ON THE GROUP OF A DIRECTED GRAPH

ROBERT L. HEMMINGER

In 1938, Frucht **(2)** proved that for any given finite group G there exists a finite symmetric graph X such that $G(X)$ is abstractly isomorphic to G. Since $G(X)$ is a permutation group, it is natural to ask the following related question: If P is a given finite permutation group, does there exist a symmetric (and more generally a directed) graph X such that $G(X)$ and P are isomorphic (see Convention below) as permutation groups? The answer for the symmetric case is negative as seen in **(3)** and more recently in **(1)**. It is the purpose of this paper to deal with this problem further, especially in the directed case. In §3, we supplement Kagno's results (**3**, pp. 516–520) for symmetric graphs by giving the corresponding results for directed graphs. These results are useful in studying which permutation groups have directed graphs, but our main results are in §1. Since forming products (in the case of permutation groups as used here, the product is a group and is in fact isomorphic to the direct product since we assume the factors have disjoint support sets) is one of the major ways of constructing new permutation groups from old ones, we have investigated the problem of relating the existence of a directed (symmetric) graph for a product to the existence of graphs for the factors. Corollary 1.1.1 shows that the solution is the "natural" one in general, but Theorem 1.3 shows that this "natural" solution is not always the correct one for fixed point free graphs.

By a directed graph X we mean a finite set $V(X)$ called the vertices of X, together with a set $E(X)$, called the edges of X, consisting of ordered pairs of distinct elements from $V(X)$. We shall indicate ordered pairs by parentheses. We say (a, b) is a symmetric edge if both (a, b) and (b, a) are edges, and we frequently distinguish symmetric edges by writing $[a, b]$ in place of (a, b). If every edge in X is a symmetric edge, then we say X is a symmetric graph. A directed graph is connected if upon considering the edges as unordered pairs it is connected as an undirected graph. The complement of a directed graph X, denoted by X^c, is the directed graph with $V(X^c) = V(X)$ and

$$E(X^c) = \{(a, b){:}a, b \in V(X) \text{ and } (a, b) \notin E(X)\}.$$

If $A \subseteq V(X)$, then the section graph of X on A is the graph with vertex set A and whose edges are all edges of X that have end points in A.

Let X be a directed graph. Then the group of automorphisms of X, denoted by $G(X)$, is the set of all permutations σ of $V(X)$ such that $(a, b) \in E(X)$ if and only if $\sigma(a, b) = (\sigma(a), \sigma(b)) \in E(X)$. Thus from the viewpoint of the group

Received November 11, 1964.

of automorphisms, a symmetric graph is the same as an undirected graph. An element in the domain of a permutation group P is in the support set of P if it is not fixed for all $\sigma \in P$. A directed graph X is called fixed point free if $V(X)$ is the support set of $G(X)$.

In §3 we follow the usual practice **(3; 4)** of listing only permutation groups having their domains and support sets equal. But these are not sufficient to study the permutation groups of a graph properly, as seen by Corollary 1.3.1. For these reasons and for ease of statement and proof of results we make the following convention.

Convention. Let P_i be a permutation group with support set S_i and let

$$P(S_i) = \{\sigma' : \sigma' = \sigma \mid S_i, \sigma \in P_i\}, \qquad i = 1, 2.$$

Then we shall say that P_1 and P_2 are *isomorphic permutation groups* if $P(S_1)$ and $P(S_2)$ are isomorphic as permutation groups in the usual sense **(6**, p. 39**)**. Thus we are viewing permutations as products of disjoint cycles, each of length at least two, and having as domain any set containing the symbols displayed in these cycles.

If for a given permutation group P there exists a directed graph X such that $G(X) = P$ (where the permutations are viewed as products of cycles as above), then we say that P has the directed graph X.

For any terms used but not defined see **(5)**.

1. Group products and their graphs. Before obtaining the theorems dealing with the existence of graphs for products of permutation groups we need two lemmas. The first is easy and well known; so we state it without proof.

LEMMA 1.1.1. *If X is a directed graph, then* (*a*) *either X or X^c is connected, and* (*b*) $G(X) = G(X^c)$.

LEMMA 1.1.2. *If X is a directed* (*symmetric*) *graph and*

$$\{T_j : j = 1, 2, \ldots, k\}$$

is a partitioning of $V(X)$, then

$$P = \{\sigma \in G(X) : \sigma(T_j) = T_j, j = 1, 2, \ldots, k\}$$

has a directed (*symmetric*) *graph.*

Proof. It is clear that P is a subgroup of $G(X)$. Let $|V(X)| = v$ and let

$$A = \{b_0, b_1, \ldots, b_{v+1}, a_0, a_{01}, a_{02}, \ldots, a_{0v}, a_{11}, a_{12}, \\ \ldots, a_{1v}, \ldots, a_{(k-1)1}, a_{(k-1)2}, \ldots, a_{(k-1)v}, a_{k1}\}$$

with $V(X) \cap A = \emptyset$. Define the graph X^* by taking $V(X^*) = V(X) \cup A$ and $E(X^*) = E(X) \cup E_1 \cup E_2$, where E_1 and E_2 consist only of symmetric edges and are as follows: E_1 consists of the symmetric edge $[a_{01}, b_1]$ and the edges in the symmetric arcs $[b_0, b_1, \ldots, b_{v+1}]$ and

$$[a_0, a_{01}, a_{02}, \ldots, a_{0v}, a_{11}, \ldots, a_{(k-1)v}, a_{k1}]$$

while E_2 consists of the symmetric edges $[t_j, a_{j1}]$ for all $t_j \in T_j, j = 1, 2, \ldots, k$.

By starting with b_0 and working out on the arcs from b_0 having vertices in A, it is easy to see that A is fixed vertexwise by $G(X^*)$; hence $G(X^*) \leqslant G(X)$. Also since a_{j1} is fixed by $G(X^*)$, the set T_j must be invariant under $G(X^*)$. Therefore $G(X^*) \subseteq P$. On the other hand, if $\sigma \in G(X)$ such that $\sigma(T_j) = T_j$, $j = 1, 2, \ldots, k$, a straightforward check of possible ordered pairs reveals that $\sigma \in G(X^*)$, so that $P \subseteq G(X^*)$. Hence $P = G(X^*)$, i.e., P has a directed graph. The symmetric case follows immediately since X^* is symmetric if and only if X is symmetric.

THEOREM 1.1. *If P_1 and P_2 are permutation groups with disjoint support sets, then $P_1 P_2$ has a directed (symmetric) graph if and only if P_1 and P_2 have directed (symmetric) graphs.*

Proof. Let P_i have support set A_i, $i = 1, 2$, with $A_1 \cap A_2 = \emptyset$. First assume that P_i has the directed graph X_i. By Lemma 1.1.1, we can assume that X_i is connected. Define the directed graph X by taking $V(X) = A_1 \cup A_2$ and

$$E(X) = E(X_1) \cup E(X_2) \cup \{(a, b): a \in A_1 \text{ and } b \in A_2\}.$$

We proceed to show that $G(X) = P_1 P_2$. Clearly $P_1 P_2 \subseteq G(X)$ since we included all edges from X_1 to X_2; so let $\sigma \in G(X)$ and let $|A_1| \leqslant |A_2|$ (the argument in the case $|A_1| \geqslant |A_2|$ is analogous). Then if $a \in A_1$ and $b \in A_2$, we see that $\rho_i(a) < |A_1| \leqslant \rho_i(b)$, where $\rho_i(x)$ is the incoming local degree at x. But σ must preserve the incoming local degree; hence $\sigma(A_1) \subseteq A_i$ for $i = 1, 2$. Therefore $\sigma(A_i) = A_i$ for $i = 1, 2$. Then clearly $\sigma_i \in P_i$, where σ_i is σ restricted to A_i; so $\sigma = \sigma_1 \sigma_2$ and $G(X) \subseteq P_1 P_2$. Hence $G(X) = P_1 P_2$; so $P_1 P_2$ has a directed graph.

One easily sees that in the event that X_1 is not isomorphic to X_2, it suffices to define $E(X) = E(X_1) \cup E(X_2)$ in the last paragraph since X_i is connected. Thus if X_1 and X_2 are symmetric graphs and X_1 is not isomorphic to X_2, then $P_1 P_2$ has a symmetric graph. But if $X_1 \cong X_2 = Y$, we can define Y^* as in the proof of Lemma 1.1.2 with the T_j being the transitivity sets of $G(Y)$. Then

$$G(Y^*) = \{\sigma \in G(Y) : \sigma(T_j) = T_j, j = 1, 2, \ldots, k\} = G(Y)$$

and clearly X_1 and Y^* are connected, Y^* is symmetric, and X_1 is not isomorphic to Y^*. So if X is defined by taking $V(X) = V(X_1) \cup V(Y^*)$ and

$$E(X) = E(X_1) \cup E(Y^*),$$

then X is symmetric and, as before, $G(X) = G(X_1)G(Y) = P_1 P_2$; so $P_1 P_2$ has a symmetric graph if P_1 and P_2 have symmetric graphs. We note, however, that X is not a fixed point free graph in this case. (See Theorem 1.3 for clarification of this difficulty.)

https://doi.org/10.4153/CJM-1966-023-2 Published online by Cambridge University Press

Conversely, assume P_1 has no directed graph but that $P = P_1 P_2$ has a directed graph X. Then $A_1 \cup A_2$ is the support set for P. Let X_1 be the section graph of X on A_1. Then $P_1 \subseteq G(X_1)$. Let $T_j, j = 1, \ldots, k$, be the transitivity classes of P_1 and let $a_j \in T_j$ and $b \in A_2$. Denote $\{(a_j, b) : a_j \in T_j\}$ by (T_j, b). Then from the observation that $(a_j, b) \in E(X)$ if and only if $(T_j, b) \subseteq E(X)$ and $(b, a_j) \in E(X)$ if and only if $(b, T_j) \subseteq E(X)$, we see that if $\sigma \in G(X_1)$ such that $\sigma(T_j) = T_j$ for $j = 1, 2, \ldots, k$, then $\sigma \in P$ and so $\sigma \in P_1$. Hence we have shown that

$$P_1 = \{\sigma \in G(X_1) : \sigma(T_j) = T_j, j = 1, 2, \ldots, k\}.$$

But then by Lemma 1.1.2 P_1 has a graph, which is a contradiction; so P has no graph. This completes the proof of the theorem. The general case is immediate by induction. We state it as a corollary.

COROLLARY 1.1.1. *If $P_1, P_2, \ldots, P_k$ are permutation groups with pairwise disjoint support sets, then $P_1 P_2 \ldots P_k$ has a directed (symmetric) graph if and only if P_i has a directed (symmetric) graph for $i = 1, 2, \ldots, k$.*

COROLLARY 1.1.2. *If P_i, $i = 1, 2, \ldots, k$, has a directed fixed point free graph, then $P_1 P_2 \ldots P_k$ has a directed fixed point free graph.*

The proof of the last corollary is immediate from Corollary 1.1.1 and the first part of the proof of Theorem 1.1. That this corollary does not hold for symmetric graphs will be seen in Theorem 1.3.

Before proceeding to the next result, which is related to Theorem 1.1, we give a definition.

DEFINITION 3.1. *Let T_j, $j = 1, 2, \ldots, k$, be a partitioning of the support set of a permutation group P. We say that P is independently transitive on T_i with respect to T_j if given a, $b \in T_i$ and $c \in T_j$ there exists $\sigma \in P$ such that $\sigma(a) = b$ and $\sigma(c) = c$.*

If $T_1, \ldots, T_k$ are the transitivity sets of P, then it is easy to see that P is independently transitive on T_i with respect to T_j, $i \neq j$, if and only if P is independently transitive on T_j with respect to T_i. Hence in that case we say that P is *independently transitive on T_i and T_j*.

THEOREM 1.2. *Let X be a directed graph and let $T_1, T_2, \ldots, T_k$ be the transitivity sets of $G(X)$, X_i the section graph of X on T_i, and X_{ij} the section graph of X on $T_i \cup T_j$. Then*

(a) *$G(X) \subseteq G(X_1)G(X_2) \ldots G(X_k)$;*

(b) *if $G(X)$ is independently transitive on T_i and T_j, $i \neq j$, then*

$$G(X_i)G(X_j) \subseteq G(X_{ij});$$

(c) *$G(X)$ is independently transitive on T_i and T_j for all i, j, $1 \leqslant i < j \leqslant k$, if and only if $G(X) = G(X_1)G(X_2) \ldots G(X_k)$.*

Proof. If $\sigma \in G(X)$ and σ_i is the restriction of σ to T_i, then $\sigma_i \in G(X_i)$ and $\sigma = \sigma_1 \sigma_2 \ldots \sigma_k$, so we have (a).

Let $a, b \in T_i$ and $c, d \in T_j$. Then there exists $\sigma \in G(X)$ such that $\sigma(c) = d$ and so, by our assumption on T_i and T_j, there exists $\tau \in G(X)$ such that $\tau(\sigma(a)) = b$ and $\tau(d) = d$. Thus $\tau\sigma(a, c) = (b, d)$. Hence either all or none of the possible edges from X_i to X_j are present. Likewise either all or none of the edges from X_j to X_i are present, but the two cases are independent. Because of this situation we see that if $\sigma \in G(X_i)$ or if $\sigma \in G(X_j)$, then $\sigma \in G(X_{ij})$. Thus $G(X_i)G(X_j) \subseteq G(X_{ij})$, which proves (b).

In (c), the sufficiency of the condition is obvious and the necessity follows by (a) and a straightforward generalization of the argument in (b) showing that $G(X_1)G(X_2) \ldots G(X_k) \subseteq G(X)$.

THEOREM 1.3. *If P_1 and P_2 are transitive permutation groups on disjoint support sets and P_i has a fixed point free symmetric graph for $i = 1, 2$, then $P_1 P_2$ has no fixed point free symmetric graph if and only if P_1 and P_2 are isomorphic permutation groups and all of their fixed point free symmetric graphs are isomorphic. In particular, if P_1 has a fixed point free symmetric graph Y, then $Y \cong Y^c$ since $G(Y) = G(Y^c)$.*

Proof. Let T_i be the support set of P_i, $i = 1, 2$. First assume that P_1 and P_2 are isomorphic permutation groups such that all of their fixed point free symmetric graphs are isomorphic. $P_1 P_2$ has a symmetric graph by Theorem 1.1; but, we now demonstrate that $P_1 P_2$ has no fixed point free symmetric graph. Suppose X was such a graph. Then T_1 and T_2 are the transitivity sets of $G(X)$, and $G(X)$ is independently transitive on T_1 and T_2. So by Theorem 1.2, $P_1 P_2 = G(X) = G(X_1)G(X_2)$, where X_i is the section graph of X on T_i. But this means that $P_i = G(X_i)$, so $X_1 \cong X_2$. Let σ be the isomorphism between X_1 and X_2. Then $\sigma \in G(X)$, but $\sigma \notin P_1 P_2$, which is a contradiction so our assumption was false and our claim true.

Conversely, if it is not the case that P_1 and P_2 are isomorphic permutation groups such that all of their fixed point free symmetric graphs are isomorphic, then P_i has a connected fixed point free symmetric graph X_i such that X_1 is not isomorphic to X_2. But then $G(X) = P_1 P_2$, where $X = X_1 \cup X_2$ is a symmetric fixed point free graph (we have already seen this in the proof of Theorem 1.1).

By taking P_1 and P_2 isomorphic to $\langle (ab)(cd) \rangle$, we see that the assumption of transitivity is necessary.

By taking P_1 and P_2 isomorphic to the dihedral group $\langle (abcde), (ab)(cd) \rangle$, we see that the theorem is not vacuous and we state this as a corollary.

COROLLARY 1.3.1. *There exist permutation groups with symmetric graphs but with no fixed point free symmetric graphs.*

https://doi.org/10.4153/CJM-1966-023-2 Published online by Cambridge University Press

2. Miscellaneous results. In this section we shall give a number of results enabling us to give a concise tabular form in §3. Some of these results are general enough to be of interest on their own.

In most cases the notation used for the permutation groups appearing in this and the next section is self-explanatory, but it can be found in **(4)**.

In the remainder of the paper we shall use X as a directed graph with

$$V(X) = \{a_1, a_2, \ldots, a_n\}.$$

THEOREM 2.1. *The following are equivalent:*
(a) $G(X) = (a_1 a_2 \ldots a_n)$*all*,
(b) $G(X)$ *is k-ply transitive*, $k \geqslant 2$,
(c) X *is either the null graph or the complete directed graph.*

THEOREM 2.2. *The groups* $G_{6,6} = \{(abcd)\text{cyc}(ef)\}\text{pos}$, $G_{6,22} = (abcdef)_{18}$, *and* $G_{6,27} = (abcdef)_{24_5}$ *have directed graphs.*

Proof. One checks that $G_{6,6}$ has the graph with edges (a, b), (b, c), (c, d), (d, a), (a, f), (b, e), (c, f), (d, e), and $[e, f]$; $G_{6,22}$ has the graph with edges (a, b), (b, c), (c, a), (d, e), (e, f), and (f, e); and $G_{6,27}$ has the graph with edges (a, b), (a, d), (b, e), (b, f), (c, b), (c, d), (d, e), (d, f), (e, a), (e, c), (f, a), and (f, c).

THEOREM 2.3. $(a_1 a_2 \ldots a_n)\text{cyc}$ *has the directed circuit* $(a_1, a_2, \ldots, a_n, a_1)$ *as a graph.*

THEOREM 2.4. *Let* $\sigma \in G(X)$ *be the rotation* $(a_1 a_2 \ldots a_n)$ *and let* τ *be the reflection* $(a_1 a_n)(a_2 a_{n-1}) \ldots (a_k a_{n-k+1})$ *with* $k = [\frac{1}{2}n]$. *Then* $\tau \in G(X)$ *if and only if* X *is symmetric.*

Proof. By repeated use of σ, every edge can be rotated into one of the form (a_1, a_i) or (a_i, a_1), $i = 2, 3, \ldots, n$. We consider only those edges of the form (a_1, a_i); those of the form (a_i, a_1) are handled in the same manner. Now $\tau(a_1, a_i) = (a_n, a_{n-i+1}) = \sigma^{n-i}(a_i, a_1)$, so $\tau(a_1, a_i) \in E(X)$ if and only if $(a_i, a_1) \in E(X)$, $i = 2, 3, \ldots, n$. Thus $\tau \in G(X)$ if and only if X is symmetric.

Both Theorem 2.3 and 2.4 are quite simple but important to the study of groups that have directed graphs but do not have symmetric graphs. From them we see that the cyclic groups $C_n = (a_1 a_2 \ldots a_n)\text{cyc}$ fall in that category. From Corollary 1.1.1, any permutation group having C_n as a direct factor also falls in that category; in fact, the tables in §3 suggest that these groups account for most of the groups in that category.

THEOREM 2.5. *If* $(a_1 a_2 \ldots a_m)\text{pos} \subseteq G(X)$ *with* $m \geqslant 4$, *then*

$$(a_1 a_2 \ldots a_m)\text{all} \subseteq G(X).$$

Proof. For $m \geqslant 4$, $(a_1 a_2 \ldots a_m)$pos is 2-ply transitive and the result follows easily.

COROLLARY 2.5.1. *$(a_1 a_2 \ldots a_n)$pos has a directed graph for $n \geqslant 2$ if and only if $n = 2$ or 3.*

Proof. $(a_1 a_2)$pos is the identity group and has the graph X_2 with

$$E(X_2) = \{(a_1, a_2)\}.$$

$(a_1 a_2 a_3)$pos $= (a_1 a_2 a_3)$cyc and has the graph X_3 with

$$E(X_3) = \{(a_1, a_2), (a_2, a_3)(a_3, a_1)\}.$$

By the theorem, $(a_1 a_2 \ldots a_n)$pos has no directed graph for $n \geqslant 4$.

THEOREM 2.6. *If $P = \langle (a_1 a_2), R\rangle$pos $\subseteq G(X)$ for some group of permutations R (not all positive) with support set $\{a_3, \ldots, a_n\}$ such that P is independently transitive on $\{a_1, a_2\}$ with respect to $\{a_3, \ldots, a_n\}$, then $\langle (a_1 a_2), R\rangle \subseteq G(X)$.*

Proof. Let $\mu = (a_1 a_2)$ and $\tau \in R$ such that $\mu\tau \in P$. Then μ acts like either $\mu\tau$ or the identity on all ordered pairs except some of those of the form (a_i, p) or (p, a_i) for $i = 1, 2$; $p \in \{a_3, \ldots, a_n\}$. But by our assumption on P, there exists $\alpha \in P$ such that $\alpha(a_1) = a_2$ and $\alpha(p) = p$; so $\mu(a_i, p) = \alpha(a_i, p)$. Thus $\mu \in P$ and the result follows.

In connection with this theorem, we note by Theorem 1.1 that $\langle (a, a_2), R\rangle$ has a graph if and only if R has a graph.

COROLLARY 2.6.1. *$\langle (a_1 a_2), (a_3, a_4), \ldots, (a_{n-1} a_n)\rangle$pos with $n = 2k$ has a directed graph if and only if $k = 1$ or 2.*

Proof. If $k > 2$, the theorem applies and the given group has no directed graph. The case $k = 1$ is trivial and

$$\langle (a_1 a_2), (a_3 a_4)\rangle\text{pos} = \{1, (a_1 a_2)(a_3 a_4)\}$$

has the directed graph X with

$$V(X) = \{a_1, a_2, a_3, a_4\} \text{ and } E(X) = \{(a_1, a_3), (a_3, a_4), (a_4, a_2)\}.$$

Note that the theorem does not apply when $k = 2$ because of the independently transitive condition.

In connection with this corollary, we note that $(a_1 a_2) \ldots (a_{n-1} a_n)$ has a symmetric graph for all even n.

THEOREM 2.7. *If $\tau = (a_1 a_2)(a_3 a_4)$ and $\sigma = (a_3 a_4)(a_5 a_6) \in G(X)$, then $\langle (a_1 a_2), (a_3 a_4), (a_5 a_6)\rangle \subseteq G(X)$.*

Proof. Let $\mu = (a_1 a_2)$. Then μ acts the same as either τ or the identity on ordered pairs with the exception of (a_1, a_3), (a_3, a_1), (a_1, a_4), (a_4, a_1), (a_2, a_3), (a_3, a_2), (a_2, a_4), and (a_4, a_2). Now

$$\mu(a_1, a_3) = (a_2, a_3) = \tau(a_1, a_4) = \tau[\sigma(a_1, a_3)] = \tau\sigma(a_1, a_3),$$

so $\mu(a_1, a_3) \in E(X)$ if and only if $(a_1, a_3) \in E(X)$. The remaining exceptions are handled in the same manner, so $\mu \in G(X)$. Therefore $\mu\tau = (a_3, a_4)$ and $\mu\tau\sigma = (a_5, a_6) \in G(X)$ and the theorem follows.

THEOREM 2.8. *If* $P = \{(a_1 a_2 \ldots a_m)\text{all } (a_{m+1} a_{m+2} \ldots a_n)\text{all}\}\text{pos} \subseteq G(X)$ *with* $3 \leqslant m < n$ *and* $5 \leqslant n$, *then* $G(X) = G_1, G_2$, *or* G_3 *where*

$$G_1 = (a_1 \ldots a_m)\text{all } (a_{m+1} \ldots a_n)\text{all}, \quad G_2 = (a_1 a_{m+1} . a_2 a_{m+2} . \ldots . a_m a_{2m})G_1$$

with $n = 2m$, *and* $G_3 = (a_1 \ldots a_n)\text{all}$.

The theorem also holds for $n = 2$, $m = 1$ but easy counterexamples can be found for the other excepted values, namely, $n = 3$, $m = 1$, or 2 and $n = 4$, $m = 1, 2$, or 3.

Proof. P has the transitivity sets $T_1 = \{a_1, \ldots, a_m\}$ and $T_2 = \{a_{m+1}, \ldots, a_n\}$, so $G(X)$ either is transitive or has the transitivity sets T_1 and T_2. In either case, $G(X)$ is independently transitive on T_1 with respect to T_2 since P is (this depends on having $m \geqslant 3$).

First assume that $G(X)$ is intransitive. Then by Theorem 1.2,

$$G(X) = G(X_1)G(X_2),$$

where X_i is the section graph of X on T_i. Thus $(a_1 \ldots a_m)\text{pos} \subseteq G(X_1)$ and $(a_{m+1} \ldots a_n)\text{pos} \subseteq G(X_2)$. If $m \geqslant 4$, then $G(X_1) = (a_1 \ldots a_m)\text{all}$ by Theorem 2.5. If $m = 3$, then $n - m \geqslant 2$; so $(a_1 a_2)(a_{m+1} a_{m+2}) \in P \subseteq G(X)$; hence $(a_1 a_2) \in G(X_1)$ and again $G(X_1) = (a_1 a_2 a_3)\text{all}$. Thus for $m \geqslant 3$, we have

$$G(X_1) = (a_1 a_2 \ldots a_m)\text{all}.$$

Now for $n - m \geqslant 2$ we have $(a_1 a_2)(a_{m+1} a_{m+2}) \in P \subseteq G(X)$ so

$$(a_{m+1} a_{m+2}) \in G(X_2)$$

and $G(X_2) = (a_{m+1} \ldots a_n)\text{all}$. If $n = m + 1$, then $G(X_2) = (a_m)\text{all}$. In any case $G(X) = G(X_1)G(X_2) = G_1$.

The other possibility is that $G(X)$ is transitive. As in the proof of Theorem 1.2 we see that either all or none of the possible edges from X_1 to X_2 are present and likewise from X_2 to X_1. But now $G(X)$ is transitive, so X is regular and these edges are present from X_1 to X_2 if and only if they are present from X_2 to X_1. So by taking complements if necessary, we can assume that none of these edges are present. Thus as in the last paragraph we get

$$(a_1 \ldots a_m)\text{all } (a_{m+1} \ldots a_n)\text{all} \subseteq G(X).$$

Hence either X_1 is the null graph or the complete m point and X_2 is the null graph or the complete $n - m$ point. By the transitivity of $G(X)$, $E(X_1)$ has an element if and only if $E(X_2)$ does. The existence of edges results in $G(X) = G_2$ and the absence of edges results in $G(X) = G_3$.

THEOREM 2.9. *$\langle (a_1 \ldots a_m)(a_{m+1} \ldots a_{2m}) \rangle$ has a directed graph.*

Proof. It has the graph X with $E(X) = A_1 \cup A_2 \cup E$, where A_1 contains the edges in the directed circuit $(a_1, a_2, \ldots, a_m, a_1)$, A_2 contains the edges in the directed circuit $(a_{m+1}, a_{m+2}, \ldots, a_{2m}, a_{m+1})$, and

$$E = \{(a_i, a_{m+i}) : i = 1, 2, \ldots, m\}.$$

THEOREM 2.10. *$G_{4,3} = (abcd)_4$ and $G_{6,9} = (abcdef)_6$ have no directed graphs.*

We omit the proof since the methods resemble those previously used in this section.

3. Groups of degree six or less and directed graphs.

We shall separate the permutation groups into three categories: (I) those that have symmetric graphs (for these graphs see (**3**, pp. 516–520)), (II) those that have directed graphs but do not have symmetric graphs, and (III) those that do not have directed graphs. A thoerem appearing in parentheses after a group listed in the second (third) category is one showing that the group has (does not have) a directed graph. It is demonstrated in (**3**, pp. 516–520) that those in (II) do not have symmetric graphs.

The groups involved will be labelled $G_{p,q}$, where $G_{p,q}$ is the qth group of degree p in the listing given in (**3**, pp. 516–520).

(I) *Groups that have symmetric graphs*:

$G_{2,1} = (ab)$;		
$G_{3,2} = (abc)\text{all}$;		
$G_{4,1} = (ab \cdot cd)$;	$G_{4,2} = (ab)(cd)$;	$G_{4,5} = (abcd)_8$;
$G_{4,7} = (abcd)\text{all}$;		
$G_{5,4} = (abcde)_{10}$;	$G_{5,5} = (abc)\text{all}(de)$;	$G_{5,8} = (abcde)\text{all}$;
$G_{6,1} = (ab \cdot cd \cdot ef)$;	$G_{6,3} = (ab \cdot cd)(ef)$;	$G_{6,5} = \{(abcd)_4(ef)\}\text{dim}$;
$G_{6,8} = (abc \cdot def)\text{all}$;	$G_{6,10} = (ab)(cd)(ef)$;	$G_{6,13} = \{(abcd)_8\text{com}(ef)\}\text{dim}$;
$G_{6,17} = (abcdef)_{12}$;	$G_{6,19} = (abcd)_8(ef)$;	$G_{6,28} = (abc)\text{all}(def)\text{all}$;
$G_{6,31} = (abcd)\text{all}(ef)$;	$G_{6,32} = (abcdef)_{48}$;	$G_{6,34} = (abcdef)_{72}$;
$G_{6,37} = (abcdef)\text{all}$.		

(II) *Groups having directed graphs but not symmetric graphs*:

$G_{3,1} = (abc)\text{cyc}$	(Th. 2.3);		
$G_{4,4} = (abcd)\text{cyc}$	(Th. 2.3);		
$G_{5,1} = (abcde)\text{cyc}$	(Th. 2.3);	$G_{5,2} = (abc)\text{cyc}(de)$	(Th. 1.1, 2.3);
$G_{6,2} = (abc \cdot def)\text{cyc}$	(Th. 2.9);	$G_{6,6} = \{(abcd)\text{cyc}(ef)\}\text{pos}$	(Th. 2.2);
$G_{6,7} = (abcdef)\text{cyc}$	(Th. 2.3);	$G_{6,11} = (abcd)\text{cyc}(ef)$	(Th. 1.1, 2.1, 2.3);
$G_{6,16} = (abc)\text{cyc}(def)\text{cyc}$	(Th. 1.1, 2.3);	$G_{6,20} = (abc)\text{all}(def)\text{cyc}$	(Th. 1.1, 2.1, 2.3);
$G_{6,22} = (abcdef)_{18}$	(Th. 2.2);	$G_{6,27} = (abcdef)_{24_5}$	(Th. 2.2).

(III) *Groups having no directed graph*:

$G_{4,3} = (abcd)_4$	(Th. 2.10);	$G_{4,6} = (abcd)\text{pos}$	(Cor. 2.5.1);
$G_{5,3} = \{(abc)\text{all}(de)\}\text{pos}$	(Th. 2.6);	$G_{5,6} = (abcde)_{20}$	(Th. 2.1);
$G_{5,7} = (abcde)\text{pos}$	(Cor. 2.5.1);		
$G_{6,4} = \{(ab)(cd)(ef)\}\text{pos}$	(Cor. 2.6.1);	$G_{6,9} = (abcdef)_6$	(Th. 2.10);

https://doi.org/10.4153/CJM-1966-023-2 Published online by Cambridge University Press

$G_{6,12}$	$= (abcd)_4(ef)$	(Th. 1.1);	$G_{6,14}$	$= \{(abcd)_8\mathrm{cyc}(ef)\}\mathrm{dim}$	(Th. 1.2);
$G_{6,15}$	$= \{(abcd)_8\mathrm{pos}(ef)\}\mathrm{dim}$	(Th. 1.2);	$G_{6,18}$	$= (abcdef)_{12_2}$	(Th. 2.7);
$G_{6,21}$	$= \{(abc)\mathrm{all}(def)\mathrm{all}\}\mathrm{pos}$	(Th. 2.8);	$G_{6,23}$	$= (abcd)\mathrm{pos}(ef)$	(Th. 1.1);
$G_{6,24}$	$= \{(abcd)\mathrm{all}(ef)\}\mathrm{pos}$	(Th. 2.6);	$G_{6,25}$	$= (\pm abcdef)_{24}$	(Th. 2.7);
$G_{6,26}$	$= (+abcdef)_{24}$	(Th. 2.7);	$G_{6,29}$	$= (abcdef)_{36}$	(Th. 2.8);
$G_{6,30}$	$= (abcdef)_{36_3}$	(Th. 2.4);	$G_{6,33}$	$= (abcdef)_{60}$	(Th. 2.7);
$G_{6,35}$	$= (abcdef)_{120}$	(Th. 2.4);	$G_{6,36}$	$= (abcdef)\mathrm{pos}$	(Cor. 2.5.1).

References

1. Chong-Yun Chao, *On a theorem of Sabidussi*, Proc. Amer. Math. Soc., *15* (1964), 291–292.
2. R. Frucht, *Herstellung von Graphen mit vorgegebener abstrakter Gruppe*, Compositio Math., *6* (1938), 239–250.
3. I. N. Kagno, *Linear graphs of degree* $\leqslant 6$ *and their groups*, Amer. J. Math., *68* (1946), 505–520. Corrections, Amer. J. Math., *77* (1955), 392.
4. G. A. Miller, *Memoir on the substitution groups whose degree does not exceed eight*, Amer. J. Math., *21* (1899), 288–337.
5. O. Ore, *Theory of graphs* (Providence, R.I., 1962).
6. H. Zassenhaus, *The theory of groups*, 2nd ed. (New York, 1958).

Vanderbilt University
Nashville, Tennessee

https://doi.org/10.4153/CJM-1966-023-2 Published online by Cambridge University Press