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The weak closure of the set of singular elements in a Banach algebra

J. D. Gray

In this note it is proved that for a certain class of infinite dimensional Banach algebras the set of singular elements (the non-units) is dense in the weak topology.

It is well known and easily proven (Rickart, [2], p. 12), that in any (complex) Banach algebra B, with identity, the set S of singular elements (the non-units) is closed in the norm topology. In some recent work of the author on a generalization of the operational calculus for Banach algebras it became important to know something of the topological nature of S when B is equipped with the weak topology. This topology has as a basis sets of the form

$$\{\xi \in B : |x^*(x)-x^*(\xi)| < \varepsilon \ ; \ x^* \in A\}$$

where $x \in B$, $\varepsilon > 0$ and A is a finite set in the dual space B^* of continuous linear functionals on B. If B (as a vector space) has finite dimension, the weak and the norm topology coincide, and so, in this case, S is closed in the weak topology.

For a certain class of algebras we have a partial converse to this result.

THEOREM. *Suppose B is an infinite dimensional, semi-simple, commutative Banach algebra with identity, for which the Gelfand map is surjective. Then S is weakly dense in B.*

Proof. To see that this is a partial converse to the above statement

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we note that S is always a proper subset of B, and so, if it is dense it is not closed. The Gelfand map is that well known homomorphism $B \to C(X)$ of B into the Banach algebra $C(X)$ of all continuous, complex-valued functions on the compact Hausdorff space X of maximal ideals of B. As B is semi-simple, this homomorphism is injective, and thus, by assumption, bijective. By the open mapping theorem we conclude that it is a homeomorphism. Therefore, by Theorem V.3.15 of Dunford and Schwartz [1], it is a homeomorphism when both B and $C(X)$ have the weak topology. Thus it suffices to prove the theorem for the algebra $C(X)$. Now a function in $C(X)$ is singular if and only if it vanishes at some point of X. The problem is then: given $f \in C(X)$ (which we may assume *not to be identically zero*), *show that every neighbourhood of* f *in the* weak topology contains a function which vanishes somewhere in X. It suffices therefore to exhibit a net $\{f_\alpha\}$ of singular elements with $\lim_\alpha f_\alpha = f$.

f may be written in a unique way as $\phi + i\psi$, where $\phi, \psi : X \to R$ are continuous. Now, as $C(X)$ is infinite dimensional, X is an infinite set, and as it is also compact, we conclude that there is a point $p \in X$ which is not isolated. Let U be a neighbourhood basis for p - indexed by some well-ordered set Γ', so $U = \{U'_\alpha : \alpha \in \Gamma'\}$. Choose an $\alpha_0 \in \Gamma'$ and define $U_{\alpha_0} = U'_{\alpha_0}$. If $\beta \geq \alpha_0$ and U_β has been defined, define inductively $U_{\beta+} = U'_{\beta+} \cap U_\beta$, where $\beta+$ is the least element of the set $\{\gamma \in \Gamma' : \gamma > \beta\}$. Then the family $\{U_\alpha : \alpha \geq \alpha_0\}$ is also a neighbourhood basis of open sets for p. Furthermore, if $\beta > \alpha \geq \alpha_0$ we have $U_\beta \supset U_\alpha$, and each U_α contains a point other than p (as p is not isolated). Next, because X is Hausdorff, we have $\bigcap_{\alpha \geq \alpha_0} U_\alpha = \{p\}$. For convenience we let Γ be the directed set $\{\alpha \in \Gamma' : \alpha \geq \alpha_0\}$. For each $\alpha \in \Gamma$ let $F_\alpha = X - U_\alpha$, so that F_α is closed. Also, let $\hat{F}_\alpha$ be the closed set $F_\alpha \cup \{p\}$. For $\alpha \in \Gamma$ we may, by construction, choose a point $p_\alpha \in U_\alpha$ in such a way that $p_\alpha \notin U_{\alpha+}$. X, being a compact Hausdorff space, is also normal, and hence, by Urysohn's Lemma, for each $\alpha \in \Gamma$, we may choose a continuous real-valued

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function g_α on X so that $0 \le g_\alpha \le 1$; $g_\alpha(p_\alpha) = 1$ and g_α vanishes on $\hat{F}_\alpha$. Now define $\phi_\alpha : X \to R$ by

$$\phi_\alpha(\lambda) = (1-g_\alpha(\lambda))\phi(\lambda) \; ; \quad \lambda \in X \, .$$

Then ϕ_α is continuous; $-\phi \le \phi_\alpha \le \phi$; $\phi_\alpha(p_\alpha) = 0$ and $\phi_\alpha|\hat{F}_\alpha = \phi|\hat{F}_\alpha$. In exactly the same manner we construct a continuous function $\psi_\alpha : X \to R$ with $-\psi \le \psi_\alpha \le \psi$; $\psi_\alpha(p_\alpha) = 0$ and $\psi_\alpha|\hat{F}_\alpha = \psi|\hat{F}_\alpha$. Write $f_\alpha = \phi_\alpha + i\psi_\alpha$ so that each f_α is a singular element of $C(X)$.

The net $\{f_\alpha : \alpha \in \Gamma\}$ will converge to f in the weak topology if for each continuous linear functional x^* on $C(X)$, and for each $\varepsilon > 0$, there is an $\alpha_1 \in \Gamma$ so that $\alpha > \alpha_1$ implies that

$$|x^*(f) - x^*(f_\alpha)| < \varepsilon \, .$$

The Riesz Representation Theorem ([1], Theorem IV.6.3) asserts the existence of an isometric isomorphism between $C(X)^*$ and the Banach space of regular, countably-additive, complex-valued measures on the Borel sets of X . Further, if μ is such a measure,

$$x^*(g) = \int_X g d\mu$$

for all $g \in C(X)$. Thus

$$|x^*(f)-x^*(f_\alpha)| \le \left|\int_{F_\alpha} (f-f_\alpha)d\mu\right| + \left|\int_{U_\alpha} (f-f_\alpha)d\mu\right| \, .$$

However, for each $\alpha \in \Gamma$, the first factor above is identically zero as f and f_α agree on F_α . As for the second factor, it is

$$\le \int_{U_\alpha} |f-f_\alpha| . d\|\mu\| \, ,$$

here $\|\mu\|$ represents the total variation of μ , and, by Theorem III.5.12 of [1], $\|\mu\|$ is also a regular (positive) measure on the Borel sets of X . Now suppose that the $\|\mu\|$-measure of the point p is zero. Then $\inf_V \|\mu\|(V) = 0$ – the infimum being taken over all open sets V

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containing p . Thus we can choose such an open set V with $\|\mu\|(V) < \varepsilon/2\|f\|$. But $\{U_\alpha\}$ is a basis for the open sets containing p , and as it is also decreasing, there is an $\alpha_1 \in \Gamma$ such that, if $\alpha > \alpha_1$ we have $U_\alpha \subset V$ and so $\|\mu\|(U_\alpha) < \varepsilon/2\|f\|$. Hence

$$\int_{U_\alpha} |f-f_\alpha| . d\|\mu\| \le \|f-f_\alpha\| \int_{U_\alpha} d\|\mu\| \le \|f-f_\alpha\| . \|\mu\|(U_\alpha) \le 2\|f\| . \|\mu\|(U_\alpha) < \varepsilon$$

provided $\alpha > \alpha_1$. Now suppose that $\{p\}$ does not have $\|\mu\|$-measure zero, then, without loss of generality, we may assume that $\|\mu\|(\{p\}) = 1$ and that $\int_{\{p\}} (f-f_\alpha)d\mu = 0$ (remember that $f(p) = f_\alpha(p)$). Then

$$\int_{U_\alpha} |f-f_\alpha| d\|\mu\| = \int_{U_\alpha - p} |f-f| d\|\mu\| \le 2\|f\| . \|\mu\|(U_\alpha - p) .$$

By regularity we may choose an open set V containing p for which $\|\mu\|(V) < 1 + \varepsilon/2\|f\|$ so that $\|\mu\|(V-p) < \varepsilon/2\|f\|$. Arguing as before we find an α_1 such that $\|\mu\|(U_\alpha - p) < \varepsilon/2\|f\|$ provided $\alpha > \alpha_1$. This completes the proof of the theorem.

The proofs of the next results follow immediately from the theorem.

COROLLARY 1. *Suppose* B *is an infinite dimensional, commutative,* B^**-algebra with identity. Then* S *is weakly dense in* B .

COROLLARY 2. *Let* X *be an infinite compact Hausdorff space. Suppose* X *contains a non-isolated point with a countable neighbourhood basis. Then* S *is weakly sequentially dense in* $C(X)$.

COROLLARY 3. *Suppose the Banach algebra* B *satisfies the conditions of the theorem. Then, in the weak topology, the group of units of* B *has empty interior.*

Finally, let B be the Banach algebra of bounded, complex-valued functions on a set Ω . As mentioned in [*2*], p. 295, the group G of units of B is open and dense in the norm topology. However, in the weak topology, G has empty interior. This follows from Corollary 3 and Theorem IV.6.18 of [*1*], according to which B is (isometrically isomorphic to) an algebra $C(X)$, for a suitable compact Hausdorff space X .

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University of New South Wales,
Kensington, New South Wales.

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