mathbb{Z})$ be a Smith diagonal matrix and $\mathbf{a} \in A^k$ such that $(D, \mathbf{a}) = 1$, where A is an* hcd *group of finite rank. Then there exists $\mathbf{c} \in A^k$ such that*

1) $\mathbf{c} \equiv \mathbf{a} \bmod D$, *i.e.* $\mathbf{c} - \mathbf{a} \in DA^k$ *and*

2) $(\mathbb{Z}_k\mathbf{a})_* = \oplus_{i=1}^k \langle c_i \rangle_*$.

Before we give a proof of Theorem 4.1, we need a few lemmas.

LEMMA 4.2. *Let A be a torsion free group, $a, b \in A$ such that $\tau(a) \leq \tau(b)$, $m \in \mathbb{Z}$, such that $(m, a) = 1$. Then there exists $c \in \langle b \rangle$ such that*

i) $c \in b + m\langle b \rangle$

ii) $\chi(a) \leq \chi(c)$

PROOF. Let $\mathcal{P}$ be the set of all primes and $E = \{p \in \mathcal{P} : h_p(a) > h_p(b)\}$, since $\tau(a) \leq \tau(b)$, E is finite and $h_p(a) = \alpha_p < \infty$ for every $p \in E$. Let

$$n = \prod_{p \in E} p^{\alpha_p - \beta_p} \quad \text{where} \quad h_p(b) = \beta_p.$$

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Now since $(m,a)=1$, $q\in\mathcal{P}$ and $q|m$ implies that q does not divide a so that $h_q(a)=0$ and $q\notin E$. It follows that $(m,n)=1$. Let $s,t\in\mathbb{Z}$ such that $sm+tn=1$ and put $c=tnb$, then $b=smb+c$ and $\chi(a)\leq\chi(c)$. ∎

LEMMA 4.3 (A. MADER). *Let $M=K\oplus L$ be a direct decomposition of R-modules. Then*

$$\phi\longmapsto(1+\phi)L=\{x+\phi(x):x\in L\}$$

defines a bijective correspondence between the maps of $\mathrm{Hom}_R(L,K)$ *and the set of complementary summands of K in M. Furthermore, $1+\phi\colon L\longrightarrow(1+\phi)L$ is an isomorphism.*

PROOF OF THEOREM 4.1. Let D be a Smith diagonal matrix and $\mathbf{a}=(a_1,\dots,a_k)\in A^k$. Since $(D,\mathbf{a})=1$ we know that $\mathbf{a}$ is distinguished. $H=(\mathbb{Z}_k\mathbf{a})_*$ is a pure subgroup of A, therefore H is itself an hcd group [5, vol. II, p. 114] of rank k. Now $\langle a_k\rangle_*$ is a pure subgroup of H and as such, we have:

$$H=\langle a_k\rangle_*\oplus H_k.$$

We put $c_k=a_k$. Now suppose that we have constructed elements $c_k,c_{k-1},\dots,c_{j+1},j\leq k-1$, such that:

$$H=C_j\oplus H_j$$

where

i) $C_j=\oplus_{i=j+1}^k\langle c_i\rangle_*$.
ii) $c_i-a_i\in d_iA$, $i=j+1,\dots,k$.
iii) $(D,\mathbf{a}^j)=1$, where $\mathbf{a}^j=(a_1,\dots,a_j,c_{j+1},\dots,c_k)$.

Then $a_j=b_j+h_j$ where $b_j\in C_j$ and $h_j\in H_j$. Now there exists $n,n_{j+1},\dots,n_k\in\mathbb{Z}$ such that

$$nb_j=\sum_{i=j+1}^k n_ic_i,\quad\text{where }\gcd(n,n_{j+1},\dots,n_k)=1.$$

We claim that $(d_j,h_j)=1$. Indeed if p is a prime divisor of both d_j and h_j, we let $\boldsymbol{\alpha}=(\alpha_i)_{i=1}^k$ where

$$\alpha_i=\begin{cases}0, & i=1,\dots,j-1\\ n, & i=j\\ -n_i, & i=j+1,\dots,k.\end{cases}$$

Then

$$\boldsymbol{\alpha}\mathbf{a}^j=na_j-nb_j=nh_j\in pG$$

and

$$\boldsymbol{\alpha}D=(0,\dots,nd_j,-n_{j+1}d_{j+1},\dots,-n_kd_k)\in p\mathbb{Z}_k$$

so that $\boldsymbol{\alpha}\in\mathrm{Ann}_p(\mathbf{a}^j)\cap K_p(D)=p\mathbb{Z}_k$, *i.e.* $\boldsymbol{\alpha}=p\boldsymbol{\beta}$ for some $\boldsymbol{\beta}\in\mathbb{Z}_k$. This means that p divides $\gcd(n,n_{j+1},\dots,n_k)$ which is a contradiction. Thus $(d_j,h_j)=1$. But $\tau(h_j)=\tau(b_j)$, we apply Lemma 4.2 to obtain an element $b_j'\in b_j+d_j\langle b_j\rangle$ such that $\chi(h_j)\leq\chi(b_j')$. We write $b_j'=b_j+d_jb_j''$, where $b_j''\in\langle b_j\rangle$. Let

$$\phi\colon\langle h_j\rangle_*\longrightarrow\langle b_j'\rangle_*\quad\text{defined by: }\phi(h_j)=b_j'.$$

ϕ is a well defined homomorphism since $\chi(h_j) \leq \chi(b'_j)$. Put $c_j = a_j + d_j b''_j$, then $c_j = h_j + b_j + d_j b''_j = h_j + b'_j = (1+\phi)(h_j)$. Thus $\langle c_j \rangle_* = (1+\phi)\langle h_j \rangle_*$. From Lemma 4.3

$$\langle h_j \rangle_* \oplus C_j = \langle c_j \rangle_* \oplus C_j.$$

Now, $\langle h_j \rangle_*$ is a summand of the hcd group H_j, say $H_j = \langle h_j \rangle_* \oplus H_{j-1}$. Put $C_{j-1} = \langle c_j \rangle_* \oplus C_j$ then $H = C_{j-1} \oplus H_{j-1}$. Clearly $c_j - a_j \in d_j A$ and $C_{j-1} = \oplus_{i=j}^{k} \langle c_i \rangle_*$. It remains to show that $(D, \mathbf{a}^{j-1}) = 1$, where $\mathbf{a}^{j-1} = (a_1, \ldots, a_{j-1}, c_j, \ldots, c_k)$. Let p be a prime and $\boldsymbol{\alpha} \in \mathrm{Ann}_p(\mathbf{a}^{j-1}) \cap K_p(D)$, then

$$\boldsymbol{\alpha} D = p\boldsymbol{\beta} \quad i.e.\ \alpha_i d_i = p\beta_i \ i = 1, \ldots, k$$

and

$$\boldsymbol{\alpha}\mathbf{a}^{j-1} \in pH \quad i.e.\ \sum_{i=1}^{j-1} \alpha_i a_i + \sum_{i=j}^{k} \alpha_i c_i \in pH$$

but

$$\boldsymbol{\alpha}\mathbf{a}^{j-1} = \boldsymbol{\alpha}\mathbf{a} + \sum_{i=j}^{k} \alpha_i (c_i - a_i) = \boldsymbol{\alpha}\mathbf{a} + \sum_{i=j}^{k} \alpha_i d_i b''_i, \quad \text{where } c_i - a_i = d_i b''_i.$$

Therefore $\boldsymbol{\alpha}\mathbf{a} \in pH$ so that $\boldsymbol{\alpha} \in \mathrm{Ann}_p(\mathbf{a}) \cap K_p(D) = p\mathbb{Z}_k$. It follows that

$$\mathrm{Ann}_p(\mathbf{a}^{j-1}) \cap K_p(D) = p\mathbb{Z}_k.$$

This completes the induction step. Now putting $\mathbf{c} = (c_1, \ldots, c_k)$ and $\mathbf{b}'' = (b''_1, \ldots, b''_k)$ we have $\mathbf{c} - \mathbf{a} = D\mathbf{b}''$ and the result follows. ∎

5. **Finite essential extension of completely decomposable groups.** In this section we restrict our attention to $f.e.e.$'s of completely decomposable groups of finite rank. Such groups have been studied in the literature under the name of almost completely decomposable groups (acd groups). A comprehensive treatment of the hitherto known facts and theories about these groups has been given by A. Mader in [10] and it is his work that prompted us to study $f.e.e.$'s. Our approach is different and could be classified as belonging to the generators and relations type. We want to make a case for our techniques by proving the following theorem.

THEOREM. *Let X be an almost completely decomposable group such that $T_{cr}(X) = \{\sigma, \tau\}$. Then X is isomorphic to a direct sum of rank one and rank two groups. In particular, if X is indecomposable, then X has rank one or rank two.*

Furthermore our approach pinpoints the major obstacle that prevents generalizations of this theorem to the case of acd's of more than 2 incomparable critical types.

We need some facts relating decompositions of the original group A and the representation of $f.e.e.$'s of A. First something quite general whose proof is straightforward.

LEMMA 5.1. *Let $A = B \oplus C$ and k be a positive integer, $k \leq \mathrm{rank}(A)$. Put $G = A \oplus \mathbb{Z}_k\mathbf{x}$, where $\mathbb{Z}_k\mathbf{x}$ is a free group of rank k. Let $\mathbf{y} = N\mathbf{x} + \mathbf{a}$ where $N \in M_k(\mathbb{Z})$, $\mathbf{a} \in A^k$ and write $\mathbf{a} = \mathbf{b} + \mathbf{c}$, $\mathbf{b} \in B^k$, $\mathbf{c} \in C^k$. Then*

$$(C + \mathbb{Z}_k\mathbf{y}) \cap (B \oplus \mathbb{Z}_k\mathbf{x}) = \mathbb{Z}_k(\mathbf{y} - \mathbf{c}) \quad \text{and} \quad C + \mathbb{Z}_k\mathbf{y} = C + \mathbb{Z}_k(\mathbf{y} - \mathbf{c}).$$

Furthermore if $\det N \neq 0$ *then* $B \cap \mathbb{Z}_k(\mathbf{y} - \mathbf{c}) = 0$ *and* $C \cap \mathbb{Z}_k\mathbf{y} = 0 = C \cap \mathbb{Z}_k(\mathbf{y} - \mathbf{c})$.

The next proposition is very important. It is in fact the key of the reason why the results do not generalize to the case of more than two non comparable critical types.

PROPOSITION 5.2. *If in Lemma 5.1 we suppose that $\mathbb{Z}_k\mathbf{y}$ is an A-high subgroup of G. Then $(N, \mathbf{b}) = 1$ if and only if $C \oplus \mathbb{Z}_k\mathbf{y}$ is a pure subgroup of G.*

PROOF. If $(N, \mathbf{b}) = 1$ then since $\det N \neq 0$, and $\mathbf{y} - \mathbf{c} = N\mathbf{x} + \mathbf{b}$ we see from Theorem 3.1 that $\mathbb{Z}_k(\mathbf{y} - \mathbf{c})$ is B-high in $B \oplus \mathbb{Z}_k\mathbf{x}$. It follows that $\mathbb{Z}_k(\mathbf{y} - \mathbf{c})$ is a pure subgroup of $B \oplus \mathbb{Z}_k\mathbf{x}$ and $C \oplus \mathbb{Z}_k\mathbf{y} = C \oplus \mathbb{Z}_k(\mathbf{y} - \mathbf{c})$ is pure in $C \oplus B \oplus \mathbb{Z}_k\mathbf{x} = G$. Conversely, if $C \oplus \mathbb{Z}_k\mathbf{y}$ is pure in G, then $\mathbb{Z}_k(\mathbf{y} - \mathbf{c})$ is pure in G since it is the intersection of two pure subgroups of G. Now $\det N \neq 0$ so that in view of Lemma 5.1, $\mathbb{Z}_k(\mathbf{y} - \mathbf{c}) \cong \mathbb{Z}_k\mathbf{y}$, and thus it is a pure subgroup of $B \oplus \mathbb{Z}_k\mathbf{x}$ of rank k. Therefore $\mathbb{Z}_k(\mathbf{y} - \mathbf{c})$ is B-high in $B \oplus \mathbb{Z}_k\mathbf{x}$ and by Theorem 3.1, $(N, \mathbf{b}) = 1$. ∎

It is important to note the cross relationship in Proposition 5.2. If

$$\mathbf{a} = \mathbf{b} + \mathbf{c}, \quad \mathbf{b} \in B^k, \ \mathbf{c} \in C^k$$

then $B \oplus \mathbb{Z}_k\mathbf{y}$ is pure in G if and only if $(N, \mathbf{c}) = 1$ and $C \oplus \mathbb{Z}_k\mathbf{y}$ is pure in G if and only if $(N, \mathbf{b}) = 1$.

Now, we come back to a completely decomposable group A of finite rank. We can write $A = \bigoplus_{i=1}^n A_i$ where A_i is an hcd of type τ_i and $\tau_i \neq \tau_j$ if $i \neq j$. If K is an *f. e. e.* of A we may assume that A is a subgroup of K and in view of Corollary 3.7, the purification $(A_i)_*$ of A_i in K is isomorphic to A_i since it is an *f. e. e.* of A_i. Thus without loss of generality we can assume that A_i is already pure in K for each i. Furthermore if k is the width of K/A and $A \subset B \subset K$ then the width of K/B is $\leq k$. It follows that in the representation $(A \oplus \mathbb{Z}_k\mathbf{x})/\mathbb{Z}_k\mathbf{y}$ of an *f. e. e.* of $A = \bigoplus_{i=1}^n A_i$ we may assume that $k < \mathrm{rank}(A)$ and that $A_i \oplus \mathbb{Z}_k\mathbf{y}$ is a pure subgroup of $A \oplus \mathbb{Z}_k\mathbf{x}$, for each i.

LEMMA 5.3. *Let A be a torsion free group of finite rank and K an f. e. e. of A of absolute width k. Let $(A \oplus \mathbb{Z}_k\mathbf{x})/\mathbb{Z}_k\mathbf{y}$ be a reduced representation of K and suppose $A = B \oplus C$ such that $B \oplus \mathbb{Z}_k\mathbf{y}$ is pure in $A \oplus \mathbb{Z}_k\mathbf{x}$. Then $k \leq \mathrm{rank}(C)$.*

PROOF. Consider the short exact sequence

$$(A \oplus \mathbb{Z}_k\mathbf{y})/B \oplus \mathbb{Z}_k\mathbf{y} \longrightarrow (A \oplus \mathbb{Z}_k\mathbf{x})/B \oplus \mathbb{Z}_k\mathbf{y} \longrightarrow A \oplus \mathbb{Z}_k\mathbf{x})/A \oplus \mathbb{Z}_k\mathbf{y}$$

the first term is isomorphic to C and the last term to $\mathbb{Z}_k\mathbf{x}/\mathbb{Z}_kD\mathbf{x}$ where $\mathbf{y} = D\mathbf{x} + \mathbf{a}$ for some Smith diagonal matrix $D \in M_k(\mathbb{Z})$ and some $\mathbf{a} \in A^k$. The middle term is a torsion free group and in fact it is an *f. e. e.* of C of relative width k. Therefore $k \leq \mathrm{rank}(C)$. ∎

In the remaining part of this section our group A is the direct sum of two hcd groups of finite rank B and C such that $\tau(B)$ and $\tau(C)$ are incomparable.

PROPOSITION 5.4. *Let $A = B \oplus C$, where B is* hcd *of rank m and C is* hcd *of rank n. Let K be an f. e. e. of A and let k be the absolute width of K over A, then $k \le \min(n, m)$. Furthermore if K has no rank 1 summand then* $\operatorname{rank}(A) = 2k$.

PROOF. From the remarks after Proposition 5.2 K can be represented in the form $(A \oplus \mathbb{Z}_k\mathbf{x})/\mathbb{Z}_k\mathbf{y}$ where k is the absolute width of K over A and $B \oplus \mathbb{Z}_k\mathbf{y}$ and $C \oplus \mathbb{Z}_k\mathbf{y}$ are both pure subgroups of $A \oplus \mathbb{Z}_k\mathbf{x}$. By Lemma 5.3, we have $k \le \operatorname{rank}(B)$ and $k \le \operatorname{rank}(C)$. Therefore $k \le \min(m, n)$. Write $\mathbf{y} = D\mathbf{x} + \mathbf{a}$ for a Smith diagonal matrix $D \in M_k(\mathbb{Z})$ and $\mathbf{a} \in A^k$, then $\mathbf{a} = \mathbf{b} + \mathbf{c}$ for some $\mathbf{b} \in B^k$ and $\mathbf{c} \in C^k$ and by Proposition 5.2, $(D, \mathbf{b}) = 1 = (D, \mathbf{c})$. Now by Lemma 3.4 both $\mathbf{b}$ and $\mathbf{c}$ are distinguished. Thus $\mathbb{Z}_k\mathbf{b}$ and $\mathbb{Z}_k\mathbf{c}$ are both of rank k. But $(\mathbb{Z}_k\mathbf{b})_*$ and $(\mathbb{Z}_k\mathbf{c})_*$ are each summands of B and C respectively, say $B = B' \oplus (\mathbb{Z}_k\mathbf{b})_*$ and $C = C' \oplus (\mathbb{Z}_k\mathbf{c})_*$. Now

$$(A \oplus \mathbb{Z}_k\mathbf{x})/\mathbb{Z}_k\mathbf{y} = (B' \oplus C' \oplus \mathbb{Z}_k\mathbf{y})/\mathbb{Z}_k\mathbf{y} \oplus \big((\mathbb{Z}_k\mathbf{b})_* \oplus (\mathbb{Z}_k\mathbf{c})_* \oplus \mathbb{Z}_k\mathbf{x}\big)/\mathbb{Z}_k\mathbf{y}.$$

If K has no summands of rank 1, the left hand factor which is isomorphic to $B' \oplus C'$ must be null. Therefore $B = (\mathbb{Z}_k b)_*$ and $C = (\mathbb{Z}_k c)_*$ thus $\operatorname{rank} A = 2k$. ∎

THEOREM 5.5. *Let $A = B \oplus C$ where B and C are* hcd *groups of non comparable types and K an f. e. e. of A. Then K is the direct sum of groups of rank two or one.*

PROOF. Let k be the absolute width of K over A, and represent K as $(A \oplus \mathbb{Z}_k\mathbf{x})/\mathbb{Z}_k\mathbf{y}$. Then as in Proposition 5.4,

$$(A \oplus \mathbb{Z}_k\mathbf{x})/\mathbb{Z}_k\mathbf{y} = \big((B' \oplus C' \oplus \mathbb{Z}_k\mathbf{y})/\mathbb{Z}_k\mathbf{y}\big) \oplus \Big(\big((\mathbb{Z}_k\mathbf{b})_* \oplus (\mathbb{Z}_k\mathbf{c})_* \oplus \mathbb{Z}_k\mathbf{x}\big)/\mathbb{Z}_k\mathbf{y}\Big).$$

The left hand summand is a completely decomposable group isomorphic to $B' \oplus C'$. So we consider the right hand summand. We have

$$\mathbf{y} = D\mathbf{x} + \mathbf{b} + \mathbf{c}, \quad \text{and} \quad (D, \mathbf{b}) = 1 = (D, \mathbf{c}).$$

By the substitution property for hcd groups (Theorem 4.1) there exists $\mathbf{b}'$ and $\mathbf{b}''$, and $\mathbf{c}'$ and $\mathbf{c}''$ such that

$$\mathbf{b}' = \mathbf{b} + D\mathbf{b}'' \quad \text{and} \quad \mathbf{c}' = \mathbf{c} + D\mathbf{c}''$$

and

$$(\mathbb{Z}_k\mathbf{b})_* = (\mathbb{Z}_k\mathbf{b}')_* = \bigoplus_{i=1}^{k} \langle b_i' \rangle_*, \quad (\mathbb{Z}_k\mathbf{c})_* = (\mathbb{Z}_k\mathbf{c}')_* = \bigoplus_{i=1}^{k} \langle c_i' \rangle_*.$$

Thus

$$(\mathbb{Z}_k\mathbf{b})_* \oplus (\mathbb{Z}_k\mathbf{c})_* \oplus \mathbb{Z}_k\mathbf{x} = (\mathbb{Z}_k\mathbf{b}')_* \oplus (\mathbb{Z}_k\mathbf{c}')_* \oplus \mathbb{Z}_k\mathbf{z}$$

where $\mathbf{z} = \mathbf{x} - \mathbf{b}'' - \mathbf{c}''$. Therefore

$$\big((\mathbb{Z}_k\mathbf{b})_* \oplus (\mathbb{Z}_k\mathbf{c})_* \oplus \mathbb{Z}_k\mathbf{x}\big)/\mathbb{Z}_k\mathbf{y} = \Big(\bigoplus_{i=1}^{k} \langle b_i' \rangle \oplus \bigoplus_{i=1}^{k} \langle c_i' \rangle \oplus \bigoplus_{i=1}^{k} \langle z_i \rangle\Big) \Big/ \bigoplus_{i=1}^{k} \langle y_i \rangle$$

where $\mathbf{z} = (z_1, \ldots, z_k)$. Now $\mathbf{y} = D\mathbf{z} + \mathbf{b}' + \mathbf{c}'$ and since $y_i + d_i z_i = b_i' + c_i'$, $i = 1, \ldots, k$, $\langle y_i \rangle \subset \langle b_i' \rangle_* + \langle c_i' \rangle_* \oplus \langle z_i \rangle$, and the right hand side is isomorphic to

$$\bigoplus_{i=1}^{k} (\langle b_i' \rangle \oplus \langle c_i' \rangle \oplus \langle z_i \rangle) / \langle y_i \rangle$$

which is a direct sum of groups of rank 2. Therefore K is the direct sum of groups of rank one and rank two. ■

In a subsequent paper we intend to present the relationship of our method and the important notions of regulating and regulator subgroups and near-isomorphism.

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