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FINITE REGULAR COVERS OF SURFACES

BY
LARRY W. CUSICK

ABSTRACT. Let $T^k = T^1 \# \ldots \# T^1$, $T^1 = S^1 \times S^1$, $U^k = \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2$, and G is a finite group. We prove (1) Every free action of G on $U^{\ell+2}$ lifts to a free action of G on the orientable two fold cover $T^{\ell+1} \to U^{\ell+2}$ and (2) The minimum k such that Z_m^ℓ can act freely on T^k is $m^\ell((\ell - 2)/2) + 1$ if $m = 2$ or ℓ is even and $m^\ell((\ell - 1)/2) + 1$ otherwise.

§0 **Introduction**. In this paper we study finite regular covers of surfaces, i.e. finite free group actions on surfaces. We shall restrict our attention to the closed compact surfaces $T^k \cong T^1 \# \ldots \# T^1$ (k times) where $T^1 \cong S^1 \times S^1$ and $U^k \cong \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2$ (k times). The two main results are proposition (2.1) which states that any free action of G on $U^{\ell+2}$ lifts to a free G action on the orientable two-fold cover $T^{\ell+1} \to U^{\ell+2}$ and proposition (2.5) which gives the minimum k such that an elementary abelian group $G \cong Z_{m_1} \times \ldots \times Z_{m_\ell}$ acts freely on T^k. Both results are consequences of proposition (1.7) that gives a sufficient condition for determining when the kernel of an epimorphism $\partial: \pi_1 U^{\ell+2} \to G$ is isomorphic to $\pi_1 T^{|G|\ell+1}$. We conjecture that this condition is also necessary.

§1 **Finite Regular Covers**. Suppose G is a finite group, of order n, acting freely on T^{m+1} with orbit space B. The natural projection map $p: T^{m+1} \to B$ is a regular covering space with resulting exact sequence

$$1 \to \pi_1 T^{m+1} \xrightarrow{p\#} \pi_1 B \xrightarrow{\partial} G \to 1$$

and G is naturally isomorphic to the group of covering transformations. Furthermore, B is a closed compact surface whose Euler characteristic, $\chi(B)$, satisfies the formula $n\chi(B) = -2m$. Consequently $B \cong T^{m/n+1}$ or $U^{2(m/n)+2}$.

Conversely, suppose we are given an epimorphism $\partial: \pi_1 B \to G$ where $B \cong T^{\ell+1}$ or $U^{\ell+2}$ and $|G| = n$, then the inclusion $\ker \partial \to \pi_1 B$ is induced by a finite regular cover $p: X \to B$, with G isomorphic to the group of covering transformations and so G acts freely on X. The Euler characteristic of X is given by

$$\chi(X) = \begin{cases} -2\ell n & \text{if } B \cong T^{\ell+1} \\ -\ell n & \text{if } B \cong U^{\ell+2}. \end{cases}$$

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To go any further we must treat the two cases of B separately.

If $B \cong T^{\ell+1}$ then B is orientable. It follows that X is a closed compact orientable surface with Euler characteristic equal to $-2\ell n$, and so $X \cong T^{\ell n+1}$. In this case the action of G on $T^{\ell n+1}$ preserves the orientation.

If $B \cong U^{\ell+2}$, then the situation is a little more interesting. For n odd we have $X \cong U^{\ell n+2}$ since no element of G, of odd order, can reverse the orientation of $T^{\ell n+1}$. In the case that n is even there are two possibilities for X, namely $X \cong U^{\ell n+2}$ or $T^{\ell n/2+1}$. It is this last case that we shall explore in a little more detail. Specifically we will address the following problem: Suppose n is even and $\partial : \pi_1 U^{\ell+2} \to G$ is an epimorphism. How might we determine $\ker \partial$? (It must be $\pi_1 U^{\ell n+2}$ or $\pi_1 T^{\ell n/2+1}$).

We begin by recalling the fundamental groups of T^k and U^k[1]:

$$(1.1) \qquad \pi_1 T^k \cong \langle \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \mid \prod_{j=1}^{k} [\alpha_j, \beta_j] = 1 \rangle$$

$$\pi_1 U^k \cong \langle \alpha_1, \ldots, \alpha_k \mid \prod_{j=1}^{k} \alpha_j^2 = 1 \rangle.$$

If $G \cong \mathbb{Z}_2$, the cyclic group of order 2 with non-trivial element τ, and $\partial : \pi_1 U^{\ell+2} \to \mathbb{Z}_2$ is an epimorphism, then we may write $\partial(\alpha_j) = \tau^{a_j}$ where each a_j is 0 or 1 and at least one a_j is 1.

(1.2) PROPOSITION. *In the above example* $\ker \partial \cong \pi_1 T^{\ell+1}$ *if, and only if* $a_1 = \ldots = a_{\ell+2} = 1$.

PROOF. We suppose $\ker \partial \cong \pi_1 X$ where $X \cong T^{\ell+1}$ or $U^{2\ell+2}$. The two-fold cover $p : X \to U^{\ell+2}$ is classified by an element $\theta \in H^1(U^{\ell+2}; \mathbb{Z}_2)$. If we let $\alpha_1^*, \ldots, \alpha_{\ell+2}^*$ represent dual classes to the Hurewicz images of $\alpha_1, \ldots, \alpha_{\ell+2}$, in $H^1(U^{\ell+2}; \mathbb{Z}_2)$ then it is not hard to show that $\theta = \sum_{j=1}^{\ell+2} a_j \alpha_j^*$. There is a long exact sequence associated to this cover ([4] or [3]):

$$(1.3) \qquad \begin{array}{ccc} H^*(U^{\ell+2}; \mathbb{Z}_2) & \xrightarrow{\ \theta\ } & H^*(U^{\ell+2}; \mathbb{Z}_2) \\ & {}_{\mathrm{tr}}\nwarrow \quad \swarrow_{p^*} & \\ & H^*(X; \mathbb{Z}_2) & \end{array}$$

where tr denotes the transfer map. Due to the naturality of tr with respect to the Steenrod squaring operations we obtain a commutative diagram:

$$\begin{array}{ccc} H^1(X; \mathbb{Z}_2) & \xrightarrow{\ \mathrm{tr}\ } & H^1(U^{\ell+2}; \mathbb{Z}_2) \\ \mathrm{Sq}^1 \downarrow & & \downarrow \mathrm{Sq}^1 \\ H^2(X; \mathbb{Z}_2) & \xrightarrow{\ \mathrm{tr}\ } & H^2(U^{\ell+2}; \mathbb{Z}_2). \end{array}$$

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Now, it is easy to show that Sq^1 is zero if $X \cong T^{\ell+1}$, whereas Sq^1 is non-zero if $X \cong U^{2\ell+2}$. In fact the product structure of $H^*(U^k; \mathbb{Z}_2)$ is given by $\alpha_i^*\alpha_j^* = 0$ when $i \neq j$ and $(\alpha_1^*)^2 = \ldots = (\alpha_k^*)^2 \neq 0$.

We proceed to prove the proposition. Suppose some $a_j = 0$. By reindexing we may assume $a_{\ell+2} = 0$. It follows that

$$\theta = \sum_{j=1}^{\ell+1} a_j\alpha_j^* \text{ and } \alpha_{\ell+2}^*\cdot\theta = 0.$$

Consequently, by exactness of (1.3), $\alpha_{\ell+2}^* = \mathrm{tr}(x)$ for some $x \in H^1(X; \mathbb{Z}_2)$. We compute

$$\begin{aligned} \mathrm{tr}\, Sq^1(x) &= Sq^1\, \mathrm{tr}(x) \\ &= Sq^1(\alpha_{\ell+2}^*) \\ &\neq 0. \end{aligned}$$

So $Sq^1(x) \neq 0$ and $X \cong U^{2\ell+2}$.

On the otherhand every non-orientable surface admits an orientable two-fold cover [1], consequently $X \cong T^{\ell+1}$ exactly when $a_1 = \ldots = a_{\ell+2} = 1$. □

(1.4) COROLLARY. *The two-fold cover* $q: T^{\ell+1} \to U^{\ell+1}$ *is unique, up to equivalence.* □

Let $\epsilon: \pi_1 U^{\ell+2} \to \mathbb{Z}_2$ be the map $\epsilon(\alpha_j) = \tau$ for $j = 1, \ldots, \ell + 2$. Note that ker ϵ consists of all words in $\pi_1 U^{\ell+2}$ of even length.

(1.5) DEFINITION. *Suppose* $\partial: \pi_1 U^{\ell+2} \to G$ *is a finite quotient, we define* $N_\partial = \partial \ker \epsilon$, *a subgroup of* G.

(1.6) LEMMA. $[G:N_\partial] = 2/[\partial^{-1}(N_\partial):\ker \epsilon]$.

PROOF.
$$\begin{aligned} [G:N_\partial] &= |G|/|N_\partial| \\ &= [\pi_1 U^{\ell+2}:\ker \partial]/[\partial^{-1}(N_\partial):\ker \partial] \\ &= [\pi_1 U^{\ell+2}:\ker \epsilon]/[\partial^{-1}(N_\partial):\ker \epsilon] \\ &= 2/[\partial^{-1}(N_\partial):\ker \epsilon]. \quad \square \end{aligned}$$

REMARK. There are only two possible values for $[G:N_\partial]$, namely 1 or 2.

The next proposition is the main result of this section.

(1.7) PROPOSITION. *If* $\partial: \pi_1 U^{\ell+2} \to G$ *is a finite quotient,* $|G| = 2n$ *and* $[G:N_\partial] = 2$ *then* $\ker \partial \cong \pi_1 T^{n\ell+1}$.

PROOF. Since $[G:N_\partial] = 2$ we must have $\partial^{-1}(N_\partial) \cong \ker \epsilon \cong \pi_1 T^{\ell+1}$ by the above lemma. Now, $\partial^{-1}(N_\partial)$ contains ker ∂ as a normal subgroup of finite index. If $\ker \partial \cong \pi_1 U^{2n\ell+2}$ this would imply that $U^{2n\ell+2}$ covers $T^{\ell+1}$, an impossibility. The only other possibility for ker ∂ is $\pi_1 T^{n\ell+1}$. □

(1.8) REMARK. We conjecture that $[G:N_\partial] = 2$ is necessary and sufficient for $\ker \partial \cong \pi_1 T^{n\ell+1}$. This has been proven for $\ell = 0$ [2].

§2 **Applications**. Our first application is to lifting finite free actions on $U^{\ell+2}$ to the orientable two-fold cover $T^{\ell+1}$

(2.1) PROPOSITION. *If G is finite group acting freely on $U^{\ell+2}$ then there exists a lifting to a free action of G on $T^{\ell+1}$ rendering the natural projection map $q:T^{\ell+1} \to U^{\ell+2}$ G-equivariant.*

PROOF. If $|G| = n$, then n divides ℓ and the orbit space $U^{\ell+2}/G$ is homeomorphic to $U^{\ell/n+2}$. Consider the pull-back diagram

$$\begin{array}{ccc} X & \xrightarrow{\bar{q}} & U^{\ell+2} \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle p} \\ T^{\ell/n+1} & \xrightarrow{q} & U^{\ell/n+2} \end{array}$$

where the bottom map is the unique two fold cover. Once we show X is connected we will be done. This is because $X = \{(u, t) \in U^{\ell+2} \times T^{\ell/n+1}: p(u) = q(t)\}$ which inherits the free G action from $U^{\ell+2}$ and $\bar{q}(u, t) = u$ is clearly equivariant. If X were connected it must be homeomorphic to $T^{\ell+1}$ since it covers $T^{\ell/n+1}$.

To show X is connected it is sufficient to show that the composition

$$\pi_1 T^{\ell/n+1} \xrightarrow{q\#} \pi_1 U^{\ell/n+2} \xrightarrow{\partial} G$$

is an epimorphism, where ∂ is the epimorphism associated to the free action of G on $U^{\ell+2}$.

We compute

$$\begin{aligned} \text{image } (\partial \circ q_{\#}) &= \partial \text{ image } q_{\#} \\ &= \partial \ker \\ &= N_\partial . \end{aligned}$$

But $[G:N_\partial] = 1$, else $\ker \partial \cong \pi_1 T^{\ell/2+1}$ by proposition (1.7). We may conclude $N_\partial = G$ and $\partial \circ q\#$ is an epimorphism. $\square$

We begin our second application by first recalling a theorem due to R. D. Anderson.

(2.2) PROPOSITION. [5] *Every finite group acts freely on some T^k.* $\square$

The above theorem is the inspiration for the following definition.

(2.3) DEFINITION. *If G is a finite group then genus (G) is the minimum k such that G acts freely on T^k.*

There are a few immediate properties.

(2.4) PROPOSITION.

(a) *If K is a subgroup of G then genus $(K) \leqslant$ genus (G).*

(b) $$genus\ (G) \equiv \begin{cases} 1 \bmod |G| \text{ if } |G| \text{ is odd} \\ 1 \bmod |G|/2 \text{ if } |G| \text{ is even} \end{cases}$$

(c) *If ℓ is the minimum number of generators for G, $|G|(\ell/2 - 1) + 1 \leqslant genus\ (G) \leqslant |G|(\ell - 1) + 1$.*

PROOF.

(a) Obvious.

(b) If G acts freely on T^k with orbit space B then $2 - 2k = \chi(B)\cdot|G|$.

(c) We will first show that G can act freely on $T^{|G|(\ell-1)+1}$, providing the upperbound on genus (G). Pick ℓ generators $\sigma_1, \ldots, \sigma_\ell$ of G. Define $\partial: \pi_1 T^\ell \to G$ by $\partial(\alpha_j) = \sigma_j$, $\partial(\beta_j) = 1$. Obviously $\partial(\Pi[\alpha_j, \beta_j]) = 1$. Thus ker $\partial \cong \pi_1 T^{|G|(\ell-1)+1}$ giving our free action. To prove the lower bound assume G acts freely on T^k with orbit space B. There are two possibilities for B, namely $B \cong T^{(k-1)/|G|+1}$ or $U^{2(k-1)/|G|+2}$. In either case $\pi_1 B$ is generated by $2(k-1)/|G| + 2$ elements. Since $\partial: \pi_1 B \to G$ is an epimorphism we must have $\ell \leqslant 2(k-1)/|G| + 2$. A little bit of algebra then gives our lower bound for genus (G). □

Let $\mathbb{Z}_m$ denote the cyclic group of order m.

(2.5) PROPOSITION. *If $G = \mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_\ell}$ where ℓ is minimal and $m = m_1 m_2 \ldots m_\ell$ then*

$$Genus\ (G) = \begin{cases} m\left(\dfrac{\ell-2}{2}\right) + 1 \text{ if some } m_i = 2 \text{ or } \ell \text{ is even.} \\ m\left(\dfrac{\ell-1}{1}\right) + 1 \text{ otherwise.} \end{cases}$$

PROOF. Let $g =$ genus (G) and write the generators of G as $\sigma_1, \ldots, \sigma_\ell$. First assume ℓ is even. Define $\partial: \pi_1 T^{\ell/2} \to G$ by $\partial(\alpha_j)$ and σ_j and $\partial(\beta_j) = \sigma_{j+\ell/2}$ for $j = 1, \ldots, \ell/2$. This is clearly an epimorphism with ker $\partial \cong \pi_1 T^{m((\ell-2)/2)+1}$. This proves $g \leqslant m((\ell-2)/2) + 1$. On the other hand proposition (2.4) (c) implies $m((\ell-2)/2) + 1 \leqslant g$. Thus $g = m((\ell-2)/2) + 1$.

Now suppose ℓ is odd. In this case we may construct an epimorphism $\partial: \pi_1 T^{(\ell+1)/2} \to G$ by $\partial(\alpha_j) = \sigma_j$ for $j = 1, \ldots, (\ell+1)/2$, $\partial(\beta_j) = \sigma_{j+(\ell+1)/2}$ for $j = 1, \ldots, (\ell-1)/2$, $\partial(\beta_{(\ell+1)/2}) = 1$. Then ker $\partial \cong T^{m((\ell-1)/2)+1}$ and consequently $g \leqslant m((\ell-1)/2) + 1$. On the other hand we have the usual lower bound $m((\ell-2)/2 + 1 \leqslant g$. Assume m is odd. Then $g \equiv 1 \bmod m$. The only integer g satisfying the above congruence and lying in the above range is $g = m((\ell-1)/2) + 1$. Now assume m is even. The congruence becomes $g \equiv 1 \bmod m/2$. There are two possibilities for g that lies in the state range, namely

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$$g = \begin{cases} m\left(\dfrac{\ell - 2}{2}\right) + 1 \text{ or} \\ \\ m\left(\dfrac{\ell - 1}{2}\right) + 1. \end{cases}$$

If G acted freely on $T^{m((\ell-2)/2)+1}$ then the orbit space B would have Euler characteristic $\chi(B) = 2 - \ell$. Since we are assuming ℓ is odd, $2 - \ell$ is odd, and therefore $B \cong U^\ell$ with an epimorphism $\partial : \pi_1 U^\ell \to G$. If all $m_i \neq 2$ then this is not possible (because when we factor this map through the abelianization of $\pi_1 U^\ell$ we obtain an epimorphism $\mathbb{Z}^{\ell-1} \times \mathbb{Z}_2 \to G$ which is a contradiction, no $m_i = 2$). We may conclude $g = m((\ell - 1)/2) + 1$ if no $m_i = 2$.

Now, for $m_1 = 2$ we shall produce a free action of G on $T^{m(\ell-2)/2+1}$. Define an epimorphism $\partial : \pi_1 U^\ell \to G$ by $\partial(\alpha_j) = \sigma_j$, $j = 1, \ldots, \ell$. We will show $\ker \partial \cong \pi_1 T^{m(\ell-2)/2+1}$ by employing proposition (1.7). $N_\partial = \partial \ker \epsilon$ is the subgroup of G generated by $\{\sigma_i\sigma_j\}_{1 \leq i < j \leq \ell}$ (recall $\ker \epsilon$ is the subgroup of $\pi_1 U^\ell$ consisting of words of even length). But for $i > 1$ we have $\sigma_i\sigma_j = (\sigma_1\sigma_i)(\sigma_1\sigma_j)$ and thus N_∂ is generated by $\{\sigma_1\sigma_j\}_{2 \leq j \leq \ell}$. We conclude that $[G:N_\partial] = 2$ and consequently $\ker \partial \simeq \pi_1 T^{m(\ell-2)/2+1}$. This completes the proof of the proposition. □

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Department of Mathematics
California State University
Fresno CA 93710