

NOTE ON TD + DC $_{\mathbb{R}}$ IMPLYING AD $^{L(\mathbb{R})}$

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Abstract. A short core model induction proof of AD $^{L(\mathbb{R})}$ from TD + DC $_{\mathbb{R}}$.

§1. Introduction. There are two known proofs that TD + DC $_{\mathbb{R}}$ imply AD $^{L(\mathbb{R})}$ both due to Woodin. The later proof involves proving the stronger result of Suslin determinacy from Turing determinacy + DC $_{\mathbb{R}}$ directly [1]. Combining that with Kechris and Woodin’s theorem that Suslin determinacy in $L(\mathbb{R})$ implies AD $^{L(\mathbb{R})}$ [2], the desired result becomes an immediate corollary.

Woodin’s original proof uses an early version of the core model induction (CMI) technique. Through the work of many set theorists, the CMI has been developed into a proper framework for proving determinacy results from non-large cardinal hypotheses such as generic elementary embeddings, forcing axioms, and the failure of fine-structural combinatorial principles. The technique as it is understood in $L(\mathbb{R})$ -like models (i.e., $L(\mathbb{R}^g)$ where \mathbb{R}^g are the reals of a symmetric collapse) can be seen as an inductive method by which one proves $J_\alpha(\mathbb{R}) \models \text{AD}$ for all α . An introduction to this as well as all terminology used in this paper can be found in Schindler and Steel’s book [7].

This paper aims to prove that TD + DC $_{\mathbb{R}}$ implies AD $^{L(\mathbb{R})}$ using modern perspectives on the core model induction in $L(\mathbb{R})$. The key lemma is a modification of a well-known theorem of Kechris and Solovay to work in the TD context. Utilizing the proof of the witness dichotomy it is sufficient to just prove the $J \mapsto M_1^{\#,J}$ step of the core model induction.

§2. Rough background.

2.1. Determinacy. Given some $A \subseteq \omega^\omega$ the Gale–Stewart game $G_\omega(A)$ is defined to be the perfect information game where two players **I** and **II** take turns playing digits $x_n \in \omega$ for ω turns as written:

I	x_0	x_2	x_4	\dots
II	x_1	x_3	\dots	

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This results in the infinite sequence $x = (x_0, x_1, \dots) \in \omega^\omega$. We say that **I** wins if $x \in A$, otherwise we say **II** wins. A player is said to have a winning strategy provided they can ensure themselves a win regardless of how their opponent plays. A game $G_\omega(A)$ is determined if a player has a winning strategy.

DEFINITION 2.1. The axiom of determinacy AD states that for every $A \subseteq \omega^\omega$, $G_\omega(A)$ is determined.

One of the earliest remarkable consequences of AD is Martin’s cone theorem. We say that $\mathbf{A} \subseteq \mathcal{D}$ is a cone if there is some $\mathbf{x} \in \mathcal{D}$ such that $\mathbf{A} = \{\mathbf{a} \mid \mathbf{a} \geq_{\mathcal{D}} \mathbf{x}\}$.

THEOREM 2.2 [4]. Assume AD. Suppose that $\mathbf{A} \subseteq \mathcal{D}$. Then either \mathbf{A} or \mathbf{A}^c contains a Turing cone.

PROOF. Consider the set $A = \{x \in \mathbb{R} \mid [x]_T \in \mathbf{A}\}$. As this is a set of reals it’s determined, so assume that player I has a winning strategy σ . Then for any $x \geq_T \sigma$ we have that $x \equiv_T \sigma * x \in A$. Therefore, $\{x \mid x \geq_T \sigma\} \subseteq A$ and $\{\mathbf{x} \mid \mathbf{x} \geq_{\mathcal{D}} [\sigma]_T\} \subseteq \mathbf{A}$. If player II has a winning strategy τ , then there is an almost identical argument that \mathbf{A}^c contains a Turing cone above $[\tau]_T$. ⊔

The axiom of Turing determinacy (TD) is the isolation of the consequence of this theorem, e.g., that every set of Turing degrees contains or is disjoint from a cone.

2.2. Core model theory. The core model theory required in a core model induction is largely summarized by the K^J existence dichotomy. This is a straightforward generalization of the typical K existence dichotomy to a hierarchy of relativized mice. These relativized mice, called hybrid mice, abstract the use of the rudimentary closure to take a one step in the constructibility hierarchy to the use of some “model operator” J with similar enough properties. Examples of model operators include:

- $x \mapsto \text{rud}(x \cup \{x\})$;
- Mouse operators, e.g., the sharp operator, the one Woodin cardinal operator;
- Hybrid mouse operators, e.g., term-relation hybrid mouse operators from self-justifying systems, strategy hybrid mouse operators.

For an exposition of operator mice in the style of this paper one can read either Chapter 1 of [7] or Sections 2.1–2.3 of [11]. For the sake of this paper, the full definition of a mouse operator is not that important.

An important property all model operators of interest have is that they “condense well.” Condensing well is a technical condition ensuring that one can develop a fine structure for structures built in terms of J , i.e., one can perform background constructions relativized to J that behave in the same manner as they do with the rudimentary closure. In particular, J condensing well implies that the models constructed in a $K^{c,J}$ construction are J -premise and there is a relativized core model theory.

THEOREM 2.3. (K^J existence dichotomy) Let Ω be a measurable cardinal. Let J be a model operator with real parameter z on H_Ω which condenses well. Let \mathcal{P} be countable model with parameter z and let $K^{c,J}(\mathcal{P}) = K^{c,J}(\mathcal{P}) \upharpoonright \Omega$. Then the following statements are true:

1. If the $K^{c,J}(\mathcal{P})$ construction reaches $M_1^{\#,J}(\mathcal{P})$, then $M_1^{\#,J}(\mathcal{P})$ is $(\omega, \Omega, \Omega + 1)$ -iterable via the unique strategy guided by $J^\#$, i.e., the sharp for L^J .

2. If the $K^{c,J}(\mathcal{P})$ construction does not reach $M_1^{\#,J}(\mathcal{P})$, then $K^{c,J}(\mathcal{P})$ is $(\omega, \Omega, \Omega + 1)$ -iterable. This implies that $K^J(\mathcal{P})$ exists and is $(\omega, \Omega, \Omega + 1)$ -iterable via the unique strategy guided by $J^\#$.

In this case, the “true” $K^J(\mathcal{P})$ is defined as in [9], the only real change being that one has to relativize all notions considered there to the model operator J .

All model operators encountered in the core model induction condense well. Additional properties possessed by all model operators encountered in the core model induction are that they relativize well and determine themselves on generic extensions.

DEFINITION 2.4. We say that a model operator J *relativizes well* if there is a formula $\varphi(x, y, z)$ such that for any $\mathcal{N}, \mathcal{N}'$ models such that $\mathcal{N} \in |\mathcal{N}'|$ and \mathcal{M} a J -premouse with base model \mathcal{N}' such that $\mathcal{M} \models \text{ZFC}^-$, then $J(\mathcal{N}) \in \mathcal{M}$ and $J(\mathcal{N})$ is the unique $x \in |\mathcal{M}|$ such that $\varphi(x, \mathcal{N}, J(\mathcal{N}'))$.

DEFINITION 2.5. We say that J *determines itself on generic extensions* if there is a formula $\phi(x, y, z)$ and some parameter $c \in HC$ such that for any countable transitive structure \mathcal{M} satisfying ZFC^- containing c and closed under J , for any generic extension $\mathcal{M}[g]$ of \mathcal{M} in V , $J \cap \mathcal{M}[g] \in \mathcal{M}[g]$ and is definable via $b = J(a)$ iff $\mathcal{M}[g] \models \phi(c, a, b)$.

§3. Kechris–Solovay theorem. The following is our primary lemma which is a modification on a theorem of Kechris and Solovay [3]. Given a set of ordinals S , OD_S -Turing determinacy (i.e., OD_S -TD) is the assertion that every set of Turing degrees ordinal definable with S as an additional parameter contains or is disjoint from a Turing cone.

LEMMA 3.1. Assume TD. For any set of ordinals S , on a Turing cone \mathcal{C} the following holds for $x \in \mathcal{C}$:

$$L[S, x] \models \text{OD}_S\text{-TD}.$$

PROOF. Assume for a contradiction that there is no cone of reals on which $L[S, x] \models \text{OD}_S\text{-TD}$. Then we can define, on a cone \mathcal{C} , the map $x \mapsto A_x$ where A_x is the least $\text{OD}_S^{L[S,x]} \equiv_T$ -invariant subset of \mathbb{R} which doesn't contain a Turing cone and whose complement does not contain a Turing cone in $L[S, x]$. Notice that A_x only depends on the S -constructibility degree of x .

It is clear from the last observation that the set $\{x \in \mathbb{R} \mid x \in A_x\}$ is \equiv_T -invariant and is well-defined on \mathcal{C} . Suppose that this set contains a Turing cone \mathcal{C}' . Consider some arbitrary $y \in \mathcal{C} \cap \mathcal{C}'$, if $w \geq_T z \geq_T y$ and $w \in L[S, z]$ then we have that $w \in A_w = A_z$. So A_z contains a Turing cone in $L[S, z]$. We reach a similar conclusion if we assume that $\{x \in \mathbb{R} \mid x \notin A_x\}$ contains a Turing cone. Contradiction. \dashv

Ordinal (Turing) determinacy has the following well-known consequence:

COROLLARY 3.2. $\text{HOD}_S^{L[S,x]} \models$ “ $\omega_1^{L[S,x]}$ is measurable” for a cone of x .

PROOF OUTLINE. This proof is standard, but for the sake of completeness it will be sketched. We work inside $L[S, x]$ and assume that $\text{OD}_S\text{-TD}$ holds. Let f be the function $f : x \mapsto \omega_1^x$ which maps x to the least x -admissible ordinal,

then we can define μ as the pushforward of the cone filter under f , i.e., $A \in \mu$ iff $f^{-1}(A)$ contains a Turing cone. Clearly μ is countably closed and, by assumption, μ restricts to an ultrafilter on $P(\omega_1) \cap \text{OD}_S$. The cone measure and the map f are both definable, so $\text{HOD}_S \cap \mu \in \text{HOD}_S$ witnesses that ω_1^V is measurable in HOD_S . \dashv

NOTE 3.3. *The proofs of Theorem 3.1 and Corollary 3.2 don't actually rely on any essential property of L that is not shared by L^J where J is some (hybrid) model operator. Strictly speaking, the L^J variants are what are used in Section 4.*

§4. The existence of $M_1^\#$. The following consequences of Δ_2^1 -TD are proven by modifying the analogous arguments from Δ_2^1 -Det in a similar fashion to the core argument of Theorem 3.1 then verifying that nothing breaks. As the modifications are relatively straightforward the proofs will not be included.

- (Martin) Assume Δ_2^1 -TD + DC, then Π_2^1 -TD.
- (Kechris–Solovay) Assume Δ_2^1 -TD + DC, then for any real y , on a Turing cone $x \in \mathcal{C}$ the following holds for $x \in \mathcal{C}$:

$$L[x, y] \models \text{OD}_y\text{-TD.}$$

- (Consequence of Kechris–Woodin [2]) Assume Δ_2^1 -TD + DC and $(\forall x \in \mathbb{R}) x^\#$ exists, then $\text{Th}(L[x])$ is fixed on a Turing cone.

We can utilize these three observations to prove the first step in our induction from a weaker hypothesis.

THEOREM 4.1. *Assume Δ_2^1 -TD + DC and $(\forall x \in \mathbb{R}) x^\#$ exists, then $M_1^\#$ exists and is ω_1 -iterable.*

PROOF. Utilizing lemma 3.1 we have that $L[x] \models \text{OD-TD}$ on a Turing cone. Let x be the base of such a cone and fix the least x -indiscernible $i_0^x < \omega_1^V$. The measure U on i_0^x given by $x^\#$ is sufficient for the construction of the Steel core model K (as described in *CMIP* [9]) in $L[x]$ below i_0^x . Suppose that the K^c construction below i_0^x in $L[x]$ does not reach an M_1 -like premouse, then there is a U -measure one set of $\alpha < i_0$ we have that $L[x] \models (\alpha^+)^K = \alpha^+$. Select such an α and let $z = \langle x, g \rangle$ where g is $L[x]$ -generic for $\text{Coll}(\omega, \alpha)$. Working in $L[z]$, K exists, is inductively definable, and ω_1 is a successor cardinal in K . As $z \geq_T x$ we have that $L[z] \models \text{OD-TD}$; therefore, $\text{HOD}^{L[z]} \models \omega_1$ is measurable. But as $K^{L[z]} \subseteq \text{HOD}^{L[z]}$ we have a contradiction. Therefore we have that the K^c construction of $L[x]$ below i_0^x reaches an M_1 -like premouse on a cone.

Utilizing the limit branch construction described in Theorem 4.16 of *HOD as a Core Model* [10] there is an ω_1 -iterable $M_1^\#$. \dashv

Note that by [5], the ω_1 -iterability of $M_1^\#$ is enough to prove Δ_2^1 -determinacy. So in fact we get an equivalence of Δ_2^1 -TD and Δ_2^1 -determinacy under $\text{ZF} + \text{DC}$.

§5. The $J \mapsto M_1^{\#,J}$ step. Following this point on the argument is identical to that of Steel and Schindler. But I will rewrite it (practically verbatim) for the sake of completeness. For the rest of this argument we will assume $\text{ZF} + \text{TD} + \text{DC} + V = L(\mathbb{R})$. Recall that every (hybrid) model operator considered in the core model induction relativizes well and determines itself on generic extensions.

THEOREM 5.1. *Let $a \in \mathbb{R}$ and let J be a (hybrid) model operator that condenses well, relativizes well, and determines itself on generic extensions, and suppose that*

$$W_x = (K^{c,J}(a))^{\text{HOD}_a^{L^{J^\#}[x]}}$$

constructed with height ω_1^V exists for a cone of x . Then there is a cone of x such that W_x cannot be $\omega_1^V + 1$ iterable above a inside $\text{HOD}_a^{L^{J^\#}[x]}$.

PROOF. Assume for a contradiction that this is not the case. Then there is a cone \mathcal{C} such that for all $x \in \mathcal{C}$ we have $L^{J^\#}[a, x] = L^{J^\#}[x]$ and

$$\omega_1^{L^{J^\#}[x]} \text{ is measurable in } \text{HOD}_a^{L^{J^\#}[x]}$$

and so we can isolate $K_x^J = (K^J(a))^{\text{HOD}_a^{L^{J^\#}[x]}}$. Furthermore, we can assume that any element of \mathcal{C} can compute the code for the parameter c which witnesses that J determines itself on generic extensions. Fixing some $x \in \mathcal{C}$ we will write K^J for K_x^J .

CLAIM 5.2. *The universe of $L^{J^\#}[x]$ is a size $< \omega_1^V$ forcing extension of $\text{HOD}_a^{L^{J^\#}[x]}$.*

PROOF. The observation we want to make is that $L^{J^\#}[x]$ is the result of adding x to $\text{HOD}_a^{L^{J^\#}[x]}$ via Vopenka forcing. Suppose that \mathbb{V} is the Vopenka forcing and τ is a name for x , then as $\text{HOD}_a^{L^{J^\#}[x]}$ is $J^\#$ -closed and contains the parameter c , it contains $L^{J^\#}(\mathbb{V}, \tau, c)$ as an inner model. Letting $g \in L^{J^\#}[x]$ be generic such that $\tau_g = x$, we have that $x \in L^{J^\#}(\mathbb{V}, \tau, a)[g]$. Using that $J^\#$ determines itself on generic extensions and relativizes well, we can then reconstruct $L^{J^\#}[x]$ inside $L^{J^\#}(\mathbb{V}, \tau, a)[g]$. So the universe of $\text{HOD}_{a,x}^{L^{J^\#}[x]}$ and $L^{J^\#}[x]$ are identical. The Vopenka forcing to add a real over $\text{HOD}_a^{L^{J^\#}[x]}$ is of size $< \omega_1^V$ as ω_1^V is inaccessible in $L^{J^\#}[x]$. \dashv

By cheapo covering and the claim we can choose some $\lambda < \omega_1^V$ above the size of the forcing such that

$$\lambda^{+K^J} = \lambda^{+\text{HOD}_a^{L^{J^\#}[x]}} = \lambda^{+L^{J^\#}[x]}.$$

Let $g \in V$ be $\text{Col}(\omega, \lambda)$ -generic over $L^{J^\#}[x]$ and let $y \in V$ be a real coding (g, x) . As J determines itself on generic extensions we have that $L^{J^\#}[y] = L^{J^\#}[x][g]$. Therefore,

$$\omega_1^{L^{J^\#}[y]} = \lambda^{+K^J} = \lambda^{+L^{J^\#}[x]}.$$

As $y \in \mathcal{C}$ we have that

$$\omega_1^{L^{J^\#}[y]} \text{ is measurable in } \text{HOD}_a^{L^{J^\#}[y]}.$$

We reach a contradiction if we can demonstrate the following claim:

CLAIM 5.3. $K^J \in \text{HOD}_a^{L^{J^\#}[y]}$.

PROOF. This claim is a verification that the proofs in Chapter 5 of CMIP [9] work given that J condenses well. K^J is still fully iterable inside $L^{J^\#}[y]$ because it has no Woodin cardinals (above a) and J condenses well so its strategy is guided by $J^\#$. This implies that K^J is still the core model above a of $L^{J^\#}[y]$, i.e., it is the common transitive collapse of $Def(W, S)$ for any W, S such that W is an ω_1^V -iterable J -weasel and ω_1^V is S -thick. Using this characterization we can conclude that $K^J \in \text{HOD}_a^{L^{J^\#}[y]}$. -|

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Utilizing this theorem we wish to show $\forall \alpha W_\alpha^*$. Suppose that for some fixed critical α , W_α^* holds, we wish to show that $W_{\alpha+1}^*$ holds. By the witness dichotomy (Theorem 3.6.1 of [7]) this means that we need to see that for all $n < \omega$, J_α^n is total on \mathbb{R} . Suppose that \mathbb{R} is closed under $J = J_\alpha^n$. To utilize Theorem 5.1 we first need to close \mathbb{R} under $J^\#$. As we're assuming DC the Martin measure ultrapower is well-founded, so as J relativizes well we have that $Ult(L^J, \mu_{Tu}) = L^J$ and $J^\#$ exists (a full proof along these lines can be found in Theorem 28 of [8]). One could avoid the use of DC by instead working inside $\text{HOD}^{L^J[x]}$ and utilizing the measurable cardinal on $\omega_1^{L^J[x]}$.

Now we can show that \mathbb{R} is closed under J_α^{n+1} . Let us fix $a \in \mathbb{R}$. By Theorem 5.1, for any (hybrid) model operator J which relativizes well and determines itself on generic extensions there is a cone of b on which

$$W_b = (K^{c,J}(a))^{\text{HOD}_a^{L^{J^\#}[b]}}$$

cannot be $\omega_1^V + 1$ iterable inside $\text{HOD}_a^{L^{J^\#}[b]}$. Let b lie in this cone, by applying the K^J existence dichotomy internal to $\text{HOD}_a^{L^{J^\#}[b]}$ we must have that the $K^{c,J}(a)$ construction reaches $M_1^{J^\#}(a)$ and $M_1^{J^\#}(a)$ is $\omega_1 + 1$ iterable in $\text{HOD}_a^{L^{J^\#}[b]}$. In summary, we can define the following map on a cone:

$$b \mapsto (M_1^{J^\#}(a))^{\text{HOD}_a^{L^{J^\#}[b]}}$$

Consider the map $f : \mathcal{D} \rightarrow \mathbb{R}$ given by

$$[b] \mapsto \text{the master code for } (M_1^{J^\#}(a))^{\text{HOD}_a^{L^{J^\#}[b]}}$$

By TD, for each $n < \omega$ the set $\{b \in \mathbb{R} : n \in f([b])\}$ either contains a cone or is disjoint from a cone. Let $n \in \mathcal{P}$ iff $n \in f([b])$ on a cone, then by countable choice for the reals $f([b]) = \mathcal{P}$ on a cone.

One can see that \mathcal{P} is ω_1 iterable in V : if \mathcal{T} is a countable tree on \mathcal{P} of limit length, then the good branch through \mathcal{T} is the one picked by the strategies of $\text{HOD}_a^{L^{J^\#}[b]}$ on a cone. Turing determinacy allows us to extend this to an $\omega_1 + 1$ -iteration strategy using the measurability of ω_1 ; therefore, \mathcal{P} is the actual $M_1^{J^\#}(a)$. From this, the map $a \mapsto J_\alpha^{n+1}(a) := M_1^{J^\#}(a)$ can be defined.

NOTE 5.4. Given a model operator J , in *The Core Model Induction* [7] the operators J^n are defined as $M_n^{J\#}$. This is not literally equal but intertranslatable with the hierarchy where $J^{n+1} = M_1^{J^n\#}$ as utilized above.

NOTE 5.5. The strongest choice principle necessary in the above argument is CC $_{\mathbb{R}}$ (which follows from TD [6]); however, the assumption $L(\mathbb{R}) \models \text{DC}$ seems to be necessary for the guts of the core model induction. In particular, it's used in both the Kechris–Woodin transfer theorem and in the A -iterability proof.

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REFERENCES

- [1] W. HUGH WOODIN, *Turing determinacy and Suslin sets*. *New Zealand Journal of Mathematics*, vol. 52 (2022), pp. 845–863.
- [2] A. S. KECHRIS and W. HUGH WOODIN, *Equivalence of partition properties and determinacy*. *Proceedings of the National Academy of Sciences*, vol. 80 (1983), no. 6, pp. 1783–1786.
- [3] A. S. KECHRIS and R. M. SOLOVAY, *On the relative consistency strength of determinacy hypothesis*. *Transactions of the American Mathematical Society*, vol. 290 (1985), no. 1, pp. 179–211.
- [4] D. A. MARTIN, *The axiom of determinateness and reduction principles in the analytical hierarchy*. *Bulletin of the American Mathematical Society*, vol. 74 (1968), no. 4, pp. 687–689.
- [5] I. NEEMAN, *Optimal proofs of determinacy II*. *Journal of Mathematical Logic*, vol. 2 (2002), no. 2, pp. 227–258.
- [6] Y. PENG and L. YU, *TD implies CCR*. *Advances in Mathematics*, vol. 384 (2021), p. 107755.
- [7] R. SCHINDLER and J. STEEL, *The core model induction*. Available at <https://ivv5hpp.uni-muenster.de/u/rds/>.
- [8] J. STEEL and S. ZOBLE, *Determinacy from strong reflection*. *Transactions of the American Mathematical Society*, vol. 366 (2014), no. 8, pp. 4443–4490.
- [9] J. R. STEEL, *The Core Model Iterability Problem*. Lecture Notes in Logic, Cambridge University Press, Cambridge, 2017.
- [10] J. R. STEEL and W. HUGH WOODIN, *HOD as a core model*, *Ordinal Definability and Recursion Theory: The Cabal Seminar, vol. 3*, Lecture Notes in Logic, Cambridge University Press, Cambridge, 2016, pp. 257–346.
- [11] T. WILSON, *Contributions to descriptive inner model theory*, Ph.D. thesis, University of California, Berkeley, 2012.

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