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THE STRUCTURE OF QUASI-FROBENIUS RINGS

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Introduction. Utilizing a matrix representation of semiperfect rings by a family of bimodules over local rings, we describe the structure of generalized quasi-Frobenius rings in two steps: a cyclic generalized quasi-Frobenius ring is a matrix ring over a cycle of Morita dualities between local rings, and an arbitrary generalized quasi-Frobenius ring is a matrix ring over a family of cyclic generalized quasi-Frobenius rings.

Our results provide a complete classification of generalized quasi-Frobenius rings, modulo the classification of local rings with Morita duality, of certain bimodules over such rings, and of certain rest families of multiplication maps. They allow with ease the construction of many examples. Ordinary quasi-Frobenius rings are handled as an immediate specialization.

Our arguments amount to a careful exploitation of a theorem by Fuller [**4**], generalized by Roux [**4**], connecting self-injectivity with Morita duality. The idea of employing this result to relate cyclic quasi-Frobenius rings to cycles of Morita dualities, and arbitrary quasi-Frobenius rings to families of cyclic ones, is due to Hannula [**5**; **6**]; similar concepts appear in Fried [**3**]. An analogous structure theory should persist for arbitrary pairs of rings with Morita duality.

1. *Matrix representation of semiperfect rings.* In [**7**] we developed a matrix representation for semiperfect rings: there is a one-to-one correspondence between the isomorphism types of semiperfect rings, and of systems (A_i, X_{ik}, μ_{ijk}) of local rings A_i, $A_i - A_k$-bimodules X_{ik} $(i \neq k)$, and bimodule homomorphisms $\mu_{ijk} : X_{ij} \otimes_{A_j} X_{jk} \rightarrow X_{ik}$ $(i \neq j \neq k)$ satisfying the associativity condition

$$\mu_{ikl}(\mu_{ijk} \otimes X_{kl}) = \mu_{ijl}(X_{ij} \otimes \mu_{jkl}) \quad \text{for } i \neq j \neq k \neq l$$

(all indices from the same finite set I).

The correspondence is realized by selecting for a given semiperfect ring R, any decomposition $1 = e_1 + \ldots + e_n$ into orthogonal indecomposable idempotents and putting $A_i = e_iRe_i$, $X_{ik} = e_iRe_k$ and μ_{ijk} the restrictions of the multiplication; and conversely by associating with the system (A_i, X_{ik}, μ_{ijk}) the set of all matrices (x_{ik}) with $x_{ik} \in X_{ik}$ (where $X_{ii} = A_i$, and μ_{ijk} is defined by the ring- or module-multiplication whenever two adjacent indices are equal).

A semiperfect ring is *basic* if no two different idempotents in a decomposition $1 = e_1 + \ldots + e_n$ into indecomposable orthogonal idempotents are iso-

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morphic; every semiperfect ring possesses a unique (up to isomorphism) basic ring to which it is Morita equivalent. A semiperfect ring with matrix representation (A_i, X_{ik}, μ_{ijk}) is basic if and only if the μ_{iki} map into the radicals of the A_i. It is ring-directly decomposable if and only if there is a partition of the index set I into two disjoint nonempty subsets I', I'' such that $X_{ik} = X_{ki} = 0$ whenever $i \in I'$, $k \in I''$.

From an arbitrary system of local rings A_i and $A_i - A_k$-bimodules X_{ik}, a basic semiperfect ring may be constructed, by taking all $\mu_{ijk} = 0$, which is clearly associative. From a family of local rings A_i only, an indecomposable basic semiperfect ring is constructable if and only if any two A_i, A_k may be joined by a chain of A_j's in which any two neighbors possess a non-zero bimodule; this question is resolved by the following observation.

LEMMA 1. *A non-zero bimodule* ${}_AX_B$ *exists for local rings* A, B *if and only if either* char A = char $B = 0$, *or* char $A = p^k$, char $B = p^l$, *or* char $A = p^k$, char $B = 0$, char $\bar{B} = p$, *where* p *is a prime number and* $\bar{B}$ *is the residue division ring of* B.

Proof. The characteristic of a local ring is either zero, or the power of a prime. A non-zero bimodule ${}_AX_B$ exists if and only if $A \otimes_{\mathbf{Z}} B \neq 0$ (since X has a projective extension as bimodule), if and only if $1 \otimes 1 \neq 0$. If the conditions of the lemma are not fulfilled, then either char$A = p^k$, char$B = q^l$ for primes $p \neq q$, or char$A = p^k$, char$B = 0$, char$\bar{B} \neq p$. In the first case $1 = p^k a + q^l b$ with a, $b \in \mathbf{Z}$, hence $1 \otimes 1 = p^k \otimes a + b \otimes q^l = 0$, while in the second case p is invertible in $\bar{B}$ hence in B, pb $= 1$, and consequently $1 \otimes 1 = 1 \otimes p^k b^k = p^k \otimes b^k = 0$.

Suppose now the conditions satisfied: if char$A$ = char$B = 0$, then the torsion subgroups TA, TB are proper ideals of A, B and A/TA, B/TB are torsionfree hence flat $\mathbf{Z}$-modules, which implies $A/TA \otimes_{\mathbf{Z}} B/TB \neq 0$. In the two other situations one has char$\bar{A}$ = char$\bar{B} = p$, hence $\bar{A}$, $\bar{B}$ are $\mathbf{Z}/(\mathrm{p})$-vectorspaces and $\bar{A} \otimes_{\mathbf{Z}} \bar{B} \neq 0$.

Remark. From a technical point of view, passing to a matrix representation allows studying a ring in three stages of vastly increasing difficulty. Fortunately many important properties of semiperfect rings may be described in terms of the subsystems (A_i) or (A_i, X_{ik}) alone; e.g. such a ring is left perfect or semiprimary if and only if all the A_i are such [**7**, Theorem 2], and it is left artinian or noetherian if and only if all the rings A_i and bimodules X_{ik} have the corresponding property.

2. Cyclic generalized quasi-Frobenius rings. A ring R is *generalized quasi-Frobenius* if it is an injective cogenerator as left- and as right-module over itself, or equivalently if $\hom_R(-, R)$ is a Morita duality. Such a ring is semiperfect; and for any indecomposable idempotent e, Re is the injective envelope of a simple module. Hence if R is basic with Jacobson radical J, and if $1 = e_1 + \ldots + e_n$ is a decomposition into indecomposable orthogonal idempotents, then socle $(Re_k) \cong Re_i/Je_i$ defines a permutation $i = \pi(k)$, which we shall call the *Nakayama permutation of* R. The corresponding permutation defined from right ideals turns out to be π^{-1} (cf., e.g., [**1**; **9**; **10**; **12**]).

Any permutation is, in a unique way, the product of cycles; if π is itself a cycle, R is called a *cyclic generalized quasi-Frobenius ring*. The numeration can then be chosen such that $\pi(k) = k + 1$ (modulo n). A cyclic generalized quasi-Frobenius ring is automatically ring-directly indecomposable.

LEMMA 2. *Let e, f be idempotents of a semiperfect basic ring R with Jacobson radical J. Then the following are equivalent:*

(1) *Re, fR are injective R-modules, and there are decompositions $e = e_1 + \ldots + e_s$, $f = f_1 + \ldots + f_s$ into indecomposable orthogonal idempotents such that the socle of Re_k is isomorphic to Rf_k/Jf_k;*

(2) *the bimodule fRe induces a Morita duality between the rings fRf and eRe, and the ring multiplication yields isomorphisms $Re \cong \hom_{fRf}(fR, fRe)$ and $fR \cong \hom_{eRe}(Re, fRe)$.*

Proof. This follows from the proofs of (3.6) and (3.3) of [**11**], and by specialization to semiperfect rings, observing that then all indecomposable idempotents are local and each idempotent is the sum of indecomposable orthogonal idempotents.

LEMMA 3. *Let (A_i, X_{ik}, μ_{ijk}) be a matrix representation of a semiperfect ring R. Then R is basic cyclic generalized quasi-Frobenius if and only if there is a cyclic permutation π of the index set, such that*

(i) *the bimodules $X_{\pi k,k}$ induce Morita dualities between $A_{\pi k}$ and A_k;*

(ii) *the $\mu_{\pi k,i,k}$ induce isomorphisms $X_{ik} \cong {}^*X_{\pi k,i}$ and $X_{\pi k,i} \cong X_{ik}{}^*$* (with the dual modules with respect to the Morita dualities);

(iii) *The μ_{iki} map into the radicals of A_i.*

Proof. (iii) is equivalent to the ring being basic. If R is a basic cyclic generalized quasi-Frobenius ring, let $e = e_k$, $f = e_{\pi k}$ in Lemma 2; then (1) holds by definition of π, hence (2) follows. Therefore we obtain (i) immediately, and (ii) by suitably restricting the isomorphisms of Lemma 2. Conversely if (i), (ii) and (iii) are satisfied, (2) of Lemma 2 hence (1) holds, demonstrating that R is two-sided selfinjective and contains all simple modules, hence is a cogenerator, i.e. generalized quasi-Frobenius. Moreover its Nakayama permutation is precisely π, hence R is cyclic.

A matrix representation of a basic cyclic generalized quasi-Frobenius ring may be described by a small set of data. By a *cycle of dualities* we shall mean a family of local rings $A_1, \ldots, A_n$ for which Morita dualities exist between A_{i+1}-left- and A_i-right-modules (i modulo n). $[r]$ is used to denote the greatest integer less or equal to the real number r, and $M^{(s)} = M^{*\cdots*}$ for the iterated right dual of a bimodule M with respect to a cycle of dualities.

Definition. *A data system* $(A_k, U_k, Y_i, \sigma_i, (Z, \tau), \mu_{ijk})$ *for a basic cyclic generalized quasi-Frobenius ring consists of*

(1) *a cycle* $A_1, \ldots, A_n$ *of dualities;*

(2) $A_{k+1} - A_k$*-bimodules* U_k *inducing these dualities;*

(3) $A_1 - A_i$*-bimodules* Y_i $(i = 2, \ldots, [\frac{1}{2}n])$ *such that* $Y_i^{(2n)} \cong Y_i$ *as bimodules, and all* $Y_i^{(s)}$ *are reflexive as right modules; and if* n *is odd, an additional* $A_1 - A_{\frac{1}{2}(n+1)}$*-bimodule* Z *such that* $Z^{(n)} \cong Z$, *and all* $Z^{(s)}$ *are right-reflexive;*

(4) *particular choices* $\sigma_i : Y_i \to Y_i^{(2n)}$ *and* $\tau : Z \to Z^{(n)}$ *of isomorphisms;*

(5) *bimodule maps* $\mu_{ijk} : X_{ij} \otimes_{A_j} X_{jk} \to X_{ik}$ $(k + 1 \neq i \neq j \neq k)$ *such that the* μ_{kjk} *map into the radicals of* A_k, *and that the associativity condition described below is satisfied.*

THEOREM 4. *There is a one-to-one correspondence between data systems, and matrix representations of basic cyclic generalized quasi-Frobenius rings with Nakayama permutation* $\pi(k) = k + 1$ *(modulo* n*).*

Proof. We construct a matrix representation (A_i, X_{ik}, μ_{ijk}) from a given data system as follows: Let A_k be the local rings of the given cycle of dualities,

$$X_{k+1,k} = U_k,\ X_{t,t+i-1} = Y_i^{(2t-2)},\ X_{i+t,t} = Y_i^{(2t-1)}\ (t = 1, \ldots, n);$$

and if n is odd,

$$X_{t,\frac{1}{2}(n-1)+t} = Z^{(2t-2)}\ (t = 1, \ldots, \tfrac{1}{2}(n+1)) \text{ and } X_{\frac{1}{2}(n+1)+1,t} = Z^{(2t-1)}\quad (t = 1, \ldots, \tfrac{1}{2}(n-1));$$

this covers all $X_{ik} (i \neq k)$. Define $\mu_{k+1,i,k}$ $(k + 1 \neq i \neq k)$ by evaluations $X_{k+1,i} \otimes X_{ik} = X_{ik}^* \otimes X_{ik} \to U_k$, unless $k = n$ and $2 \leqq i \leqq [\frac{1}{2}(n + 1)]$, in which case

$$\mu_{1in} : X_{1i} \otimes X_{in} \xrightarrow{\sigma_i \otimes 1} Y_i^{(2n)} \otimes Y_i^{(2n-1)} \to U_n \text{ and}$$

$$\mu_{1,\frac{1}{2}(n+1),n} : X_{1,\frac{1}{2}(n+1)} \otimes X_{\frac{1}{2}(n+1),n} \xrightarrow{\tau \otimes 1} Z^{(n)} \otimes Z^{(n-1)} \to U_n.$$

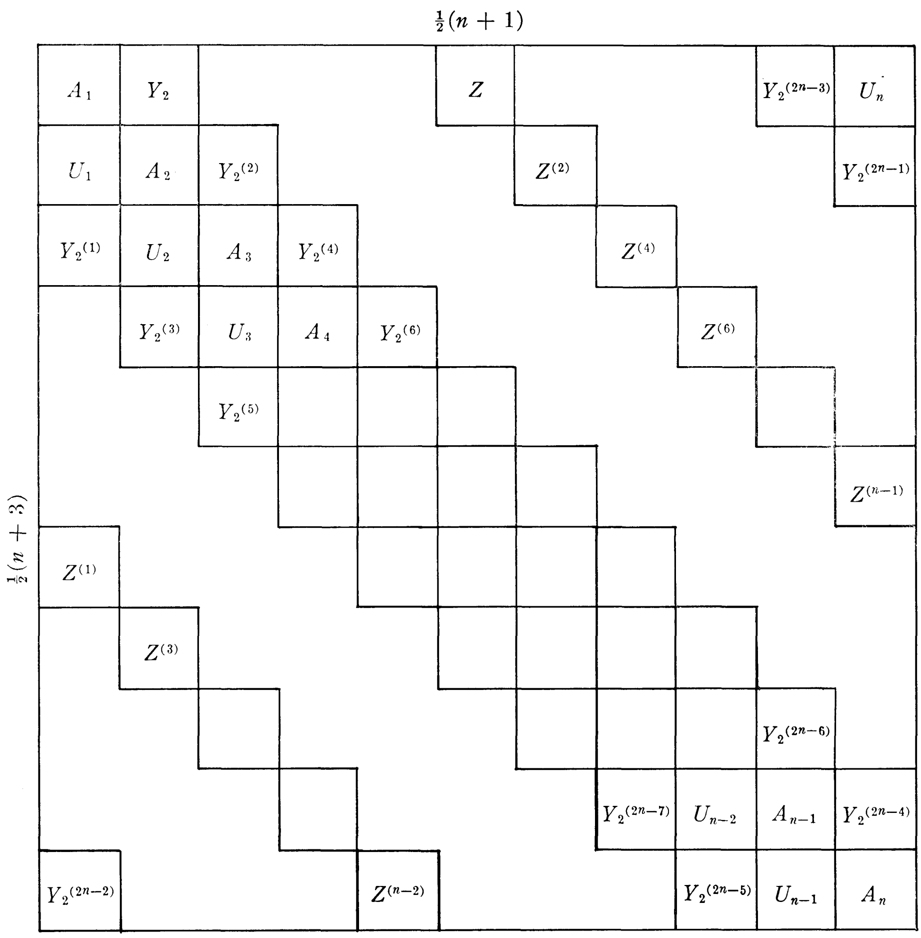
The associativity condition referred to in (5) of the definition is meant to require that the rest-family μ_{ijk} is associative together with the so-constructed $\mu_{k+1,i,k}$. This matrix representation satisfies (i), (ii) and (iii) of Lemma 3.

Conversely, from a given matrix representation (A_i, X_{ik}, μ_{ijk}) of a basic cyclic generalized quasi-Frobenius ring with Nakayama permutation $\pi(k) = k + 1$ (modulo n), a data system is derived by taking $U_k = X_{k+1,k}$, $Y_i = X_{1i}$ (and $Z = X_{1,\frac{1}{2}(n+1)}$ if n is odd). From (ii) of Lemma 3 it follows that all iterated (left and) right duals of these modules are reflexive, and that $Y_i \cong Y_i^{(2n)}$ and $Z^{(n)} \cong Z$, by isomorphisms which are composites of maps induced by suitable μ's; we take for σ_i and τ these composites.

The first construction, followed by the second one, gives the original data system back. The composition in the opposite order does not precisely reproduce the original matrix representation, but one canonically isomorphic to it, under a family of bimodule isomorphisms which are composites of maps induced by suitable μ's. The verification of the details, which requires checking the commutativity of diagrams involving these composites, is somewhat involved but straightforward.

The figure should give an impression of the matrix representation derived from a data system. It shows the case of odd n; for even n the diagonal containing the Z's is deleted. It becomes even more striking if extended periodically over the plane.

Remarks. (1) If one starts with an arbitrary cycle of dualities and bimodules inducing them, it is always possible to find bimodules Y_i and Z satisfying (3) of the definition, namely $Y_i = Z = 0$. If this or any other family of bimodules Y_i and Z satisfying (3) is given, one may pick any σ_i and τ, define the $\mu_{k+1,i,k}$ as described, and then find a rest family of μ_{ijk}'s satisfying (5), namely



$\mu_{ijk} = 0$. (That this is actually an associative family is not obvious but can be checked without difficulties). In particular a basic cyclic generalized quasi-Frobenius ring can be constructed from any cycle of dualities.

(2) We do not intend discussing here the isomorphism problem for data systems, i.e. when two data systems lead to isomorphic rings. This can be done in principal, but the results are rather complicated to formulate and do not seem to be particularly useful.

(3) As an example, Hannula [**6**] determined all cyclic basic quasi-Frobenius rings for $n = 2$, over a local quasi-Frobenius ring A, as the rings $S_2(\rho, u)$, i.e., 2×2-matrix rings over A with a modified multiplication given by the automorphism ρ of A and the element $u \in \operatorname{rad} A$, subject to the restrictions $u^\rho = u$ and $xu = ux^\rho$ for all $x \in A$. One verifies that $S_2(\rho, u)$ and $S_2(\rho', u')$ are isomorphic if and only if there is an automorphism α and an invertible element a of A such that $u' = \alpha(u)a$ and $\rho'(x) = a^{-1}(\alpha\rho\alpha^{-1})(x)a$.

(4) We describe those basic cyclic generalized quasi-Frobenius rings R which are finite dimensional algebras over a field K. Any such algebra has self-duality, induced by the K-dual A^+. A cycle of dualities is therefore the repetition of one local algebra A, and all the U_k may be taken to be A^+, except $U_n = {}_\rho A^+$ where ρ is an arbitrary K-algebra automorphism of A (cf. Remarks (3) and (4) at the end of the paper); moreover the conjugacy class of ρ in ΓA is an invariant of R.

A module is K-reflexive if and only if it is finite dimensional; and then there are natural isomorphisms $Y_i^* \cong Y_i^+$, $Y_i^{**} \cong Y_i^{++} \cong Y_i$, hence $Y_i^{(2n-2i)} \cong Y_i$ but $Y_i^{(2n-2i+1)} \cong \hom_A(Y_i, {}_\rho A^+) \cong {}_\rho(Y_i^+)$, hence

$$Y_i^{(2n-1)} = Y_i^{(2n-2i+1)(2i-2)} \cong Y_i^{(2n-2i+1)} \cong {}_\rho(Y_i^+),$$

and finally

$$Y_i^{(2n)} \cong \hom_A({}_\rho(Y_i^+), {}_\rho A^+) \cong {}_\rho(Y_i^+)_\rho.$$

Moreover, if n is odd, $Z^{(n-1)} \cong Z$ and $Z^{(n)} \cong \hom_A(Z, {}_\rho A^+) \cong {}_\rho(Z^+)$. The eligible bimodules Y_i and Z are therefore all finite dimensional modules satisfying $Y_i \cong {}_\rho(Y_i)_\rho$ and $Z \cong {}_\rho(Z^+)$. In particular if ρ is the identity, all finite dimensional Y_i are eligible, and all finite dimensional Z with $Z \cong Z^+$.

No attempt is made to classify the possible rest families of multiplication maps μ_{ijk} $(k + 1 \neq i \neq j \neq k)$.

3. Arbitrary generalized quasi-Frobenius rings. Let R be an arbitrary basic generalized quasi-Frobenius ring, $1 = e_1 + \ldots + e_m$ a decomposition into indecomposable orthogonal idempotents, and $\pi = \pi_1 \ldots \pi_n$ the cycle decomposition of the corresponding Nakayama permutation. Let ϵ_k be the sum of the idempotents e_j permuted by the cycle π_k. Then the decomposition $1 = \epsilon_1 + \ldots + \epsilon_n$ into orthogonal (but not necessarily indecomposable) idempotents leads to a matrix representation (A_k, X_{ik}, μ_{ijk}) as before, i.e., e.g., $A_k = \epsilon_k R \epsilon_k$. Though the A_k are not necessarily local, this representation is still unique up to isomorphism, since any other decomposition $1 = e_1' + \ldots + e_m'$, is isomorphic to the given one under an inner automorphism of the semiperfect ring R, hence leads to essentially the same π and an isomorphic decomposition $1 = \epsilon_1' + \ldots + \epsilon_n'$.

LEMMA 5. *With the notation just introduced, the A_k are basic cyclic generalized quasi-Frobenius rings, and the μ_{kik} map into the radicals of A_k and induce isomorphisms of X_{ik} with the duals $^*X_{ki}$ and X_{ki}^*. Conversely any matrix representation with these properties represents a basic generalized quasi-Frobenius ring, whose Nakayama permutation is the product of the cycles belonging to the A_k.*

These subrings A_k (which are unique up to isomorphism) will be called the *cyclic constituents* of the generalized quasi-Frobenius ring.

Proof. (1) of Lemma 2 holds for $e = f = \epsilon_k$, hence so does (2). We obtain a duality of A_k with itself induced by A_k, demonstrating that A_k is generalized quasi-Frobenius. Moreover the isomorphism $R\epsilon_k \cong \hom_{\epsilon_k R \epsilon_k}(\epsilon_k R, \epsilon_k R \epsilon_k)$ produces by restriction isomorphisms

$$X_{ik} = \epsilon_i R \epsilon_k \cong \hom_{A_k}(\epsilon_k R \epsilon_i, A_k) = {}^*X_{ki};$$

and analogously $X_{ik} \cong X_{ki}^*$.

Conversely if a matrix representation has the properties of Lemma 7, then (2) of Lemma 2 holds obviously for $e = f = \epsilon_k$, hence (1) follows, proving that R is self-injective on both sides and contains all simple modules, i.e. is generalized quasi-Frobenius.

Clearly R is basic if and only if the A_k are basic, and the μ_{kik} map onto the radicals of A_k. The R-monomorphism $Re_{\pi k}/Je_{\pi k} \rightarrow Re_k$ onto the socle of Re_k yields by restriction an A_k-monomorphism $A_k e_{\pi k}/\mathrm{rad} A_k e_{\pi k} \rightarrow A_k e_k$, from which one deduces (keeping in mind that one already knows R and all the A_k to be generalized quasi-Frobenius) that the Nakayama permutation π of R restricts to the Nakayama permutations of the A_k. Therefore in the first argument these are cycles, and in the converse one π is the product of the given cycles of the A_k.

Definition. A data system $(A_k, X_{ik}, \sigma_{ik}, \mu_{ijk})$ for an arbitrary basic generalized quasi-Frobenius ring consists of

(1) *basic cyclic generalized quasi-Frobenius rings A_k;*

(2) *A_i - A_k-bimodules X_{ik} $(i < k)$ which are as left- and as right-modules reflexive and annihilated by the socles of A_i and A_k;*

(3) *bimodule isomorphism $\sigma_{ik} : X_{ik}^* \rightarrow {}^*X_{ik}$ satisfying $x\alpha(y) = \sigma_{ik}(\alpha)(x)y$ for all $x, y \in X_{ik}$ and $\alpha \in X_{ik}^*$;*

(4) *bimodule maps $\mu_{ijk} : X_{ij} \otimes X_{jk} \rightarrow X_{ik}$ $(i \neq j \neq k \neq i)$ such that the associativity condition described below is satisfied.*

THEOREM 6. *There is a one-to-one correspondence between data systems, and matrix representations of basic generalized quasi-Frobenius rings over a family of basic cyclic generalized quasi-Frobenius rings.*

Proof. We construct a matrix representation from a data system by setting (for $i < k$) $X_{ki} = X_{ik}^*$, $\mu_{kik} : X_{ki} \otimes X_{ik} = X_{ik}^* \otimes X_{ik} \rightarrow A_k$ and

$$\mu_{iki} : X_{ik} \otimes X_{ki} = X_{ik} \otimes X_{ik}^* \xrightarrow{1 \otimes \sigma_{ik}} X_{ik} \otimes {}^*X_{ik} \rightarrow A_i$$

by evaluations. These μ's satisfy the necessary associativity conditions among each other because of $x\alpha(y) = \sigma_{ik}(\alpha)(x)y$; and the additional associativity condition referred to in the definition is meant to require the rest family μ_{ijk} to form an associative family together with these μ_{kik} and μ_{iki}.

Conversely from a given matrix representation as in Lemma 5, we derive a data system by taking σ_{ik} to be the composed isomorphism $X_{ik}^* \cong X_{ki} \cong {}^*X_{ik}$, which is easily checked to satisfy $x\alpha(y) = \sigma_{ik}(\alpha)(x)y$. Since A_k is generalized quasi-Frobenius, the left- and right-socles and the left- and right annihilators of the radical of A_k all coincide, and therefore μ_{kik} as defined above maps into the radical of A_k if and only if for all $\alpha \in X_{ik}^*$, $0 = \alpha(X_{ik})\mathrm{soc}\, A_k = \alpha(X_{ik}\mathrm{soc}\, A_k)$. Since A_k is a cogenerator, this means $X_{ik}\ \mathrm{soc}\ A_k = 0$.

Starting from a data system and applying both constructions one obtains the data system back. Starting from a matrix representation, one obtains a matrix representation canonically isomorphic to the original one, by isomorphisms induced by the μ's.

Remarks. The question, for which families of basic cyclic generalized quasi-Frobenius rings A_k a data system exists which leads to an *indecomposable* basic generalized quasi-Frobenius ring, is not trivial. An immediate though somewhat surprising result is that division rings cannot be built into any such data system, in a non-trivial fashion: indeed if $A = A_1$ is a division ring, then $A_1 = \mathrm{soc}\ A_1$ hence (2) of the definition can hold only for $X_{1k} = 0 = X_{k1}$.

Quite contrary to this solitary behavior of the division rings, arbitrary basic cyclic quasi-Frobenius rings which are not division rings, can be built together into a data system, if we restrict attention to the classical case of finite dimensional algebras over a field. (What happens in general we were unable to determine; though the following proof may be somewhat generalized, it depends on some sort of finiteness condition to keep the tensor products small.)

PROPOSITION 7. *For any finite family of basic cyclic quasi-Frobenius rings A_k which are finite dimensional algebras over a field K and not division algebras, there is a ring-directly indecomposable basic quasi-Frobenius algebra whose cyclic constituents are the A_k.*

Proof. For any basic cyclic generalized quasi-Frobenius ring A which is not a division ring, $\mathrm{soc}\ A \subset \mathrm{rad}\ A$: indeed otherwise $\mathrm{soc}\ A$ would contain a non-zero indecomposable idempotent e, hence eA, Ae would be simple, hence $eA(1 - e) = 0 = (1 - e)Ae$, producing a decomposition $A = eAe \oplus (1 - e)A(1 - e)$ which cannot exist.

Next we show that for arbitrary algebras A and B as in the proposition, $X = \bar{B} \otimes_K \bar{A}$ where $\bar{A} = A/\text{rad } A$ and $\bar{B} = B/\text{rad } B$, is a non-zero bimodule satisfying (2) and (3) of the definition. Clearly X is non-zero and finite dimensional over K hence reflexive; and $\text{rad}B\, X = 0 = X\, \text{rad}A$ hence $\text{soc}B\, X = 0 = X\, \text{soc}A$, proving (2). Furthermore we have a bimodule isomosphism

$$X^* = \hom(\bar{B} \otimes \bar{A}_A, A_A) \cong \hom_K(\bar{B}, \hom_A(\bar{A}, A)) \cong \hom_K(\bar{B}, \text{soc}A)$$
$$\cong \hom_K(\bar{B}, \bar{A}),$$

utilizing $\text{soc}A \cong \bar{A}$ as left-A-modules, since both are the direct sum of all isomorphism types of simple left-A-modules. Similarly we get ${}^*X \cong \hom_K(\bar{A}, \bar{B})$ which in turn is isomorphic to $\hom_K(\bar{B}^+, \bar{A}^+)$ where $^+$ denotes the K-dual. By [**2**, Proposition 5] $\bar{A} \cong \bar{A}^+$ and $\bar{B} \cong \bar{B}^+$ as bimodules, hence altogether $X^* \cong {}^*X$, producing the isomorphism required for (3) of the definition. For $\alpha \in X^*$ we have $0 = \alpha(X \text{ rad } A) = \alpha(X) \text{ rad } A$ hence $\alpha(X)$ lies in the left annihilator $\text{soc } A$ of $\text{rad } A$, whence $X\alpha(X) \subset X \text{ soc } A \subset X \text{ rad } A = 0$. Therefore $x\alpha(y) = 0$, and similarly $\sigma(\alpha)(x)y = 0$, implying trivially the needed equality $x\alpha(y) = \sigma(\alpha)(x)y$.

It remains to construct for any finite family of such algebras A_k and these bimodules $X_{ik} = \bar{A}_i \otimes {}_K\bar{A}_k$, a rest family of maps μ_{ijk} satisfying the required associativity. But $\mu_{ijk} = 0$ works, utilizing in the critical case $(i = k \neq j \neq l \neq k)$ that all μ_{iji} map into $\text{rad } A_i$ and that $\text{rad } A_i X_{il} = 0$.

4. Concluding remarks. (1) The structure of ordinary quasi-Frobenius rings is obtained from our considerations as a special case: A generalized quasi-Frobenius ring R is actually quasi-Frobenius, if and only if its cyclic constituents are artinian (i.e. quasi-Frobenius), if and only if its local constituents are artinian. This follows from [**7**, Theorem 2] and the fact that semiprimary rings with Morita duality are automatically artinian [**10**, Theorem 3]. Slight simplifications arise since for an artinian ring with duality, a module is reflexive if and only if it is of finite length.

(2) If a Morita duality exists between two local rings A and B, then they determine each other up to isomorphism, each being isomorphic to the endomorphism ring of the injective envelope of the simple module of the other. Therefore, if a local ring A belongs to a cycle of dualities, it creates a unique shortest cycle, and any other cycle containing it arises as a repetition of this shortest cycle.

All local rings with duality known to us presently, have a "self-duality," i.e. they determine a shortest cycle of length one. In particular this is the case for all finite dimensional algebras over a field, all generalized quasi-Frobenius rings, and all commutative rings with duality [**8**, Theorem 3].

Open questions. Does there exist a local ring which creates a shortest cycle of dualities of length greater than one? Does there exist a local ring with duality, not creating any cycle at all?

(3) If a bimodule ${}_BU_A$ induces a Morita duality between rings B and A, then it also induces an isomorphism between the groups of automorphisms modulo inner automorphisms ΓB and ΓA of B and A, by $\tau \leftrightarrow \sigma$ if and only if $U \cong {}_\tau U_\sigma$, where ${}_\tau U_\sigma$ is the module whose multiplications have been modified by the automorphisms τ and σ. Therefore, if A creates a shortest cycle of dualities $A = A_1, A_2, \ldots, A_m$ and if these dualities are induced by bimodules ${}_{A_{i+1}}U_{i\,A_i}$, the composition of the group isomorphisms leads to an automorphism of ΓA, which we shall call the *twist* of the cycle. It depends on the choice of the U_i; but since these are unique up to modifications of their multiplications by automorphisms of the rings, the twist is unique up to an inner automorphism of ΓA. Open question: *Does there exist a cycle of dualities with non-trivial twist?* (In the three cases mentioned above, the twist is easily checked to be trivial).

(4) Interest in the twist arises in connection with the isomorphism problem for data systems of basic cyclic generalized quasi-Frobenius rings: To discuss this question briefly, select for each isomorphism type of a shortest cycle of dualities a particular ring A appearing in it, and for each isomorphism type of a pair of rings with duality a particular bimodule V inducing such a duality. Then any basic cyclic generalized quasi-Frobenius ring possesses a data system whose cycle of dualities is a repetition of a shortest cycle starting with the distinguished $A_1 = A$, and whose bimodules U_k are the distinguished V's up to the last one, which is $U_n = {}_\rho V_m$, modified by an element ρ of ΓA. Furthermore the possible alternative modifications are precisely by $t^p(\alpha)t^k(\rho)\alpha^{-1}$ ($k \in \mathbf{Z}$, $\alpha \in \Gamma A$), where t is the twist, and p is the number of repetitions of the shortest cycle (i.e., $n = pm$).

In particular if the twist is trivial, one obtains as invariants of the basic cyclic generalized quasi-Frobenius ring the isomorphism type of A, the number p of repetitions of the shortest cycle created by A, and the conjugacy class of ρ in ΓA.

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