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On quasiconformal non-equivalence of gasket Julia sets and limit sets

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Abstract. This paper studies quasiconformal non-equivalence of Julia sets and limit sets. We proved that any Julia set is quasiconformally different from the Apollonian gasket. We also proved that any Julia set of a quadratic rational map is quasiconformally different from the gasket limit set of a geometrically finite Kleinian group.

Key words: quasiconformal non-equivalence, Julia sets, limit sets, Apollonian gasket, circle packings
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1. *Introduction*

It is a central question in quasiconformal geometry to classify fractal sets up to quasiconformal homeomorphisms. For fractal sets that emerge in conformal dynamics, ample evidence suggests that it is possible to quasiconformally distinguish Julia sets and limit sets. It is summarized as the following conjecture in [**LLMM23**].

Conjecture 1.1. [**LLMM23**] Let J be the Julia set of a rational map and Λ be the limit set of a Kleinian group. Suppose that J and Λ are connected, and not homeomorphic to a circle or a sphere. Then, J is not quasiconformally homeomorphic to Λ.

In this paper, we will study this quasiconformal non-equivalence phenomenon for some classes of Julia set and limit set. Our first result is the following theorem.

THEOREM 1.2. *No Julia set of a rational map is quasiconformally homeomorphic to the Apollonian gasket.*

Remark 1.3. We remark that there exist Julia sets that are homeomorphic to the Apollonian gasket (see Figure 1). The Apollonian gasket is the limit set of the Apollonian group [**GLMWY05**]. In fact, for the limit set of any geometrically finite Kleinian group that

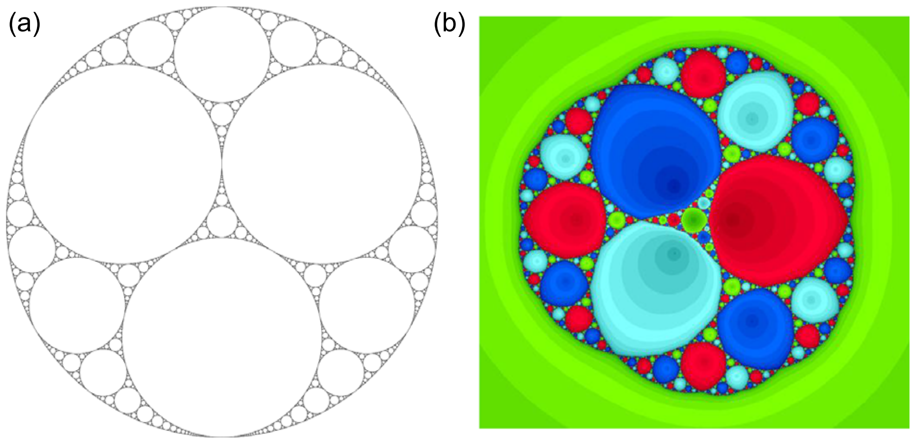

FIGURE 1. (a) The Apollonian gasket and (b) the Julia set that is homeomorphic to an Apollonian gasket. The two sets are not quasiconformally homeomorphic as Fatou components touch at an angle in panel (b) (colour online).

is homeomorphic to the Apollonian gasket, it is quasiconformally homeomorphic to the Apollonian gasket (see [**McM90**] or §2). Therefore, Theorem 1.2 can be restated as no Julia set is quasiconformally homeomorphic to an Apollonian gasket limit set of a geometrically finite Kleinian group.

More generally, we define a *gasket* K as a closed subset of $\hat{\mathbb{C}}$ so that:

(1) each complementary component is a Jordan domain;
(2) any two complementary components touch at most at 1 point;
(3) no three complementary components have a common boundary point;
(4) the *contact graph* (or the *nerve*), obtained by assigning a vertex to complementary component and an edge if two touch, is connected.

The Apollonian gasket provides one example of gaskets. Gaskets also arise naturally as Julia sets of rational maps and limit sets of Kleinian groups (see §2), and appear frequently in the literature. For example, any *polyhedral circle packing* defined in [**KN19**] gives a gasket limit set. Many rational maps have gasket Julia sets that are homeomorphic to gasket limit sets (see [**LLM22a**, **LLM22b**]). Our second result shows the following theorem.

THEOREM 1.4. *No Julia set of a quadratic rational map is quasiconformally homeomorphic to a gasket limit set of a geometrically finite Kleinian group.*

For higher degree rational maps, the combinatorics become more complicated. However, we believe it is possible to generalize our method to higher degrees and prove Conjecture 1.1 in the gasket case.

1.1. *Historical background.* Our study of quasiconformal non-equivalence between Julia sets and limit sets is motivated by the study of rigidity of quasisymmetries of a Sierpiński carpet in [**BKM09**, **BM13**, **Mer14**, **BLM16**]. Using the rigidity, it is proved in [**BLM16**, Corollary 1.2] that a Sierpiński carpet Julia set of a post-critically finite rational map has a finite quasisymmetry group. Since the quasisymmetry group is a quasiconformal

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invariant and the limit set has a large conformal symmetry group, we have immediately the following theorem (cf. [**Mer14**, Corollary 1.3]).

THEOREM 1.5. [**BLM16**, Corollary 1.3] *No Julia set of a post-critically finite rational map is quasiconformally homeomorphic to a Sierpiński carpet limit set of a Kleinian group.*

This result is generalized in [**QYZ19**] to *semi-hyperbolic* rational maps.

However, using David surgery, it is proved in [**LLMM23**] that there exists some rational map f so that:

- the Julia set $\mathcal{J}$ is homeomorphic to the Apollonian gasket; and
- the quasisymmetry group $\mathcal{QS}(\mathcal{J})$ equals the homeomorphism group $\mathrm{Homeo}(\mathcal{J})$.

The homeomorphism $\Phi : \mathcal{J} \longrightarrow \Lambda$ between the Julia set $\mathcal{J}$ and the Apollonian gasket Λ is *not* quasiconformal. Yet, it induces an isomorphism

$$\Phi_* : \mathcal{QS}(\mathcal{J}) \longrightarrow \mathcal{QS}(\Lambda) = \mathrm{Conf}(\Lambda).$$

Thus, in this case, the quasisymmetry groups do not distinguish the Julia set $\mathcal{J}$ and the Apollonian gasket Λ. The same phenomenon can also occur for other general gaskets (see [**LLMM23**] for more details).

Although the Sierpiński carpet and gasket both arise as a limit set of an acylindrical hyperbolic 3-manifold [**McM90**], the two cases are quite different. We use the combinatorics of the gaskets to study the quasiconformal non-equivalence problem.

1.2. *Strategy and techniques.* In §2, we recall a characterization of finitely generated Kleinian groups with gasket limit sets (see Theorem 2.3). If the Kleinian group is geometrically finite, then the corresponding hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ is *acylindrical* in the sense of Thurston. It follows that Γ is quasiconformally conjugate to a Kleinian group with a totally geodesic convex hull boundary. Thus, Λ is quasiconformally homeomorphic to an infinite circle packing (see Corollary 2.4).

Note that the degree of tangency at the intersection point of two complementary components of Λ is a quasiconformal invariant. Thus, a necessary condition for a gasket Julia set $\mathcal{J}$ to be quasiconformally homeomorphic to Λ is that:

- the boundary of each Fatou component contains no cusps;
- two Fatou components are tangent to each other if they touch.

We shall call such a Julia set $\mathcal{J}$ a *fat gasket* (see Figure 2).

Let $\mathcal{G}$ be the contact graph of a fat gasket Julia set $\mathcal{J}$ of f, which we shall also call the *Fatou graph* of f. We show that the local quasiconformal structure puts a global constraint on $\mathcal{G}$. In particular, we prove the following theorem.

THEOREM 1.6. *The Fatou graph $\mathcal{G}$ of a fat gasket Julia set is bipartite.*

We remark that Theorem 1.6 allows one to show that many gasket Julia sets are quasiconformally different from gasket limit sets. For example, Theorem 1.2 follows immediately from it as the contact graph of the Apollonian gasket contains a cycle of length 3, so it is not bipartite.

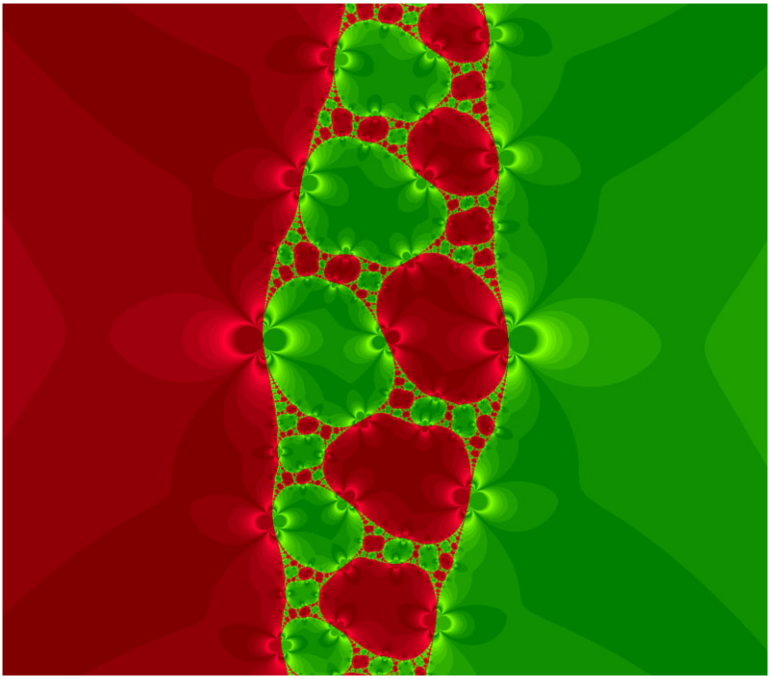

FIGURE 2. An example of a fat gasket Julia set. The Fatou graph is bipartite as one can see by the coloring of the Fatou set (colour online).

However, there are plenty of rational maps with fat gasket Julia set and we use Thurston theory of rational maps to obtain a characterization (see Theorem 4.1). In particular, for degree 2, we have the following theorem.

THEOREM 1.7. *Let f be a quadratic rational map with a fat gasket Julia set. Then either:*

(1) *the Fatou graph is a tree; or*
(2) *the Fatou graph is not a tree and f is a root of a captured type hyperbolic component with an attracting cycle of period 2.*

Moreover:

- *in case (1), if f is geometrically finite, then f is a mating of the fat basilica with a Misiurewicz polynomial;*
- *in case (2), any root of a captured type hyperbolic component with an attracting cycle of period 2 has a fat gasket Julia set.*

Remark 1.8. We recall that a hyperbolic component $\mathcal{H}$ of a quadratic rational map is called captured type if only one critical point is in the immediate basin of a periodic point. A rational map f is called a *root* of $\mathcal{H}$ if $f \in \partial\mathcal{H}$ and the dynamics on the Julia set $\mathcal{J}(f)$ is topologically conjugate to that of $g \in \mathcal{H}$.

We also recall that a quadratic polynomial $f(z) = z^2 + c$ is called a Misiurewicz polynomial if the critical point 0 is strictly pre-periodic. The Julia set of a Misiurewicz polynomial is a dendrite. We remark that not every Misiurewicz polynomial f is mateable with the fat basilica. In fact, f is mateable with the fat basilica if and only if f is not in the

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1/2-limb of the Mandelbrot set [Tan92]. We also remark even if f is mateable with the fat basilica, the Fatou graph of their mating may not be a tree. In this case, by Theorem 1.7, the mating is not a fat gasket Julia set.

To prove Theorem 1.4, we first note that the contact graph of a gasket limit set of a geometrically finite Kleinian group is not a tree (see Theorem 2.3). Thus, we can assume the Fatou graph $\mathcal{G}$ is not a tree. We then show the homeomorphism group Homeo$(\mathcal{G})$ is very restrictive (see §6). This allows us to distinguish $\mathcal{G}$ from the contact graph of a gasket limit set.

1.3. *Notes and discussions.* The Apollonian gasket or the Apollonian circle packing and its arithmetic, geometric, and dynamical properties have been extensively studied in the literature (for a non-exhaustive list, see e.g. [GLMWY03, GLMWY05, BF11, KO11, OS12, BK14, Zha22]).

More recently, there have been many new and exciting developments in the study of other gasket or circle packings coming from the limit set of Kleinian groups [KN19, KK23, BKK24, LLM22a].

It is shown in [LLM22a] that many of these gaskets are homeomorphic to Julia sets of rational maps. In the same paper, many other pairs of a homeomorphic Julia set and limit set are constructed. It would be interesting to study the quasiconformal non-equivalence problem for such Julia sets and limit sets.

1.4. *Structure of the paper.* We study Kleinian groups with gasket limit sets in §2. The induced dynamics on the Fatou graph of a rational map with a fat gasket Julia set is studied in §3, where Theorem 1.6 is proved. The realization of rational maps with fat gasket Julia sets are studied in §4. In particular, Theorem 1.7 is proved there. Finally, the combinatorics of Fatou graphs for quadratic fat gasket Julia sets are studied in §5 and Theorem 1.4 is proved in §6.

2. *Kleinian groups with gasket limit set*

In this section, we recall some results on geometrically finite Kleinian groups, especially those with gasket limit sets. We refer to [LZ23, Appendix B] for details. Throughout the section, let $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$ be a Kleinian group, $\Lambda \subset \hat{\mathbb{C}}$ its limit set, and $\Omega = \hat{\mathbb{C}} - \Lambda$ its domain of discontinuity. Up to a finite-index subgroup, we may assume Γ is torsion-free. We also denote by $M = \Gamma \backslash \mathbb{H}^3$ the corresponding hyperbolic three-manifold.

2.1. *Geometric finiteness and acylindricity.* Recall that the *convex core* of M is given by

$$\mathrm{core}(M) := \Gamma \backslash \mathrm{Cvx\ Hull}(\Lambda),$$

where Cvx Hull(Λ) denotes the convex hull of Λ in $\mathbb{H}^3$. We say M (and the corresponding Kleinian group Γ) is *geometrically finite* if the unit neighborhood of core(M) has finite volume.

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Let (N, P) be a *pared manifold*, where N is a compact oriented 3-manifold with boundary and $P \subset \partial N$ is a submanifold consisting of incompressible tori and annuli. See [Thu86] for a precise definition in arbitrary dimension.

Set $\partial_0 N = \partial N - P$. We say (N, P) is *acylindrical* if $\partial_0 N$ is incompressible, and every cylinder

$$f : (S^1 \times [0, 1], S^1 \times \{0, 1\}) \to (N, \partial_0 N)$$

whose boundary components $f(S^1 \times \{0\})$ and $f(S^1 \times \{1\})$ are essential curves of $\partial_0 N$ can be homotoped rel boundary into ∂N.

For a geometrically finite M, let $\mathrm{core}_\epsilon(M)$ be the convex core of M minus ϵ-thin cuspidal neighborhoods for all cusps. Here, ϵ is chosen small enough, say smaller than the Margulis constant in dimension 3. Let $P \subset \partial\, \mathrm{core}_\epsilon(M)$ be the union of boundaries of all cuspidal neighborhoods. Then, $(\mathrm{core}_\epsilon(M), P)$ is a pared manifold, and we say M (and the corresponding Kleinian group Γ) is acylindrical if $(\mathrm{core}_\epsilon(M), P)$ is.

One can recognize acylindricity from the limit set in the geometrically finite case. The following characterization is well known; see for example [LZ23, Proposition B.2].

PROPOSITION 2.1. (Characterization of geometrically finite acylindrical Kleinian groups) *Suppose Γ is non-elementary and geometrically finite of infinite volume. Then, Γ is acylindrical if and only if any connected component of the domain of discontinuity Ω is a Jordan domain, and the closures of any pair of connected components share at most one point. Moreover, any common point of the closures of two connected components is a parabolic fixed point, and any rank-1 parabolic fixed point arises this way.*

One ingredient in the proof is the following lemma, essentially from [Mas74, Theorem 3].

LEMMA 2.2. *Let Γ be any non-elementary Kleinian group, and Ω_1, Ω_2 two connected components of its domain of discontinuity Ω. Let Γ_i be the stabilizer of Ω_i in Γ and assume $X_i := \Gamma_i \backslash \Omega_i$ is a Riemann surface of finite type. Suppose $\overline{\Omega_1} \cap \overline{\Omega_2}$ consists of one point p. Then:*

(1) *p is a parabolic fixed point;*
(2) *let σ_i be a curve on X_i that is not null-homotopic in $\overline{M} := \Gamma \backslash \{\mathbb{H}^3 \cup \Omega\}$. Then, σ_1 and σ_2 are not homotopic in $\overline{M}$.*

A *quasiconformal deformation* of Γ is a discrete and faithful representation $\xi : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ that preserves parabolics, induced by a quasiconformal map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ (that is, $\xi(\gamma) = f \circ \gamma \circ f^{-1}$ for all $\gamma \in \Gamma$). In particular, the limit sets of Γ and $\xi(\Gamma)$ are quasiconformally homeomorphic via f.

The *quasiconformal deformation space* of Γ is defined by

$$\mathcal{QC}(\Gamma) := \{\xi : \Gamma \to \mathrm{PSL}(2, \mathbb{C}) \text{ is a quasiconformal deformation}\}/\sim,$$

where $\xi \sim \xi'$ if they are conjugate by a Möbius transformation. This space is naturally identified with the quasi-isometric deformation space of the corresponding hyperbolic manifold M (see e.g. [Sul81]).

When Γ is acylindrical, there exists a unique $\Gamma' \in \mathcal{QC}(\Gamma)$ so that the convex core of $M' = \Gamma' \backslash \mathbb{H}^3$ has totally geodesic boundary [**McM90**, Corollary 4.3]. Thus, the limit set of any geometrically finite and acylindrical Kleinian group is quasiconformally homeomorphic to a circle packing.

2.2. *Gasket limit sets.* The following characterization of Kleinian groups with gasket limit sets was proved in [**LZ23**].

THEOREM 2.3. *Suppose the limit set Λ of a finitely generated Kleinian group Γ is a gasket and let $\mathcal{G}$ be its contact graph. Then, the corresponding hyperbolic 3-manifold M is homeomorphic to the interior of a compression body N. If furthermore M is geometrically finite, then $\mathcal{G}$ is not a tree, M is acylindrical, and the compression body N has empty or only toroidal interior boundary components.*

Here, a compact orientable irreducible manifold with boundary $(N, \partial N)$ is a *compression body* if the inclusion of one boundary component $\partial_e N$ induces a surjection on π_1; we refer to this component as the *exterior* boundary. All other boundary components of N are incompressible; we refer to them as *interior* boundary components. In fact, when Γ contains no rank-2 parabolic fixed points, the compression body N has no interior boundary component, so it is a handlebody. However, when $\mathcal{G}$ is a tree, M is homeomorphic to $S \times \mathbb{R}$ for some surface S of finite type, and has one geometrically infinite end.

The following result then follows from this characterization and discussion in the previous subsection.

COROLLARY 2.4. *Let Λ be a gasket limit set of a geometrically finite Kleinian group Γ. Then, Λ is quasiconformally homeomorphic to an infinite circle packing.*

In particular, to prove our main result, Theorem 1.4, we may restrict our attention to Kleinian groups with circle packing limit sets.

3. *Induced dynamics on the Fatou graph*

Recall that a gasket Julia set $\mathcal{J}$ is a fat gasket if:

- the boundary of each Fatou component contains no cusps;
- two Fatou components are tangent to each other if they touch.

In this section, we shall prove that dynamics of a rational map with a fat gasket Julia set is restricted.

THEOREM 3.1. *Let f be a rational map with a fat gasket Julia set. Let $\mathcal{G}$ be the Fatou graph for f. Then, f induces a simplicial map*

$$f_* : \mathcal{G} \longrightarrow \mathcal{G},$$

and there exists a unique fixed edge $E_0 \subseteq \mathcal{G}$ of f_* so that every edge is eventually mapped to E_0.

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Here, we recall that a map is called *simplicial* if it maps an edge to an edge. It follows that the two boundary points ∂E_0 are either both fixed, or form a periodic cycle of period 2. Thus, Theorem 1.6 follows immediately from Theorem 3.1.

Proof of Theorem 1.6. After passing f_* to the second iterate if necessary, we may assume that the boundary points $\partial E_0 = \{x, y\}$ are both fixed. Then, we can divide the vertex set into two groups U_x, U_y depending on whether the vertex is eventually mapped to x or y. This division gives the bipartite structure of the graph $\mathcal{G}$. □

The proof of Theorem 3.1 consists of several lemmas.

LEMMA 3.2. *Let f be a rational map with a fat gasket Julia set. Then, no critical point is on the boundary of a Fatou component.*

Proof. Suppose for contradiction that f has a critical point c on the boundary of a Fatou component U. Let $V = f(U)$. Then, $f(c) \in \partial V$. Let $e > 1$ be the multiplicity of the critical point c, and μ be the number of Fatou components attached at $f(c)$. Then, there are $e\mu$ Fatou components attached at c. Since at most 2 Fatou components are attached to c, $e = 2$ and $\mu = 1$. Since the two Fatou components touch tangentially at c and f behaves like z^2 near c, the boundary of V has a cusp at $f(c)$. This is a contradiction and the lemma follows. □

We have the following as an immediate corollary.

COROLLARY 3.3. *Let f be a rational map with a fat gasket Julia set. Then, f induces a simplicial map*

$$f_* : \mathcal{G} \longrightarrow \mathcal{G}.$$

Moreover, every edge of $\mathcal{G}$ is pre-periodic.

Proof. Since vertices in $\mathcal{G}$ are in bijective correspondence with Fatou components of f, f naturally induces a map f_* on the vertex set of $\mathcal{G}$. By Lemma 3.2, there are no critical points on the boundary of a Fatou component. Thus, if v, w are adjacent in $\mathcal{G}$, then $f_*(v) \neq f_*(w)$. Hence, $f_*(v)$ and $f_*(w)$ are adjacent. Therefore, we can extend the map f_* to $\mathcal{G}$ by sending an edge $E = [v, w]$ to the edge $[f_*(v), f_*(w)]$. By construction, the induced map is simplicial.

Since each vertex is pre-periodic, by construction, each edge is also pre-periodic. □

Let x be a parabolic fixed point of f. Let q be the smallest positive integer so that $(f^q)'(x) = 1$. Then, near x, we have

$$f^q(z) = (z - x) + a(z - x)^{k+1} + \cdots .$$

We call k the *parabolic index* of x and $k + 1$ the *multiplicity* of the parabolic fixed point. Note that q must divide k (see [Mil06, Ch. 7]).

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LEMMA 3.4. *Let f be a rational map with a fat gasket Julia set. Let x be a common boundary point of two Fatou components U, V. Then, x is eventually mapped to a unique parabolic fixed point with parabolic index* 2.

Proof. Assume first that x is periodic. Suppose for contradiction that x is a repelling periodic point. Then, the two Fatou components must touch at an angle at x. This is a contradiction. Thus, x is a parabolic fixed point.

Note that the parabolic index at x is at most 2, as there are at most two attracting petals at x. Suppose for contradiction that the parabolic index is 1. Then, there is only one attracting petal at x. Interchanging the role of U and V if necessary, we assume that U contains the attracting petal of x. Then, ∂U has a cusp at x, which is a contradiction. Thus, the parabolic index at x is 2. We claim the following.

CLAIM 3.5. *There is a unique periodic point, which is necessarily a fixed point, on the intersection of the boundaries of two Fatou components.*

Proof of the claim. Suppose for contradiction that there are two distinct periodic points $x_i \in \partial U_i \cap \partial V_i$, $i = 1, 2$, where $U_i \neq V_i$ are different Fatou components. After passing to an iterate, we may assume that U_i and V_i are fixed. Let u_i, v_i be the corresponding vertices in $\mathcal{G}$. Since the Fatou graph $\mathcal{G}$ is connected, there is a path containing u_1, u_2, v_1, v_2. Let $\mathcal{C}$ be the shortest path containing u_1, u_2, v_1, v_2 with respect to the edge metric on $\mathcal{G}$. Note that in particular, $\mathcal{C}$ must be a simple path. Since u_i, v_i are fixed by f_*, $f_*(\mathcal{C})$ still contains u_1, u_2, v_1, v_2. Since we assume $\mathcal{C}$ has the shortest length, $f_*(\mathcal{C})$ is a simple path. By induction, $f_*^n(\mathcal{C})$ is a simple path for any n.

Since every vertex is pre-periodic, replace $\mathcal{C}$ by $f_*^k(\mathcal{C})$ for some k if necessary, we assume that all the vertices on $\mathcal{C}$ are fixed. Since $x_1 \neq x_2$, the length of $\mathcal{C}$ is at least 2. Let a, b, c be three consecutive vertices on $\mathcal{C}$ and let U_a, U_b, U_c be the corresponding Fatou components. Let $x_{ab} \in \partial U_a \cap \partial U_b$ and $x_{bc} \in \partial U_b \cap \partial U_c$. Then, x_{ab} and x_{bc} are both fixed points. By the previous argument, x_{ab} and x_{bc} are parabolic fixed points with parabolic index 2. Thus, U_b contains both the attracting petals for x_{ab} and x_{bc}, which is a contradiction. □

By Corollary 3.3, every common boundary point of two Fatou components is pre-periodic. By the previous claim, it is eventually mapped to the unique parabolic fixed point with parabolic index 2. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. By Corollary 3.3, f induces a simplicial map $f_* : \mathcal{G} \longrightarrow \mathcal{G}$. By Lemma 3.4, every edge of $\mathcal{G}$ is eventually mapped to a fixed edge E_0. □

4. *Construction of fat gasket Julia set*

In this section, we prove the realization theorem of a fat gasket when there are no critical points on the Julia set. We will then consider the quadratic case and prove Theorem 1.7.

Let $\mathcal{G}$ be a *simple plane graph*, that is, an embedded simple graph in $\hat{\mathbb{C}}$. Two simple plane graphs are said to be isomorphic if there is a homeomorphism of $\hat{\mathbb{C}}$ inducing the

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graph isomorphism. Let $F : \mathcal{G} \longrightarrow \mathcal{G}$ be a map. We say F is a *simplicial branched covering* on $\mathcal{G}$ if:

- F is a simplicial map;
- F is the restriction of a branched covering $\widetilde{F}$ on $\hat{\mathbb{C}}$ with critical points contained in the vertex set of $\mathcal{G}$.

We say F has degree d if the branched covering $\widetilde{F}$ has degree d. We apply Thurston theory of rational maps to prove the following key result of this section.

THEOREM 4.1. *Let $\mathcal{G}$ be a connected simple plane graph. Let $F : \mathcal{G} \longrightarrow \mathcal{G}$ be a degree d simplicial branched covering. Let $E_0 = [a, b]$ be an edge of $\mathcal{G}$. Suppose that:*

- *$\mathcal{G}$ is backward invariant;*
- *E_0 is fixed and any edge E of $\mathcal{G}$ is eventually mapped to E_0;*
- *both a, b are contained in critical cycles;*
- *$\mathcal{G} - \mathrm{Int}(E_0)$ is connected.*

Then, $F : \mathcal{G} \longrightarrow \mathcal{G}$ can be realized as the dynamics on a Fatou graph of a rational map with fat gasket Julia set.

Remark 4.2. We remark that the first three conditions are necessary by Theorem 3.1. When there are no critical points on the Julia set, then it is not hard to see that if $\mathcal{G} - \mathrm{Int}(E_0)$ is disconnected, then there exists a simple closed curve γ separating the post-critical points in the two components of $\mathcal{G} - \mathrm{Int}(E_0)$, and γ is a Levy cycle. So, the topological branched covering is Thurston obstructed. Therefore, the last condition is also necessary when no critical points are on the Julia set.

4.1. *Finite core.* In this subsection, we shall see that this infinite simple plane graph in Theorem 4.1 can be constructed from its finite core.

Let $F : \mathcal{G} \longrightarrow \mathcal{G}$ be a degree d simplicial branched covering as in Theorem 4.1. Let $\mathcal{H} \subseteq \mathcal{G}$ be a finite connected graph containing all critical values and the fixed edge E_0. Since every edge is eventually mapped to E_0, there exists a constant N so that $F^N(\mathcal{H}) = E_0$.

Define $\mathcal{G}^0 = \bigcup_{k=0}^{N} F^k(\mathcal{H})$ and $\mathcal{G}^1 = F^{-1}(\mathcal{G}^0)$. We shall call $F : \mathcal{G}^1 \longrightarrow \mathcal{G}^0$ a *finite core* of $F : \mathcal{G} \longrightarrow \mathcal{G}$. We remark that since the subgraph $\mathcal{H}$ in the construction is not unique, finite cores are not unique.

LEMMA 4.3. *Let $F : \mathcal{G} \longrightarrow \mathcal{G}$ be a degree d simplicial branched covering as in Theorem 4.1. Let $F : \mathcal{G}^1 \longrightarrow \mathcal{G}^0$ be its finite core. Then:*

- *$\mathcal{G}^0 \subseteq \mathcal{G}^1$;*
- *$\mathcal{G}^1$ is connected;*
- *$\mathcal{G} = \bigcup_{k=0}^{\infty} \mathcal{G}^k$, where $\mathcal{G}^k = F^{-1}(\mathcal{G}^{k-1})$;*
- *$\mathcal{G}^1 - \mathrm{Int}(E_0)$ is connected.*

Proof. Since $F(\mathcal{G}^0) \subseteq \mathcal{G}^0$, we have that $\mathcal{G}^0 \subseteq \mathcal{G}^1$.

Since $\mathcal{G}^0$ contains all critical values, $\mathcal{G}^1$ contains all critical points. Since $\mathcal{G}^1 = \widetilde{F}^{-1}(\mathcal{G}^0)$, $\widetilde{F}$ is a homeomorphism between a face of $\mathcal{G}^1$ and a face of $\mathcal{G}^0$. Therefore, each face of $\mathcal{G}^1$ is simply connected, so $\mathcal{G}^1$ is connected.

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Applying the above argument inductively, we have a sequence of finite connected simple plane graphs

$$\mathcal{G}^0 \subseteq \mathcal{G}^1 \subseteq \mathcal{G}^2 \subseteq \cdots ,$$

where $\mathcal{G}^k = F^{-1}(\mathcal{G}^{k-1})$. Since every edge of $\mathcal{G}$ is eventually mapped to E_0, we have $\mathcal{G} = \bigcup_{k=0}^{\infty} \mathcal{G}^k$.

Suppose $\mathcal{G}^1 - \mathrm{Int}(E_0)$ is not connected. Then, let U be the unique open face of $\mathcal{G}^1$ whose boundary contains E_0. Note that U has access to two sides of E_0. Let $V = \widetilde{F}(U)$. Note that ∂V is a union of edges in $\mathcal{G}^0$. Since E_0 is fixed and U is a face of $\mathcal{G}^1$, we have $U \subseteq V$. In particular, V has access to two sides of E_0. Since $\widetilde{F} : U \longrightarrow V$ is a homeomorphism, $\widetilde{F}^{-1}|_{U \longrightarrow V}(U) \subseteq U$ is a face of $\mathcal{G}^2$ with access to two sides of E_0. Therefore, $\mathcal{G}^2 - E_0$ is not connected. Thus, inductively, we have $\mathcal{G}^k - \mathrm{Int}(E_0)$ is not connected for all k. This implies $\mathcal{G} - \mathrm{Int}(E_0)$ is not connected, which is a contradiction. □

Conversely, one can easily show that if we start with two finite connected simple plane graphs $\mathcal{G}^0 \subseteq \mathcal{G}^1$, a degree d simplicial branched covering

$$F : \mathcal{G}^1 \longrightarrow \mathcal{G}^0,$$

and an edge $E_0 = [a, b]$ of $\mathcal{G}^0$ so that:

- E_0 is fixed and any edge E of $\mathcal{G}^1$ is eventually mapped to E_0;
- $\mathcal{G}^1 - \mathrm{Int}(E_0)$ is connected,

then we can construct a degree d branched covering F on the union

$$F : \mathcal{G} = \bigcup_{k=0}^{\infty} \mathcal{G}^k \longrightarrow \mathcal{G},$$

where $\mathcal{G}^k = \widetilde{F}^{-1}(\mathcal{G}^{k-1})$ so that it satisfies the assumptions for Theorem 4.1.

4.2. *Thurston theory of rational maps.* In this subsection, we shall briefly summarize some basics of Thurston theory of rational maps.

A post-critically finite branched covering of a topological 2-sphere $\mathbb{S}^2$ is called a *Thurston map*. We denote the post-critical set of a Thurston map f by $P(f)$. Two Thurston maps f and g are *equivalent* if there exist two orientation-preserving homeomorphisms $h_0, h_1 : (\mathbb{S}^2, P(f)) \to (\mathbb{S}^2, P(g))$ so that $h_0 \circ f = g \circ h_1$, where h_0 and h_1 are isotopic relative to $P(f)$.

A set of pairwise disjoint, non-isotopic, essential, simple, closed curves Σ on $\mathbb{S}^2 \setminus P(f)$ is called a *curve system*. A curve system Σ is called f-stable if for every curve $\sigma \in \Sigma$, all the essential components of $f^{-1}(\sigma)$ are homotopic rel $P(f)$ to curves in Σ. We associate to an f-stable curve system Σ the *Thurston linear transformation*

$$f_\Sigma : \mathbb{R}^\Sigma \longrightarrow \mathbb{R}^\Sigma$$

defined as

$$f_\Sigma(\sigma) = \sum_{\sigma' \subseteq f^{-1}(\sigma)} \frac{1}{\deg(f : \sigma' \to \sigma)} [\sigma']_\Sigma,$$

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where $\sigma \in \Sigma$ and $[\sigma']_\Sigma$ denotes the element of Σ isotopic to σ', if it exists. The curve system is called *irreducible* if f_Σ is irreducible as a linear transformation. It is said to be a *Thurston obstruction* if the spectral radius $\lambda(f_\Sigma) \geq 1$.

We refer the readers to [**DH93**] for the definition of hyperbolic orbifold, but this is the typical case as any map with more than four postcritical points has hyperbolic orbifold. Thurston's topological characterization of rational maps says the following.

THEOREM 4.4. [**DH93**, Theorem 1] *Let f be a Thurston map which has hyperbolic orbifold. Then, f is equivalent to a rational map if and only if f has no Thurston's obstruction. Moreover, if f is equivalent to a rational map, the map is unique up to Möbius conjugacy.*

An arc λ in $\mathbb{S}^2 \setminus P(f)$ is an embedding of $(0, 1)$ in $\mathbb{S}^2 \setminus P(f)$ with end-points in $P(f)$. It is said to be *essential* if it is not contractible in $\mathbb{S}^2$ fixing the two end-points. A set of pairwise non-isotopic essential arcs Λ is called an *arc system*. The Thurston linear transformation f_Λ is defined in a similar way and we say that it is irreducible if f_Λ is irreducible as a linear transformation.

For a curve system Σ (respectively, an arc system Λ), we set $\widetilde{\Sigma}$ (respectively, $\widetilde{\Lambda}$) as the union of those components of $f^{-1}(\Sigma)$ (respectively, $f^{-1}(\Lambda)$) which are isotopic relative to $P(f)$ to elements of Σ (respectively, Λ). We will use $\Sigma \cdot \Lambda$ to denote the minimal intersection number between them. We will be using the following theorem excerpted and paraphrased from [**PT98**, Theorem 3.2].

THEOREM 4.5. [**PT98**, Theorem 3.2] *Let f be a Thurston map, Σ an irreducible Thurston obstruction in $(\mathbb{S}^2, P(f))$, and Λ an irreducible arc system in $(\mathbb{S}^2, P(f))$. Assume that Σ intersect Λ minimally, then either:*

- $\Sigma \cdot \Lambda = 0$*; or*
- $\Sigma \cdot \Lambda \neq 0$ *and for each $\lambda \in \Lambda$, there is exactly one connected component λ' of $f^{-1}(\lambda)$ such that $\lambda' \cap \widetilde{\Sigma} \neq \emptyset$. Moreover, the arc λ' is the unique component of $f^{-1}(\lambda)$ that is isotopic to an element of Λ.*

With this preparation, we are ready to show the following lemma.

LEMMA 4.6. *Let $F : \mathcal{G} \longrightarrow \mathcal{G}$ be a degree d simplicial branched covering satisfying the conditions in Theorem 4.1. Let $\widetilde{F}$ be the corresponding Thurston map on $\mathbb{S}^2 \cong \hat{\mathbb{C}}$. Then, $\widetilde{F}$ is equivalent to a rational map f.*

Moreover, the induced dynamics on the Fatou graph of f is conjugate to $F : \mathcal{G} \longrightarrow \mathcal{G}$.

Proof. It is easy to verify that f has hyperbolic obifold. Thus, by Theorem 4.4, it suffices to show there are no Thurston obstructions.

Suppose for contradiction that there is a Thurston obstruction Σ. After passing to a subset, if necessary, we may assume that Σ is irreducible.

Let $\Lambda = \{E_0\}$. Then, Λ is an irreducible arc system. Isotoping Σ, we may assume that Σ intersects Λ minimally. Let $\widetilde{\Sigma}_n$ be the union of those components of $f^{-n}(\Sigma)$ which are isotopic to elements of Σ.

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We claim that $\widetilde{\Sigma}_n$ does not intersect $f^{-n}(E_0) - E_0$. Indeed, applying Theorem 4.5 to f^n, we are led to the following two cases. In the first case, $\Sigma \cap E_0 = \emptyset$, so $f^{-n}(\Sigma) \cap f^{-n}(E_0) = \emptyset$ and the claim follows. In the second case, since $f(E_0) = E_0$, we conclude that E_0 is the unique component of $f^{-n}(E_0)$ that is isotopic to E_0. Thus, the only component of $f^{-n}(E_0)$ intersecting $\widetilde{\Sigma}_n$ is E_0, and the claim follows.

Since $\mathcal{G} - E_0$ is connected, there exists a finite graph $\mathcal{H} \subseteq \mathcal{G}$ so that $\mathcal{H} - E_0$ is a connected graph containing the post-critical set. Since every edge is eventually mapped to E_0, there exists N so that $\mathcal{H} \subseteq f^{-N}(E_0)$. Therefore, by the claim, $\widetilde{\Sigma}_N$ does not intersect $\mathcal{H} - E_0$. Thus, Σ does not intersect $\mathcal{H} - E_0$. This forces Σ to be empty, which is a contradiction.

The moreover part follows directly from a standard pull-back argument. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. By Lemma 4.6, there exists a post-critically finite rational map f whose induced dynamics on the Fatou graph is conjugate to $F : \mathcal{G} \longrightarrow \mathcal{G}$.

To get a fat gasket from the post-critically finite rational map f, we can perform a standard *simple pinching* deformation (see [**CT18**] or [**HT04**]) to create a double parabolic fixed point at the common boundary of the two fixed Fatou components of f. Let us denote this new map $\tilde{f}$. Note that $\tilde{f}$ is geometrically finite and $\tilde{f} : J(\tilde{f}) \longrightarrow J(\tilde{f})$ is topologically conjugate to $f : J(f) \longrightarrow J(f)$.

By the local Fatou coordinate at the double parabolic fixed point, it is easy to see the two fixed Fatou of $\tilde{f}$ are tangent to each other. By inductively pulling back this tangent point, we see that $J(\tilde{f})$ is a fat gasket. Since $\tilde{f}$ and f have the same Julia dynamics, the induced dynamics of $\tilde{f}$ on the Fatou graph is conjugate to $F : \mathcal{G} \longrightarrow \mathcal{G}$. □

4.3. *The quadratic case.* We will now consider the quadratic case and prove Theorem 1.7.

Proof of Theorem 1.7. Let $\mathcal{G}$ be the Fatou graph. By Theorem 3.1, there exists a unique fixed edge $E_0 = [a, b] \subseteq \mathcal{G}$. Note that a, b are not fixed, as any quadratic rational map with two fixed Fatou components has its Julia set homeomorphic to a circle. Thus a, b form a periodic 2-cycle under f_*.

Note that the Fatou components $U_a \cup U_b$ contain exactly one critical point. Let $c \notin U_a \cup U_b$ be the other critical point. We have two cases.

Case 1: If c is on the Julia set, then it is easy to see that the Fatou graph $\mathcal{G}$ is a tree, as c is not on the boundary of a Fatou component.

Case 2: If c is contained in the Fatou set, then f is a geometrically finite rational map. So there exists a post-critically finite rational map g with topologically conjugate dynamics on the Julia sets (see [**CT18**, Theorem 1.3]). Note that g is hyperbolic as it is post-critically finite and there are no critical points on the Julia set. Since f has a fat gasket Julia set, it has a parabolic fixed point at $\partial U_a \cap \partial U_b$. Thus, f is a root of the hyperbolic component containing g. Since every edge is eventually mapped to E_0 by Theorem 3.1, c is eventually mapped to $U_a \cup U_b$. Therefore, f is a root of a captured type hyperbolic component with an attracting cycle of period 2. This proves the first part of Theorem 1.7.

We now prove the moreover part. Suppose f is in Case 1 and is geometrically finite. Let g be the corresponding post-critically finite rational map. We can connect the two points in the critical 2-cycle of g using two internal rays, and denote this by I. We can choose a small neighborhood U of I, so that $\gamma = \partial U$ is a simple closed curve, $g^{-1}(\gamma)$ is isotopic to γ relative to the post-critical points of g, and $g : g^{-1}(\gamma) \longrightarrow \gamma$ is a two-to-one map. Thus, γ is an equator and g is a mating of the Basilica with a dendrite polynomial. Therefore, f is a mating of the fat Basilica with a dendrite polynomial.

Let g be a quadratic post-critically finite rational map of captured type with a critical 2-cycle. Let U_a, U_b be the Fatou components for the critical 2-cycle. Note that the internal fixed rays γ_a, γ_b of U_a and U_b must land at a common fixed point. Indeed, otherwise, they land at a 2-cycle x_a, x_b, but a quadratic rational map can have at most one 2-cycle, which is a contradiction. Thus, ∂U_a intersects ∂U_b. Let E_0 be the union of γ_a and γ_b together with their common landing point. Inductively, it is easy to see that $\mathcal{G}_n := g^{-n}(E_0)$ is a connected simple graph. Let $\mathcal{G} = \bigcup_{n=0}^{\infty} \mathcal{G}_n$. Then, it is not hard to see that $g : \mathcal{G} \longrightarrow \mathcal{G}$ satisfies the assumptions in Theorem 4.1. Indeed, the condition that $\mathcal{G} - \mathrm{Int}(E_0)$ is connected follows as g has no Levy cycle. Thus, similar to the proof of Theorem 4.1, if f is a root of the hyperbolic component containing g, then f has a fat gasket. □

5. *Captured type in* $\mathrm{Per}_2(0)$

To understand the moduli space of quadratic rational maps, in [**Mil93**], Milnor defines one-complex-dimensional slices, called $\mathrm{Per}_n(0)$ curves. A map f is in $\mathrm{Per}_n(0)$ if it has a super-attracting periodic n-cycle. By Theorem 1.7, we are interested in maps of captured type in $\mathrm{Per}_2(0)$ (see Figure 3). This space has been studied extensively in the literature. In particular, it can be interpreted as a partial mating of the filled Julia set of a Basilica and the Mandelbrot set (see[**Wit88**, **Luo95**, **Dud11**]).

In this section, we will study the combinatorics of Fatou graph $\mathcal{G}$ of a post-critically finite rational map of captured type in $\mathrm{Per}_2(0)$.

5.1. *The canonical finite core and critical loop.*

Let f be the post-critically finite map of a captured type hyperbolic component in $\mathrm{Per}_2(0)$. Let $\mathcal{G}$ be the Fatou graph of f and E_0 be the unique fixed edge of $\mathcal{G}$.

For simplicity, we shall regard $\mathcal{G}$ as a graph embedded in the dynamical plane of f, where a vertex is the center of the corresponding Fatou component and an edge is a union of two internal rays together with their common landing point. With this identification, we shall also denote the induced map on $\mathcal{G}$ simply by $f : \mathcal{G} \longrightarrow \mathcal{G}$.

Let $c \in \mathcal{G}$ be the vertex associated to the strictly pre-periodic critical point. Let q be the pre-period of c. Then, $f^q(c) \in \partial E_0$. We define

$$\mathcal{G}^0 := \bigcup_{i=0}^{q-1} f^{-i}(E_0) \quad \text{and} \quad \mathcal{G}^1 := f^{-1}(\mathcal{G}^0).$$

By induction, it is easy to see that $\bigcup_{i=0}^{k} f^{-i}(E_0)$ is a tree for all $k \leq q - 1$. Thus, we have the following lemma.

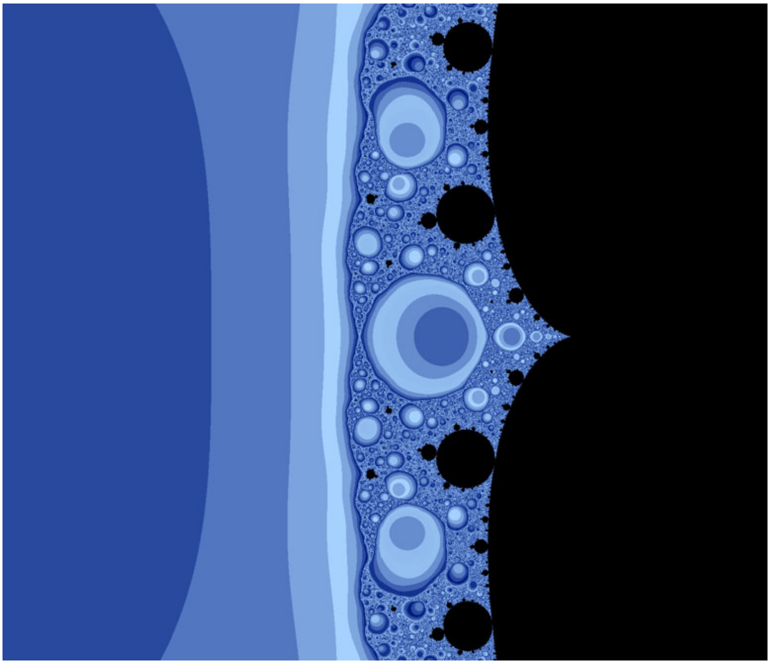

FIGURE 3. The bifurcation locus of $\mathrm{Per}_2(0)$ (colour online).

LEMMA 5.1. *The graph $\mathcal{G}^0$ is a tree containing the post-critical set of $\mathcal{G}$.*

Thus, $f : \mathcal{G}^1 \longrightarrow \mathcal{G}^0$ is a finite core of $f : \mathcal{G} \longrightarrow \mathcal{G}$ as introduced in §4.1. We call this the *canonical core* for *f*.

By Lemma 4.3, $\mathcal{G}^1 - \mathrm{Int}(E_0)$ is connected. Thus, $\mathcal{G}^1$ has at least one simple closed curve that contains E_0. Since $\mathcal{G}^0$ is a tree, it is easy to see that $\mathcal{G}^1$ contains exactly one simple closed curve $\mathcal{C}$. We call this loop $\mathcal{C}$ the *critical loop* for *f*. The length of the critical loop is an even number $2l$ for some $l \geq 2$. We call l the *critical distance* for *f*. Since $\mathcal{G}^0$ is a tree, it is easy to show the following lemma.

LEMMA 5.2. *Let $\mathcal{C}$ be the critical loop. Then, $\mathcal{C}$ contains E_0 and the critical vertices of $\mathcal{G}$. Moreover, $f(\mathcal{C})$ is a simple path of length l and f maps each of the two components of $\hat{\mathbb{C}} - \mathcal{C}$ homeomorphically to $\hat{\mathbb{C}} - f(\mathcal{C})$.*

Since $f(E_0) = E_0$, we have $E_0 \subseteq \mathcal{C} \cap f(\mathcal{C})$. Depending on the intersection $\mathcal{C} \cap f(\mathcal{C})$, we classify the post-critically finite map of captured type in $\mathrm{Per}_2(0)$ into three types (see Figures 4, 5 and 6):

Type I: $\mathcal{C} \cap f(\mathcal{C}) = E_0$;
Type IIA: $E_0 \subsetneq \mathcal{C} \cap f(\mathcal{C}) \subsetneq f(\mathcal{C})$;
Type IIB: $\mathcal{C} \cap f(\mathcal{C}) = f(\mathcal{C})$.

We shall call a map *f* of Type II if it is either Type IIA or Type IIB.

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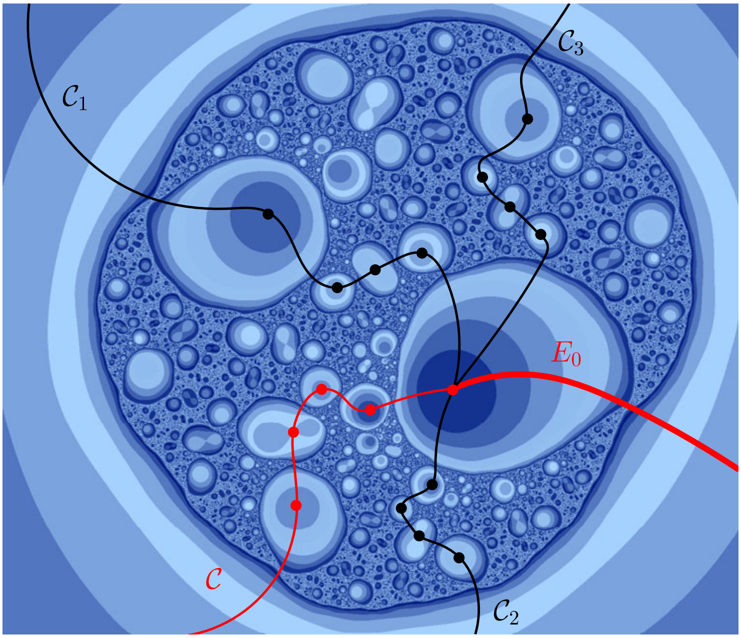

FIGURE 4. The shortest anchored simple closed curves for a Type I map (colour online).

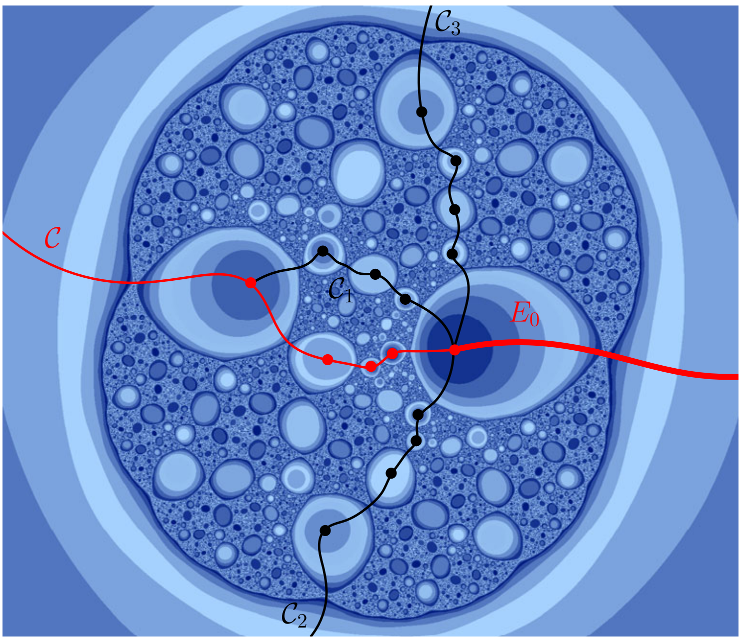

FIGURE 5. The shortest anchored simple closed curves for a Type IIA map (colour online).

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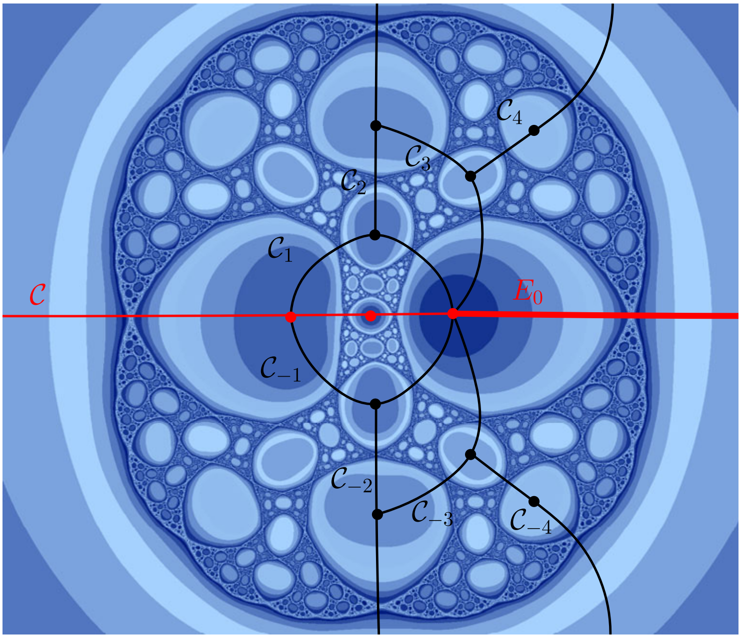

FIGURE 6. The shortest anchored simple closed curves for a Type IIB map (colour online).

5.2. *Simple closed curves in $\mathcal{G}$.* In this subsection, we prove that the critical loop has the shortest length among all simple closed curves of $\mathcal{G}$.

PROPOSITION 5.3. *Let f be a post-critically finite map of captured type in* $\mathrm{Per}_2(0)$. *Let l be the critical distance for f. Then, any simple closed curve in $\mathcal{G}$ has length $\geq 2l$.*

The proof of the proposition consists of several lemmas.

LEMMA 5.4. *Let $\mathcal{K} \subseteq \mathcal{G}$ be a simple closed curve. Then, either $\mathcal{K}$ contains both critical points, or the image $f(\mathcal{K})$ contains a simple closed curve of length $\leq$ length of $\mathcal{K}$.*

Proof. If $\mathcal{K}$ contains at most one critical point, f is locally injective at all but at most one vertex of $\mathcal{K}$. Thus, $f(\mathcal{K})$ contains a simple closed curve. It has smaller length as f is a simplicial map. □

LEMMA 5.5. *Let k be the distance between the two critical points in the graph metric of $\mathcal{G}$. Then, any simple closed curve in $\mathcal{G}$ has length $\geq 2k$. Moreover, there exists a simple closed curve passing through the two critical points with length $2k$.*

Proof. Since every edge of $\mathcal{G}$ is eventually mapped to E_0, $f^n(\mathcal{K}) = E_0$ for sufficiently large n. Thus, inductively applying Lemma 5.4, we conclude that there exists a simple closed curve containing both critical points with length $\leq$ length of $\mathcal{K}$. This implies that the length of $\mathcal{K}$ is at least $2k$.

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For the moreover part, let $\gamma \subseteq \mathcal{G}$ be a path connecting the two critical points with length k. By the minimality of γ, it is easy to see that $f(\gamma)$ is a simple path of length k. Let $\mathcal{L} = f^{-1}(f(\gamma))$. Then, $\mathcal{L}$ is a simple closed curve of length $2k$. □

LEMMA 5.6. *The critical distance l equals the distance between the two critical points in the graph metric of $\mathcal{G}$.*

Proof. Let k be the distance between the two critical points in the graph metric of $\mathcal{G}$ and let γ be a path connecting the two critical points with length k. Denote the periodic critical point by a and the strictly pre-periodic critical point by c.

Denote the vertices of γ in order by $v_0 = a, v_1, \ldots, v_k = c$. We claim the following.

CLAIM 5.7. *For any $i \neq k-1$ and any $j \geq 1$, $f^j(v_i) \neq c$.*

Proof of the claim. Since $v_k = c$ is strictly pre-periodic, $f^j(v_k) \neq c$ for any $j \geq 1$. Since f is simplicial, we have $d(f^j(a), f^j(v_i)) \leq d(a, v_i) \leq i$ for any $j \geq 1$. Since $d(a, f^j(a)) \leq 1$, we conclude that for any $i \leq k-2$, we have

$$d(a, f^j(v_i)) \leq d(f^j(a), f^j(v_i)) + 1 \leq i + 1 \leq k - 1.$$

Since $d(a, c) = k$, this proves the claim. □

If $\gamma \subseteq \mathcal{C}$, then we have $l = k$.

Otherwise, γ and $\mathcal{C}$ bound some simple closed curve $\mathcal{K}$. Since $\mathcal{C} = f^{-1}(f(\mathcal{C}))$, $f(\mathcal{K})$ contains a simple closed curve. Thus, inductively applying Lemma 5.4, we can conclude that there exists some vertex v of $\mathcal{K}$ and some iterate $j \geq 1$ so that $f^j(v) = c$. Since no vertex in $\mathcal{C}$ is mapped to c by f^j for any $j \geq 1$, together with the previous claim, there exists j so that $f^j(v_{k-1}) = c$. Therefore, $d(f(a), c) = k - 1$.

Let α' be a path of length $k-1$ connecting $f(a)$ and c, and let $\alpha = \alpha' \cup E_0$. Note that α is a path of length k connecting the two critical points that contains E_0. Denote the vertices of α in order by $w_0 = a, w_1, \ldots, w_k = c$. Since $f(w_1) = a$, we have that for any $i \leq k-1$ and any $j \geq 1$,

$$d(a, f^j(w_i)) \leq d(f^j(w_1), f^j(w_i)) + 1 \leq i \leq k - 1.$$

Thus, for any $i \leq k-1$ and any $j \geq 1$, $f^j(w_i) \neq c$. Therefore, the same argument as in the previous paragraph implies that $\alpha \subseteq \mathcal{C}$, so $l = k$. □

Proof of Proposition 5.3. Combining Lemmas 5.5 and 5.6, we have the result. □

5.3. *Shortest anchored simple closed curves.* In this subsection, we further analyze the structures of shortest simple closed curves in $\mathcal{G}$. More precisely, we call a simple closed curve that contains E_0 an *anchored simple closed curve*. We will study these shortest anchored simple closed curves.

We will use the following lemma.

LEMMA 5.8. *Let $\mathcal{K}$ be a shortest anchored simple closed curve. Then, there exists $j \geq 0$ so that $f^j(\mathcal{K})$ is the critical loop $\mathcal{C}$.*

Proof. We claim the following.

CLAIM 5.9. *If $\mathcal{K} \neq \mathcal{C}$, then $f(\mathcal{K})$ is a shortest anchored simple closed curve as well.*

Proof of the claim. If $\mathcal{K}$ does not contain both critical points, by minimality of the length of $\mathcal{K}$, then $f(\mathcal{K})$ is a shortest anchored simple closed curve.

Otherwise, let $\gamma \subseteq \mathcal{K}$ be a path of length l that connects the two critical points so that $\gamma \nsubseteq \mathcal{C}$. By minimality and Proposition 5.3, γ is disjoint from $\mathcal{C}$ other than at the two end points. Since $\mathcal{K}$ is anchored, we have $E_0 \subseteq \mathcal{K} - \text{Int}(\gamma) \subseteq \mathcal{C}$. Therefore, $f(\mathcal{K})$ is a simple closed curve. By minimality of the length of $\mathcal{K}$, it is a shortest anchored simple closed curve. □

The lemma follows by inductively applying the above claim. □

Let $\mathcal{C}_1 \neq \mathcal{C}_2$ be two shortest anchored simple closed curves. We say they are *siblings* if $E_0 \subsetneq \mathcal{C}_1 \cap \mathcal{C}_2$, that is, their intersection contains more than E_0. We say they are *non-siblings* if $E_0 = \mathcal{C}_1 \cap \mathcal{C}_2$. We will prove the following proposition.

PROPOSITION 5.10. *Let f be the post-critically finite map of captured type in* $\text{Per}_2(0)$*. Then, there are infinitely many shortest anchored simple closed curves, and each is mapped to the critical loop $\mathcal{C}$ by some iterates of f.*

Moreover:

- *if f is Type I, then no shortest anchored simple closed curve has siblings;*
- *if f is Type IIA, then the critical loop has one sibling, and other than these two, none has any siblings;*
- *if f is Type IIB, then the critical loop has two siblings, each of which has exactly three siblings, and every other shortest anchored simple closed curve has exactly two siblings.*

Proof. Let $\mathcal{K}$ be a shortest anchored simple closed curve. By Lemma 5.8, $\mathcal{K}$ is mapped to the critical loop $\mathcal{C}$ by some iterates of f. To prove there are infinitely many such curves, we will consider three cases.

Case 1: If f is Type I, then $\mathcal{C} \cap f(\mathcal{C}) = E_0$. Note that f maps each of the two components of $\hat{\mathbb{C}} - \mathcal{C}$ homeomorphically to $\hat{\mathbb{C}} - f(\mathcal{C})$. Thus, by pulling back, it is easy to verify that $f^{-1}(\mathcal{C})$ is a figure-eight curve, and one component, denoted by $\mathcal{C}_1$, is a shortest anchored simple closed curve. Since $\mathcal{C} \cap f(\mathcal{C}) = E_0$, we have $\mathcal{C} \cap \mathcal{C}_1 = E_0$, so they are not siblings. Similarly, $f^{-1}(\mathcal{C}_1)$ is a figure-eight curve and contains a shortest anchored simple closed curve, denoted by $\mathcal{C}_1$. Note that $\mathcal{C}_1$ and $\mathcal{C}_2$ are on two sides of the critical loop $\mathcal{C}$ (see Figure 4).

Let U_1 and U_2 be the regions bounded by $\mathcal{C}_1$ and $\mathcal{C}_2$ that do not intersect the critical loop $\mathcal{C}$, and let U_0 be the region bounded by $\mathcal{C}$ that does not contain $\mathcal{C}_1$. Then, $f : U_1 \longrightarrow U_0$ and $f : U_2 \longrightarrow U_1$ are homeomorphisms. Thus, by inductively pulling back, we get a sequence of shortest anchored simple closed curves $\mathcal{C}_n$, where $f(\mathcal{C}_n) = \mathcal{C}_{n-1}$, and none has a sibling.

Case 2: If f is Type IIA, then $E_0 \subsetneq \mathcal{C} \cap f(\mathcal{C}) \subsetneq f(\mathcal{C})$. Similarly as in Case 1, $f^{-1}(\mathcal{C})$ is a figure-eight curve and contains a shortest anchored simple closed curve, denoted by

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$\mathcal{C}_1$. Since $\mathcal{C} \cap f(\mathcal{C})$ contains more than E_0, $\mathcal{C}$ and $\mathcal{C}_1$ are siblings (see Figure 5). Similarly, $f^{-1}(\mathcal{C}_1)$ contains one shortest anchored simple closed curve, denoted by $\mathcal{C}_2$. Note that $\mathcal{C}_1$ and $\mathcal{C}_2$ are on two sides of the critical loop $\mathcal{C}$. Moreover, $\mathcal{C}_1$ and $\mathcal{C}_2$ are not siblings, and neither are $\mathcal{C}$ and $\mathcal{C}_2$. Then, the same pull back argument as in Case 1 gives a sequence of shortest anchored simple closed curves $\mathcal{C}_n$, where $f(\mathcal{C}_n) = \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ has no sibling for all $n \geq 2$.

Case 3: If f is Type IIB, then $\mathcal{C} \cap f(\mathcal{C}) = f(\mathcal{C})$. Note that in this case, $\mathcal{C}$ contains both critical values. Thus, $f^{-1}(\mathcal{C})$ is a union of four simple paths connecting the critical points. Note that f maps each one of the four components of $\hat{\mathbb{C}} - f^{-1}(\mathcal{C})$ homeomorphically to a component of $\hat{\mathbb{C}} - \mathcal{C}$. It is easy to verify that $f^{-1}(\mathcal{C})$ contains three shortest anchored simple closed curves, one of them is the critical loop. Denote them by $\mathcal{C}_1, \mathcal{C}_{-1}$, and $\mathcal{C}$. Therefore, $\mathcal{C}$ has two siblings (see Figure 6).

Let U_1 and U_{-1} be the regions bounded by $\mathcal{C}_1$ and $\mathcal{C}_{-1}$ that do not contain the critical loop $\mathcal{C}$, and let $\mathcal{V}_1$ and $\mathcal{V}_{-1}$ be the regions bounded by $\mathcal{C}$ that contains $\mathcal{C}_1$ and $\mathcal{C}_{-1}$, respectively. Then, $f : U_1 \longrightarrow V_{-1}$ and $f : U_{-1} \longrightarrow V_1$ are homeomorphisms. Thus, by inductively pulling back, we get a sequence of shortest anchored simple closed curves $\mathcal{C}_n$, $n \in \mathbb{Z}$, where $f(\mathcal{C}_n) = \mathcal{C}_{-\operatorname{sgn}(n)(|n|-1)}$ for $|n| \geq 1$. Here, $\operatorname{sgn}(n)$ represents the sign of n, and $\mathcal{C}_0 = \mathcal{C}$. Moreover, $\mathcal{C}_n$ has two siblings $\mathcal{C}_{n-1}$ and $\mathcal{C}_{n+1}$, and in addition $\mathcal{C}_1$ and $\mathcal{C}_{-1}$ are siblings. There are no other pairs of siblings, which gives the count in the proposition. □

6. *Quasiconformal non-equivalence between quadratic gasket Julia set and limit set*

In this section, we shall prove Theorem 1.4. We shall proceed with the proof by contradiction and suppose that there exists a quadratic rational map f whose Julia set is quasiconformally homeomorphic to a geometrically finite gasket limit set. By Theorem 2.3, the contact graph of a geometrically finite gasket limit set is not a tree, and f must have a fat gasket Julia set (see Corollary 2.4). Thus, by Theorem 1.7, f is a root of a captured type hyperbolic component with an attracting cycle of period 2. Since the Julia set of f is homeomorphic to the post-critically finite center of the corresponding hyperbolic component, Theorem 1.4 follows immediately from the following theorem.

THEOREM 6.1. *Let f be a post-critically finite of captured type in* $\mathrm{Per}_2(0)$. *Then, its Julia set J is not homeomorphic to the limit set of any geometrically finite Kleinian group.*

Proof. Suppose for contradiction that J is homeomorphic to the limit set Λ of some geometrically finite Kleinian group. Then, the Fatou graph $\mathcal{G}$ of f is homeomorphic to the contact graph of Λ.

Let $E_0 = [a, b]$ be the unique fixed edge of $\mathcal{G}$ by the induced map of f, where a is a critical point of f. By Lemma 2.2, there exists a subgroup $K \subseteq \mathrm{Homeo}(\mathcal{G})$ isomomorphic to $\mathbb{Z}$ that fixes E_0. Note that any element $g \in K$ must send a shortest anchored simple closed curve to another one. Note that $g \in K$ preserves sibling-ship. Thus, by Proposition 5.10, if f is Type IIA or Type IIB, any element $g \in K$ must fix the critical loop, which is a contradiction.

Therefore, it remains to consider the case that f is Type I. Note that we can label the shortest anchored simple closed curves by $\mathcal{C}_n$, $n \geq 0$, where $\mathcal{C}_0 = \mathcal{C}$ is the critical loop, and $f : \mathcal{C}_{i+1} \longrightarrow \mathcal{C}_i$ (see Figure 4).

Note that $\mathcal{C}_i$ divides the plane into infinitely many crescent-shaped regions, which we call gaps. The boundary of any gap contains $\{a, b\} = \partial E_0$. Denote the region bounded by $\mathcal{C}_0$ and $\mathcal{C}_1$ by R_0. Inductively, we define $R_n, n \in \mathbb{Z}$ as the adjacent gap to R_{n-1} in the counterclockwise direction viewed at the critical point a.

Note that if $n \geq 1$, then $f : R_n \longrightarrow R_{-n}$ is a homeomorphism, and if $n \leq -2$, then $f : R_n \longrightarrow R_{-n-1}$ is a homeomorphism. Thus, f induces an isomorphism between $\mathcal{G} \cap R_n$ and $\mathcal{G} \cap R_{-n}$ (or $\mathcal{G} \cap R_n$ and $\mathcal{G} \cap R_{-n-1}$, respectively). Therefore, for $n \geq 2$, $f^2 : R_n \longrightarrow R_{n-1}$ is a homeomorphism fixing a, b, and for $n \leq -2$, $f^2 : R_n \longrightarrow R_{n+1}$ is a homeomorphism fixing a, b. Moreover, f^2 induces an isomorphism on the corresponding subgraphs.

Since there exists a subgroup $K \subseteq \mathrm{Homeo}(\mathcal{G})$ isomomorphic to $\mathbb{Z}$ that fixes E_0, the subgraph in any gap R_k is isomorphic to $\mathcal{G} \cap R_n$ for some sufficiently large n. By the observation in the previous paragraph, we thus conclude that any subgraphs in any two gaps are homeomorphic, and the homeomorphism can be chosen fixing a, b. In particular, there exist homeomorphisms $g : R_0 \longrightarrow R_1$ and $h : R_{-1} \longrightarrow R_0$ so that:

- $g(a) = a$, $g(b) = b$ and $h(a) = a$, $h(b) = b$;
- g, h induce isomorphisms between of the corresponding subgraphs.

Therefore, by considering $\tau := h \circ f \circ g : R_0 \longrightarrow R_0$, we conclude that the subgraph $\mathcal{G} \cap R_0$ is symmetric under an orientation-preserving map that interchanges a and b. The proof of the theorem is complete with the following proposition. □

PROPOSITION 6.2. *Let f be a Type I post-critically finite of captured type in* $\mathrm{Per}_2(0)$. *Then, the subgraph $\mathcal{G} \cap R_0$ is not symmetric under an orientation-preserving map τ that interchanges a and b.*

The rest of the section is dedicated to the proof of this proposition. We shall suppose by contradiction that there exists such a symmetry τ.

We first set up some notation. Denote the boundary of E_0 by $a_0 = a$ and $b_0 = b$, where the critical point corresponds to a_0. Label the vertices on the critical loop $\mathcal{C}_0$ by $a_0, a_1, \ldots, a_{2l-2}, b_0$, and the vertices on the loop $\mathcal{C}_1$ by $b_0, b_1, \ldots, b_{2l-2}, a_0$ (see Figure 7). Note that we have the following dynamics:

- $f(b_i) = a_i$;
- $f(a_1) = a_0 = f(b_0)$;
- $f(a_0) = b_0$.

We also remark that the two critical points of f are a_0 and a_l.

A path $\gamma = [x_0, \ldots, x_k]$ in $\mathcal{G}$ is called a *local geodesic* if x_i, x_j are not adjacent in $\mathcal{G}$ if $|i - j| \geq 2$. Note that in particular, if x, y are adjacent in $\mathcal{G}$, then there are no other local geodesics other than the edge $[x, y]$. We also remark that given any path γ, one can construct a local geodesic by replacing $[x_i, x_{i+1}, \ldots, x_j]$ with $[x_i, x_j]$ if x_i, x_j are adjacent in $\mathcal{G}$.

Since f is post-critically finite, using the expansion of the dynamics away from the post-critical set, it is easy to show that for any pair of vertices x, y, there are only finitely many local geodesics of length k.

We define an R_0-*arc* as a local geodesic α that connects two boundary vertices in ∂R_0 so that:

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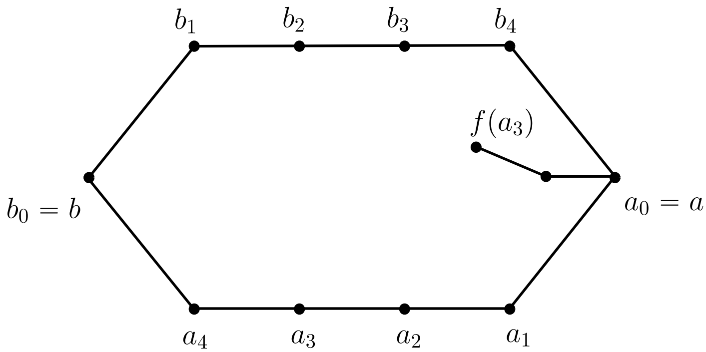

FIGURE 7. An illustration of the gap R_0.

- $\alpha \subseteq \overline{R_0}$;
- $\text{Int}\, \alpha \cap \mathcal{C}_0 = \emptyset$.

We first prove the following lemma.

LEMMA 6.3. *There exists an R_0-arc that connects a_l to a_0 so that it has a lift under f that connects b_l and a_1.*

Proof. By pulling back the critical loops using f and taking the associated local geodesic, one can show that there exists some local geodesic γ so that:

- $\text{Int}\, \gamma \subseteq \text{Int}\, R_0$;
- γ connects a_l to b_i for some $i = 1, \ldots, 2l-2$;
- no interior vertex of γ is adjacent to a_0.

Let γ be the one with the shortest length and suppose that γ connects a_l with b_i.

We claim that $\text{Int}\, \gamma \cap f(\mathcal{C}_0) = \emptyset$. Suppose not. Then, there exists a strictly shorter arc δ connecting a_l to $v \in f(\mathcal{C}_0)$. Note that $v \notin \{a, b\}$. Since f maps the gap R_0 homeomorphically to its image, we can find a lift δ' that connects b_l and a_j for some $j = 1, \ldots, 2l-2$. Then, $\tau(\delta')$ is a local geodesic that connects a_l to b_j with $\text{Int}\, \tau(\delta') \subseteq \text{Int}\, R_0$ and no interior vertex of $\tau(\delta')$ is adjacent to a_0. This is a contradiction to the minimality of γ.

It is easy to verify that R_0 contains $f(a_l)$. Let $\alpha = \gamma \cup [b_i, b_{i+1}] \cup \cdots \cup [b_{2l-2}, a_0]$. Since no interior vertex of γ is adjacent to a_0, we have that α is a local geodesic. Thus, α is an R_0-arc that connects a_l and a_0. By the claim, the closed loop

$$\alpha \cup [a_0, a_1] \cup \cdots \cup [a_{l-1}, a_l]$$

separates the critical values $f(a_l)$ and $b_0 = f(a_0)$ (see Figure 7). Since the arc $[a_0, a_1] \cup \cdots \cup [a_{l-1}, a_l]$ has a lift connecting b_l and b_0, we conclude that the arc α has a lift connecting b_l and a_1. □

Let N be the length of the shortest R_0-arc that connects a_l to a_0 with a lift under f that connects b_l and a_1. Let A be the collection of all R_0-arcs that connects a_l to a_0 of length $\leq N$. Let $B := \{\tau(\alpha) : \alpha \in A\}$. Note that A and B are finite sets with the same cardinality.

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Let K be the length of the shortest local geodesic γ as in the proof of Lemma 6.3. Then, the proof of Lemma 6.3 gives that

$$N \leq K + 2l - 1. \tag{6.1}$$

We will now prove the following lemma.

LEMMA 6.4. *Let $\beta \in B$. Then, $f(\beta) \in A$.*

Proof. Since β is an arc connecting b_l and b_0, $f(\beta)$ is an arc connecting a_l and a_0.

We will now verify that $f(\beta)$ satisfies the definition of an R_0-arc. Since $\beta = \tau(\alpha)$ for some $\alpha \in A$, $\operatorname{Int} \beta \cap \mathcal{C}_1 = \emptyset$. Thus, $\operatorname{Int} f(\beta) \cap \mathcal{C}_0 = \emptyset$.

Suppose that $f(\beta) \nsubseteq \overline{R_0}$. Since β connects b_l to b_0 in R_0, and f sends locally the region bounded between $[b_1, b_0]$ and $[b_0, a_{2l-2}]$ in R_0 to the region bounded by $[a_1, a_0]$ and $[a_0, f(a_{2l-2})]$ in R_0 (see Figure 7), we can decompose $f(\beta)$ as $f(\beta) = \gamma_1 \cup \gamma_2 \cup \gamma_3$, where γ_1 connects a_l to b_i for some i, γ_2 connects b_i to b_j for some j, and γ_3 connects b_j to a_0. Moreover, $\operatorname{Int} \gamma_1, \operatorname{Int} \gamma_3 \subseteq \operatorname{Int} R_0$.

We claim the following.

CLAIM 6.5. *The length $l(\gamma_1) \geq K$.*

Proof of the claim. If no interior vertex of γ_1 is adjacent to a_0, then $l(\gamma_1) \geq K$ by the minimality of the definition for K.

Otherwise, let $\beta_1 \subseteq \beta$ be the preimage of γ_1 in β. Since β is a local geodesic connecting b_l to b_0, no interior vertex of β_1 is adjacent to b_0. Since $f^{-1}(a_0) = \{b_0, a_1\}$, there exists some interior vertex $x \in \beta_1$ that is adjacent to a_1. Consider the truncation $\beta_1' \subseteq \beta_1$ that connects b_l and x, and $\widetilde{\beta_1} = \beta_1' \cup [x, a_1]$. Then, $\tau(\widetilde{\beta_1})$ satisfies that:

- $\operatorname{Int} \tau(\widetilde{\beta_1}) \subseteq \operatorname{Int} R_0$;
- $\tau(\widetilde{\beta_1})$ connects a_l to b_1;
- no interior vertex of $\tau(\widetilde{\beta_1})$ is adjacent to a_0.

Thus, by minimality of K, we have

$$K \leq l(\tau(\widetilde{\beta_1})) = l(\widetilde{\beta_1}) = l(\beta_1') + 1 \leq l(\beta_1) = l(\gamma_1). \qquad \square$$

By Proposition 5.3, any simple closed curve in $\mathcal{G}$ has length $\geq 2l$. Thus, for any two vertices v, w, there exists at most one path connecting v, w with length $< l$. Moreover, if there is a path connecting v, w with length l, then all the other paths have length $\geq l$. Since b_i, b_j, a_0 all lie on a shortest loop $\mathcal{C}_1$, we have that the lengths $l(\gamma_2), l(\gamma_3) \geq l$. So,

$$l(\beta) = l(\gamma_1) + l(\gamma_2) + l(\gamma_3) \geq K + 2l > N,$$

which is a contradiction to equation (6.1).

We now show that $f(\beta)$ is a local geodesic. Suppose not. Let δ be the associated local geodesic. Since β is a local geodesic and is contained in $\overline{R_0}$, the (non-simple) closed loop by concatenating $f(\beta)$ (from a_l to a_0) with δ (from a_0 to a_l) separates the two critical values $b_0, f(a_l)$ with winding number 1. Thus, δ is an R_0-arc with a lift connecting b_l and a_1. Note that $l(\delta) < l(\beta) \leq N$, which is a contradiction to the minimality of N.

Since length $l(f(\beta)) = l(\beta) \leq N$, $f(\beta) \in A$. $\square$

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We are ready to prove Proposition 6.2.

Proof of Proposition 6.2. By Lemma 6.4, we have an induced map $f_* : B \longrightarrow A$. This map is clearly injective as f is injective on Int R_0. Since A, B have the same cardinality, f_* is also surjective. However, by the definition of N, there exists $\alpha \in A$ whose lift connects b_l to a_1, so this arc α is not in the image of the induced map f_*. This is a contradiction and the proposition follows. □

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