ON A CLASS OF SINGULAR DIFFERENTIAL OPERATORS

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In the considerable literature on linear operators in L_2 or L_p arising from ordinary differential operators it has always been assumed that the coefficient of the highest order derivative appearing does not vanish in the interior of the interval under consideration. If this coefficient vanishes at one or both endpoints of the interval, or if one or both of the endpoints is infinite the differential operator is said to be singular. In this paper we shall allow this leading coefficient to vanish in the interior of the interval, and show that the theory of such operators can sometimes be reduced to a consideration of several operators of the well-known type. We shall also indicate how those which cannot be so reduced should be dealt with.

A cursory examination of the problem leads one to the conclusion that the major change from the known situation will be in the definition of appropriate domains for such operators, and thus in the construction of appropriate boundary conditions.

Thus in § 1 we shall define the domains of basic minimal and maximal operators associated with a given differential expression, and thus define a class of operators arising from a differential expression. This class of operators will be the subject of the rest of the paper, and in § 2 we show how these operators' domains are determined by suitable boundary conditions. In § 3 we restrict to a narrower class of differential expressions in order to obtain more detailed information. For this restricted class we have a problem, which differs from the known case only in the nature of the boundary conditions. In § 4 we show that operators of this restricted class, which are formally self-adjoint, give rise to L_2 expansion theorems, and in § 5 we consider a few examples.

1. Differential operators and adjoints. We shall consider operators on $L_p(I)$ $(1 \leqslant p \leqslant \infty)$ for any interval $I = [a, b]$ (where a or b or both may be infinite), which are generated by expressions

$$(1.1) \qquad \tau = \sum_{j=0}^{n} p_j(x) D^{n-j},$$

where $D = d/dx$ and p_j is a complex-valued function belonging to $C^{n-j}(I)$. We define the adjoint differential expression by

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$$\tau^* = \sum_{j=0}^{n} q_j(x)D^{n-j} \tag{1.2}$$

$$q_j = \sum_{k=0}^{j} (-1)^{n-k}C_{n-j}^{n-k}D^{j-k}p_k.$$

Note that τ^* is an operator of the same type as τ, and that it is the conjugate of the usual Lagrange adjoint. This modification is made for convenience in dealing with Banach space adjoints, and when we consider formally self-adjoint operators in § 4 we shall return to the more usual notation.

When one assumes that $p_0(x) \neq 0$ on the interior of I, the definition of minimal and maximal operators on $L_p(I)$, which are associated with τ, is relatively direct (see, for example, Rota **(8)**). However, as we do not wish to make this assumption some modifications must be made. Denoting by $K_n(I)$ the space of all n times continuously differentiable functions which vanish outside a compact subset of I we define the operators $T^\wedge(\tau, p, I)$ on $L_p(I)$ for $1 \leqslant p \leqslant \infty$ by $T^\wedge(\tau, p, I)f = \tau f$ for $f \in K_n(I)$. Note that τf is continuous on I, and vanishes outside a compact subset if I is unbounded, so $\tau f \in L_p(I)$.

Now for any $p \geqslant 1$ $T^\wedge(\tau, p, I)$ has an adjoint $T_1(\tau^*, q, I)$ on $L_q(I)$ with domain $D_1(\tau^*, q, I)$, and $T_1(\tau^*, q, I)$ has an adjoint $T_0(\tau, p, I)$ on $L_p(I)$ with domain $D_0(\tau, p, I)$. Clearly, $T_0(\tau, p, I)$ is an extension of the closure of $T^\wedge(\tau, p, I)$. It will, in fact, be the closure unless $p = \infty$.

We shall now show how $T_1(\tau^*, q, I)$ is related to τ^*, and give a characterization of $D_1(\tau^*, q, I)$. First we define

$$\overset{*}{\tau_0}f = p_0(x)f(x) \tag{1.3}$$

$$\overset{*}{\tau}_{k+1}f = D(\overset{*}{\tau_k}f) + (-1)^{k+1}p_{k+1}(x)f(x),\ k = 0, 1, \ldots, n-1.$$

THEOREM 1.1. *$D_1(\tau^*, q, I) = \{f \in L_q(I) | \tau_k{}^*f$ is absolutely continuous on any compact subset of $I, k = 0, 1, \ldots, n-1;$ $\tau_n{}^*f \in L_q(I)\}$. Also, for $f \in D_1(\tau^*, q, I)$, $T_1(\tau^*, q, I)f = (-1)^n\tau_n{}^*f$, and if $f \in C^n$ on a neighbourhood of x_0 then $T_1(\tau^*, q, I)f = \tau^*f$ on that neighbourhood.*

Proof. By definition $f \in D_1(\tau^*, q, I)$ if and only if $f \in L_q(I)$, and there is $f_1 \in L_q(I)$ such that for any $g \in K_n(I)$ we have

$$\int_I (f\tau g - f_1 g)dx = 0. \tag{1.4}$$

Given such a g there is a compact interval $[c, d] \subset I$ outside of which g is zero. Thus if $\phi = D^n g$ we have

$$D^k g = \int_c^x \frac{(x-\xi)^{n-k-1}}{(n-k-1)!}\phi(\xi)d\xi,\ k = 0, 1, \ldots, n-1; \tag{1.5a}$$

$$\int_c^d x^k\phi(x)dx = 0,\ k = 0, 1, \ldots, n-1; \tag{1.5b}$$

and may rewrite (1.4) in the form

$$0 = \int_c^d \Big[p_0(x)f(x)\phi(x) + \sum_{j=1}^{n} p_j(x)f(x) \int_c^x \frac{(x-\xi)^{j-1}}{(j-1)!}\phi(\xi)d\xi$$
$$- f_1(x) \int_c^x \frac{(x-\xi)^{n-1}}{(n-1)!}\phi(\xi)d\xi \Big] dx.$$

When we interchange the order of integration in each term which involves two integrals this becomes

$$0 = \int_c^d \phi(x) \Big[p_0(x)f(x) + \sum_{j=1}^{n} \int_x^d \frac{(\xi - x)^{j-1}}{(j-1)!} p_j(\xi)f(\xi)d\xi$$
$$- \int_x^d \frac{(\xi - x)^{n-1}}{(n-1)!} f_1(\xi)d\xi \Big] dx$$
$$= \int_c^d \phi(x)\psi(x)\, dx.$$

Now we may choose any $\phi(x)$ fulfilling (1.5b) and define $g \in K_n(I)$ by (1.5a) (for $k = 0$) inside $[c, d]$, and by 0 outside $[c, d]$. Thus $\psi(x)$ is a function in $L_q(c, d)$ which is annihilated by all functions ϕ fulfilling (1.5b), and it must be equal almost everywhere to a polynomial of degree $n - 1$.

Thus we have

$$(1.6) \quad p_0(x)f(x) + \sum_{j=1}^{n} \int_x^d \frac{(\xi - x)^{j-1}}{(j-1)!} p_j(\xi)f(\xi)d\xi$$
$$- \int_x^d \frac{(\xi - x)^{n-1}}{(n-1)!} f_1(\xi)d\xi = P_{n-1}(x) \qquad \text{a.e. on } [c, d].$$

At this point in the known case, one alters f on a set of measure 0 so that (1.6) holds everywhere on $[c, d]$. We may do this except at points where $p_0(x)$ is zero, and shall assume that this has been done. Thus (1.6) holds except at a subset A of measure 0 of $\mathfrak{N}_0 = \{x | p_0(x) = 0\}$. Thus for any $x_0 \in A$ we have

$$\lim_{x \to x_0} p_0(x)f(x)$$

existing, where the limit is taken along a sequence of points not in A. As $f \in L_q(I)$ it cannot be infinite on an open set, so if x_0 is an interior point of $\mathfrak{N}_0$, or if it is the limit of interior points of $\mathfrak{N}_0$, this limit is 0. In any case we see that (1.6) can fail to hold only at points where $p_0(x) = 0$ and $f(x)$ is infinite, so the definition of the product was in doubt in any case. We define its value to be this limit, so that (1.6) will hold everywhere in $[c, d]$.

Thus $\tau_0^* f$ is absolutely continuous on $[c, d]$. If $\tau_k^* f$ exists and is absolutely continuous on $[c, d]$ then (1.6) yields

$$\tau_k^* f + (-1)^k \sum_{j=k+1}^{n} \int_x^d \frac{(\xi - x)^{j-k-1}}{(j-k-1)!} p_j(\xi)f(\xi)d\xi$$
$$- (-1)^k \int_x^d \frac{(\xi - x)^{n-k-1}}{(n-k-1)!} f_1(\xi)d\xi = D^k P_{n-1}(x) \quad \text{on} \quad [c, d].$$

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Thus

$$D(\tau_k^* f) + (-1)^{k+1} p_{k+1}(x) f(x) + (-1)^{k+1} \sum_{j=k+2}^{n} \int_x^d \frac{(\xi - x)^{j-k-2}}{(j-k-2)!} p_j(\xi) f(\xi) d\xi$$

$$- (-1)^{k+1} \int_x^d \frac{(\xi - x)^{n-k-2}}{(n-k-2)!} f_1(\xi) d\xi = D^{k+1} P_{n-1}(x) \quad \text{a.e. on } [c, d].$$

However, this means that the derivative $\tau_{k+1}^* f$ of an absolutely continuous function

$$\tau_k^* f + (-1)^k \int_x^d p_{k+1}(\xi) f(\xi) d\xi$$

is equal a.e. to an absolutely continuous function. It must thus be equal everywhere, and $\tau_{k+1}^* f$ exists and is absolutely continuous on $[c, d]$.

Therefore $\tau_k^* f$, for $k = 0, 1, \ldots, n-1$ are absolutely continuous on $[c, d]$ and $\tau_n^* f = (-1)^n f_1$ a.e. on $[c, d]$. As we could begin with any compact $[c, d] \subset I$ this completes the proof of the description of $D_1(\tau^*, q, I)$, and of the action of $T_1(\tau^*, q, I)$ on this domain. The fact that $T_1(\tau^*, q, I) f = \tau^* f$ if $f \in C^n$ follows from an easy computation.

For $1 \leqslant p < \infty$ it follows (see, for example, Rota **(8)**) that $T_0(\tau, p, I)$ is the closure of $T^\wedge(\tau, p, I)$, and is thus a restriction of $T_1(\tau, p, I)$. For $p = \infty$ the graph of $T_0(\tau, \infty, I)$ is the closure of the graph of $T^\wedge(\tau, \infty, I)$ in the $L_1(I) \oplus L_1(I)$ topology of $L_\infty(I) \oplus L_\infty(I)$. Thus $T_0(\tau, \infty, I)$ will, in general, be a proper extension of the closure of $T^\wedge(\tau, \infty, I)$. However, the graph of $T_1(\tau, \infty, I)$ is also closed in the $L_1(I) \oplus L_1(I)$ topology of $L_\infty(I) \oplus L_\infty(I)$, and contains the graph of $T^\wedge(\tau, \infty, I)$. Thus again $T_0(\tau, \infty, I) \subset T_1(\tau, \infty, I)$ and we have

$$T_0^*(\tau, p, I) = T_1(\tau^*, q, I) \qquad 1 \leqslant p \leqslant \infty, q = \frac{p}{p-1}$$

$$T_1^*(\tau, p, I) = T_0(\tau^*, q, I) \qquad 1 \leqslant p \leqslant \infty, q = \frac{p}{p-1}$$

$$T_0(\tau, p, I) \subset T_1(\tau, p, I) \qquad 1 \leqslant p \leqslant \infty.$$

We shall consider closed operators T on $L_p(I)$ such that $T_0(\tau, p, I) \subset T \subset T_1(\tau, p, I)$. These will be called differential operators associated with τ, or τ-operators. Clearly the adjoint of a τ-operator on $L_p(I)$ is a τ^*-operator on $L_q(I)$.

2. τ-Operators and boundary conditions. The direct way of specifying a τ-operator T on $L_p(I)$ is to give its domain $D(T)$ as a subspace of $D_1(\tau, p, I)$, which contains $D_0(\tau, p, I)$. We note that under the norm $||f||_{\tau, p} = ||T_1(\tau, p, I) f||_p + ||f||_p$ (called the τ-norm on $L_p(I)$), $D_1(\tau, p, I)$ is a Banach space, and $D_0(\tau, p, I)$ is a closed subspace. In order that T be closed $D(T)$ must also be a closed subspace of $D_1(\tau, p, I)$ under this norm. Clearly we could also specify $D(T)$ by giving the subspace of $D_1(\tau, p, I)/D_0(\tau, p, I)$ onto which it projects under the natural projection of $D_1(\tau, p, I)$ onto $D_1(\tau, p, I)/D_0(\tau, p, I)$.

As the image of an $f \in D_1(\tau, p, I)$ in the space $D_1(\tau, p, I)/D_0(\tau, p, I)$ represents the portion of f not in $D_0(\tau, p, I)$ it also represents the way in which f fails to be appropriately zero at the "boundary" of I. Thus we shall call this projection of f the *boundary value* (or *boundary values*) of f, and call the space $D_1(\tau, p, I)/D_0(\tau, p, I)$ the *space of boundary values* for τ on $L_p(I)$. This makes it natural to define a *boundary condition* for τ on $L_p(I)$ as an element of $[D_1(\tau, p, I)/D_0(\tau, p, I)]^*$, which is thus the space of boundary conditions for τ on $L_p(I)$.

Clearly, a boundary condition F for τ on $L_p(I)$ is completely specified by a linear functional on $D_1(\tau, p, I)$ which vanishes on $D_0(\tau, p, I)$ and is continuous in the τ-norm on $L_p(I)$. We shall also denote this functional by F.

THEOREM 2.1. *If F is a boundary condition for τ on $L_p(I)$ there is $g \in D_1(\tau^*, q, I)$ such that for any $f \in D_1(\tau, p, I)$*

$$F(f) = \int_I [T_1(\tau, p, I)f(x)g(x) - f(x)T_1(\tau^*, q, I)g(x)]\, dx.$$

Thus the space of boundary conditions for τ on $L_p(I)$ is isomorphic to the space of boundary values for τ^* on $L_q(I)$.

Proof. We see immediately that

$$\langle f, g\rangle = \int_I [T_1(\tau, p, I)fg - fT_1(\tau^*, q, I)g]\, dx$$

is a bilinear form on $D_1(\tau, p, I) \times D_1(\tau^*, q, I)$, which satisfies the following conditions:

$\langle f, g\rangle = 0$ for all $f \in D_1(\tau, p, I)$ if and only if $g \in D_0(\tau^*, q, I)$,
$\langle f, g\rangle = 0$ for all $g \in D_1(\tau^*, q, I)$ if and only if $f \in D_0(\tau, p, I)$,
$|\langle f, g\rangle| \leqslant ||f||_{\tau,p}||g||_{\tau^*,q}$.

Thus $\langle f, g\rangle$ induces a continuous, non-singular bilinear form on $[D_1(\tau, p, I)/D_0(\tau, p, I)] \times [D_1(\tau^*, q, I)/D_0(\tau^*, q, I)]$, and if F is any non-zero boundary condition for τ on $L_p(I)$ there is $f \in D_1(\tau, p, I)/D_0(\tau, p, I)$ for which $F(f_1) \neq 0$. Now for any $f \in D_1(\tau, p, I)/D_0(\tau, p, I)$ we have $f = F(f)f_1/F(f_1) + f_0$ where $F(f_0) = 0$. If $\mathfrak{N}$ consists of all $g \in D_1(\tau^*, q, I)/D_0(\tau^*, q, I)$, which annihilate the null-space of F, there must exist $g_1 \in \mathfrak{N}$ such that $\langle f_1, g_1\rangle \neq 0$ or $\mathfrak{N}$ would annihilate all of $D_1(\tau, p, I)/D_0(\tau, q, I)$ and $\langle f, g\rangle$ would be singular. Then $F(f) = \langle f, g_2\rangle$ where $g_2 = F(f_1)g_1/\langle f_1, g_1\rangle$.

The mapping $F \to g_2$ is clearly an isomorphism so the proof is complete.

This representation of boundary conditions allows us to show in what sense boundary values and boundary conditions are related to the boundary of the interval I.

THEOREM 2.2. *If* $p_0(x) \neq 0$ *on* $[x_1, x_2] \subset I$, *where* $a < x_1 < x_2 < b$, *the value of* $\langle f, g\rangle$ *depends only on the values of* f *and* g *on the set* $I - [x_1, x_2]$.

Proof. Since the continuous function $p_0(x)$ is non-zero on the closed interval $[x_1, x_2]$ there is a larger closed interval $[x_1', x_2']$ such that $a < x_1' < x_1 < x_2 < x_2' < b$, on which $p_0(x) \neq 0$. Thus for $f \in D_1(\tau, p, I)$, $g \in D_1(\tau^*, q, I)$ we have

$$\langle f, g\rangle = \int_{I-[x_1',x_2']} [T_1(\tau, p, I)fg - fT_1(\tau^*, q, I)g]\, dx \\ + \int_{[x_1',x_2']} [T_1(\tau, p, I)fg - fT_1(\tau^*, q, I)g]\, dx.$$

However on $[x_1', x_2']$ it is clear that f and g have absolutely continuous derivatives up to order $n - 1$ and $T_1(\tau, p, I)f = \tau f$, $T_1(\tau^*, q, I)g = \tau^* g$ on this interval. Thus the second term above is given by

$$\int_{x_1'}^{x_2'} [\tau f\, g - f\, \tau^* g]\, dx = [fg](x_2') - [fg](x_1'),$$

where

$$[fg](x) = \sum_{j=0}^{n-1} \sum_{k=1}^{n-j} (-1)^{k-1} D^{n-j-k} f(x)\, D^{k-1}(p_j(x) g(x)).$$

Thus $\langle f, g\rangle$ is given by an expression fulfilling the requirements of the theorem.

COROLLARY 2.1. $\langle f, g\rangle$ *depends only on the values of* f *and* g *in the neighbourhood of* $\mathfrak{N}_0 = \{x | p_0(x) = 0\}$ *and in the neighbourhood of the endpoints of* I.

THEOREM 2.3. *If* $p_0(x) \equiv 0$ *on* $[x_1, x_2] \subset I$ *then the value of* $\langle f, g\rangle$ *depends only on the values of* f *and* g *at points outside* $[x_1, x_2]$, *at its endpoints, and in the neighbourhood of points inside* (x_1, x_2) *where* $p_1(x)$ *is zero.*

Proof. We may assume that $p_0(x)$ is not identically zero on any interval containing $[x_1, x_2]$, although this does not alter the proof. The conditions which the restriction of $f \in D_1(\tau, p, I)$ to $[x_1, x_2]$ must satisfy are: $\tau_0 f = q_0 f = 0$, $\tau_1 f = - q_1 f$ absolutely continuous, etc. Thus it is clear that on $[x_1, x_2]$ we are really dealing with the operator

$$\tau_{[x_1,x_2]} = \sum_{j=1}^{n} p_j(x) D^{n-j},$$

and we may apply Theorem 2.2 to the boundary form $\langle f, g\rangle$ for this operator to obtain the desired result.

This reduction process can clearly be repeated to arrive at a complete characterization of the points in I at which the values of f and g are relevant to the values of $\langle f, g\rangle$. We use the notations $\mathfrak{N}_k = \{x | p_j(x) = 0, j = 0, 1, \ldots, k\}$ and $\mathfrak{N}_{k0}$ for the interior of $\mathfrak{N}_k$, and combine our results in the following theorem.

THEOREM 2.4. *The value of* $\langle f, g\rangle$ *depends only on the values of* f *and* g *in the neighbourhood of the set* $\mathfrak{B} = \{x \in I | x$ *is an endpoint of* I, *or there exists an integer* k *between* 0 *and* $n - 1$ *such that* $x \in \mathfrak{N}_k$, $x \notin \mathfrak{N}_{k0}\}$.

Thus for the differential operators τ and τ^* the set $\mathfrak{B}$ plays the role of the boundary of the interval I. The boundary values of a function in $D_1(\tau, p, I)$ or $D_1(\tau^*, q, I)$ depend only on the values of the function in a neighbourhood of $\mathfrak{B}$. Thus there are boundary conditions for τ on $L_p(I)$, which depend on values of the function away from the endpoints of I. This is the main significant difference which arises in our general class of operators.

Another remark might be made at this stage. If $f \in C^n(I) \cap L_p(I)$ and $g \in C^n(I) \cap L_q(I)$ it is easily seen that $\langle f, g\rangle$ depends only on the values of f and g near the endpoints of I. If f_n and g_m are sequences of such functions which converge in $L_p(I)$ and $L_q(I)$ to f_0 and g_0 respectively, and if τf_n and $\tau^* g_m$ converge in $L_p(I)$ and $L_q(I)$ to f_0^* and g_0^* respectively, it is clear that $f_0 \in D_1(\tau, p, I)$, $g_0 \in D_1(\tau^*, q, I)$, $T_1(\tau, p, I)f_0 = f_0^*$, $T_1(\tau^*, q, I)g_0 = g_0^*$, and that as n and m approach ∞ $\langle f_n, g_m\rangle$ converges to $\langle f_0, g_0\rangle$. Thus it is clear that $\langle f_0, g_0\rangle$ depends only on the values of f_0 and g_0 in the neighbourhood of the endpoints of I. Thus $T_1(\tau, p, I)$ and $T_1(\tau^*, q, I)$ are not given, in general, by the closures of τ and τ^* on $C^n(I) \cap L_p(I)$ and $C^n(I) \cap L_q(I)$ respectively.

3. Regular operators. In order to obtain more specific results it seems to be necessary to restrict the class of operators somewhat. The natural restriction to make, and one which we shall make throughout the remainder of this paper, is that $\mathfrak{B}$ should be finite, consisting of $\{x_0, x_1, \ldots, x_m\}$, where x_0 and x_m are the endpoints of I, and either or both may be infinite. We shall denote by I_j the interval $[x_{j-1}, x_j]$ $j = 1, \ldots, m$.

It is necessary to restrict somewhat further than this however. The *essential spectrum* of an operator T is the set $\{\lambda | T - \lambda$ *does not have closed range*$\}$, and we shall define the essential spectrum of τ to be the essential spectrum of T_0 (τ, p, I) and denote it by $\sigma_e(\tau, p, I)$. The *essential point spectrum* of τ is the point spectrum of $T_0(\tau, p, I)$ and will be denoted by $P\sigma_e(\tau, p, I)$. The essential resolvent set $\rho_e(\tau, p, I)$ is the complement of

$$\sigma_e(\tau, p, I) \cup P\sigma_e(\tau, p, I),$$

and we shall say that τ is a *regular* operator if $\mathfrak{B}$ is finite and if $\rho_e(\tau, p, I)$ is non-empty.

THEOREM 3.1. *If I^1 and I^2 are two subintervals of I such that $I^1 \cap I^2$ is a single point and $I^1 \cup I^2 = I$, then*

$$\sigma_e(\tau, p, I) = \sigma_e(\tau, p, I^1) \cup \sigma_e(\tau, p, I^2)$$

and

$$P\sigma_e(\tau, p, I) \supset P\sigma_e(\tau, p, I^1) \cup P\sigma_e(\tau, p, I^2).$$

Proof. If $\hat{x}$ is the point common to I^1 and I^2 it is finite, so for $f \in D_1(\tau, p, I)$, $D_1(\tau, p, I^1)$, or $D_1(\tau, p, I^2)$, it follows that $\tau_k f$ for $k = 0, 1, \ldots, n-1$ has a finite limit at $\hat{x}$. Thus if $f \in D_0(\tau, p, I^1)$ and $T_0(\tau, p, I^1)f = \lambda f$ it follows that for any $g \in C^n(I^1)$, which is zero on a neighbourhood of x_0, we must have

$$0 = \int_{x_0}^{\hat{x}} [T_0(\tau, p, I^1) fg - f\tau^* g]\, dx$$
$$= \lim_{\epsilon \to 0^+} \sum_{k=1}^{n} (-1)^{n-k+1} \tau_{n-k} f(\hat{x} - \epsilon)\, g^{(k-1)}(\hat{x} - \epsilon).$$

Thus for $k = 0, 1, \ldots, n-1$ the limit of $\tau_k f(x)$ as $x \to \hat{x}$ is zero, and the function f_1 which is equal to f on I^1 and is zero on I^2 belongs to $D_0(\tau, p, I)$ and has the property that $T_0(\tau, p, I) f_1 = \lambda f_1$. Thus

$$P\sigma_e(\tau, p, I^1) \subset P\sigma_e(\tau, p, I),$$

and the proof for I^2 is precisely the same.

For the other part of the theorem we note that as the null-space of $T_0(\tau, p, I) - \lambda$ is contained in the direct sum of the null-spaces of $T_1(\tau, p, I_j) - \lambda$, which are finite dimensional, the former must be finite dimensional also. Similarly the null-spaces of $T_0(\tau, p, I^1) - \lambda$ and $T_0(\tau, p, I^2) - \lambda$ must be finite dimensional. Thus results of Rota **(8)** yield our conclusion.

We might note that these results depend only on $\mathfrak{B}$ being finite, and not on $\rho_e(\tau, p, I)$ being non-empty. There are several immediate consequences of Theorem 3.1 which we list as corollaries.

COROLLARY 3.1. *If $\mathfrak{B}$ is finite*

$$\sigma_e(\tau, p, I) = \bigcup_{j=1}^{m} \sigma_e(\tau, p, I_j)$$

and

$$P\sigma_e(\tau, p, I) \supset \bigcup_{j=1}^{m} P\sigma_e(\tau, p, I_j),$$

and if $\rho_e(\tau, p, I)$ is non-empty, we have $\rho_e(\tau, p, I_j)$ non-empty for $j = 1, \ldots, m$.

COROLLARY 3.2. *$\sigma_e(\tau, p, I)$ is closed and equal to the essential spectrum of any τ-operator if $\rho_e(\tau, p, I)$ is non-empty. Also $\sigma_e(\tau, p, I)$ coincides with $\sigma_e(\tau^*, q, I)$ if $1 \leqslant p \leqslant \infty$.*

Corollary 3.1 is obvious and Corollary 3.2 follows from corresponding results of Rota **(8)** for the usual case. One cannot state that $P\sigma_e(\tau, p, I)$ is closed, and it is quite possible for $P\sigma_e(\tau, p, I_j)$ to be empty for $j = 1, \ldots, m$ and yet have $P\sigma_e(\tau, p, I)$ non-empty.

THEOREM 3.2. *If τ is regular on $L_p(I)$ or if $\mathfrak{B}$ is finite and $\rho_e(\tau, p, I_j)$ is non-empty for $j = 1, 2, \ldots, m$ then $D_1(\tau, p, I)/D_0(\tau, p, I)$ is finite dimensional.*

Proof. We note that by Corollary 3.1 τ regular on $L_p(I)$ implies that $\rho_e(\tau, p, I_j)$ is non-empty for $j = 1, 2, \ldots, m$. It is the latter which is the necessary hypothesis here, and it does not imply that $\rho_e(\tau, p, I)$ is non-empty.

Now for each j there is an integer k_j such that $p_\alpha(x) \equiv 0$ on I_j for $\alpha < k_j$ and $p_{k_j}(x) \not\equiv 0$ for x in the interior of I_j. Thus every $f \in D_1(\tau, p, I)$ belongs,

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when restricted to I_j, to the domain of an operator of order $n - k_j$, which we shall still denote by τ. On I_j τ satisfies all the hypotheses required by Rota **(8)**, so $D_1(\tau, p, I_j)/D_0(\tau, p, I_j)$ is finite dimensional. Let π_j be the projection of $D_1(\tau, p, I_j)$ onto $D_1(\tau, p, I_j)/D_0(\tau, p, I_j)$. If we define the transformation π from $D_1(\tau, p, I)$ to

$$\sum_{j=1}^{m} \oplus D_1(\tau, p, I_j)/D_0(\tau, p, I_j)$$

by $\pi f = (\pi_1 f, \pi_2 f, \ldots, \pi_m f)$ we see immediately that the null space of π is contained in $D_0(\tau, p, I)$ so that $D_1(\tau, p, I)/\mathrm{D}_0(\tau, p, I)$ is isomorphic to the quotient of $\pi D_1(\tau, p, I)$ by $\pi D_0(\tau, p, I)$, which must be finite dimensional.

The previous discussion has essentially consisted of proofs that a regular operator possesses properties very similar to those of the usual case. In fact they possess sufficiently similar properties to carry over unchanged the results on extensions proved by Rota in **(8)**.

However, further questions naturally arise. It would simplify the problem immensely if any τ-operator had the property of commuting with the projections P_j defined by $P_j f = \chi_j f$ where χ_j is the characteristic function of I_j. Even if this is not true one might wish to examine the nature of the operator $\hat{T}_j$ on $L_p(I_j)$ arising from a τ-operator T by $\hat{T}_j P_j f = P_j T f$ for $f \in D(T)$. One can hardly expect that $\hat{T}_j$ will even be closed in general, for all the functions f in its domain have $\tau_{n-1} f$ absolutely continuous on compact subintervals of I_j which contain the end point of I_j, which is interior to I. This condition is not satisfied by all $f \in D_1(\tau, p, I_j)$ in general. For the particular case when T is $T_1(\tau, p, I)$ one might hope that the closure of $\hat{T}_{1j}(\tau, p, I)$ would be $T_1(\tau, p, I_j)$.

THEOREM 3.3. *The second adjoint of* $\hat{T}_{1j}(\tau, p, I)$ *is* $T_1(\tau, p, I_j)$.

Proof. We shall assume that the terms of τ with coefficients identically zero on I_j have been omitted and thus that $p_0(x) \neq 0$ except at x_{j-1} and x_j. Thus, if g belongs to the domain of the adjoint of $\hat{T}_{1j}(\tau, p, I)$ it has absolutely continuous derivatives up to order $n - 1$ on any compact interval properly contained in I_j. Thus for any $f \in D_1(\tau, p, I)$ and C^∞-functions ϕ_1 and ϕ_2, such that $\phi_1 + \phi_2 \equiv 1$,

$$\phi_1 \equiv 1 \qquad \text{for} \qquad x \leqslant \frac{3x_{j-1} + x_j}{4},$$

and

$$\phi_1 \equiv 0 \qquad \text{for} \qquad x \geqslant \frac{3x_j + x_{j-1}}{4};$$

we have

$$0 = \langle f, g \rangle_{I_j} = \langle \phi_1 f, g \rangle_{I_j} + \langle \phi_2 f, g \rangle_{I_j}.$$

As $\phi_1 f$ and $\phi_2 f$ both belong to $D_1(\tau, p, I)$ we must also have $0 = \langle \phi_1 f, g \rangle_{I_j} = \langle \phi_2 f, g \rangle_{I_j}$, and thus

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$$0 = \lim_{\epsilon\to 0+} \sum_{k=1}^{n} (-1)^{n-k+1}\tau_{n-k}[\phi_1(x_{j-1}+\epsilon)f(x_{j-1}+\epsilon)]D^{k-1}g(x_{j-1}+\epsilon)$$

$$= \lim_{\epsilon\to 0+} \sum_{k=1}^{n} (-1)^{n-k+1}\tau_{n-k}f(x_{j-1}+\epsilon)D^{k-1}g(x_{j-1}+\epsilon),$$

and

$$0 = \lim_{\epsilon\to 0+} \sum_{k=1}^{n} (-1)^{n-k+1}\tau_{n-k}[\phi_2(x_j-\epsilon)f(x_j-\epsilon)]D^{k-1}g(x_j-\epsilon)$$

$$= \lim_{\epsilon\to 0+} \sum_{k=1}^{n} (-1)^{n-k+1}\tau_{n-k}f(x_j-\epsilon)D^{k-1}g(x_j-\epsilon).$$

To show that these conditions imply that $g \in D_0(\tau^*, q, I_j)$ we must show that if these conditions hold for $f \in D_1(\tau, p, I)$ they also hold for $f \in D_1(\tau, p, I_j)$. If x_{j-1} or x_j is an endpoint of I this implication follows immediately at that endpoint. For, let $f \in D_1(\tau, p, I_j)$ and $x_{j-1} = x_0$. Thus the function $\hat{f}$ which is $\phi_1 f$ in I_j and 0 outside I_j belongs to $D_1(\tau, p, I)$ and

$$\lim_{\epsilon\to 0+} \sum_{k=1}^{n} (-1)^{n-k+1}\tau_{n-k}f(x_{j-1}+\epsilon)g(x_{j-1}+\epsilon)$$
$$= \lim_{\epsilon\to 0+} \sum_{k=1}^{n} (-1)^{n-k+1}\tau_{n-k}\hat{f}(x_{j-1}+\epsilon)g(x_{j-1}+\epsilon) = 0.$$

The proof if $x_j = x_m$ is similar. Otherwise both x_j and x_{j-1} are finite. We shall treat only the condition at x_{j-1} as the treatment of the other is similar. If $x^0 \in I$ then

$$\tau_{n-k}f(x) = \sum_{\nu=0}^{k-1} \tau_{n-k+\nu}f(x^0)\frac{(x-x^0)^\nu}{\nu!}$$
$$+ \int_{x^0}^{x} \left\{ \frac{(x-\xi)^{k-1}}{(k-1)!}\tau_n f(\xi) - \left[\sum_{\nu=0}^{k-1} (-1)^{n-\nu} q_{n-\nu}(\xi)\frac{(x-\xi)^{k-1-\nu}}{(k-1-\nu)!} \right] f(\xi) \right\} d\xi,$$

which clearly exists for $x = x_{j-1}$. Thus we can use this expression for $x^0 = x_{j-1}$ whether $f \in D_1(\tau, p, I)$ or $D_1(\tau, p, I_j)$. Now if

$$\lim_{\epsilon\to 0+} \epsilon D^k g(x_{j-1}+\epsilon) \neq 0$$

g can hardly belong to $L_q(I_j)$ so

$$\lim_{\epsilon\to 0+} \sum_{k=1}^{n} (-1)^{n-k+1}\tau_{n-k}f(x_{j-1}+\epsilon)D^{k-1}g(x_{j-1}+\epsilon)$$
$$= \lim_{\epsilon\to 0+} \sum_{k=1}^{n} (-1)^{n-k+1} \left\{ \tau_{n-k}f(x_{j-1}) + \int_{x_{j-1}}^{x_{j-1}+\epsilon} [K_1^{(k)}(x_{j-1}+\epsilon, \xi)\tau_n f(\xi) \right.$$
$$\left. + K_2^{(k)}(x_{j-1}+\epsilon, \xi)f(\xi)]d\xi \right\} D^{k-1}g(x_{j-1}+\epsilon).$$

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It is also easy to verify that $g \in L_q(I_j)$ implies that

$$\lim_{\epsilon\to 0+} D^{k-1}g(x_{j-1}+\epsilon)\int_{x_{j-1}}^{x_{j-1}+\epsilon}[K_1^{(k)}(x_{j-1}+\epsilon,\xi)\tau_n f(\xi)+K_2^{(k)}(x_{j-1}+\epsilon,\xi)f(\xi)]d\xi = 0.$$

Thus the condition reduces to

$$0 = \lim_{\epsilon\to 0+}\sum_{k=1}^{n}(-1)^{n-k+1}\tau_{n-k}f(x_{j-1})D^{k-1}g(x_{j-1}+\epsilon).$$

We must show that if this holds for all $f \in D_1(\tau, p, I)$ it also holds for all f in $D_1(\tau, p, I_j)$. This amounts to showing that the admissible values of the vector $(\tau_0 f(x_{j-1}), \tau_1 f(x_{j-1}), \ldots, \tau_{n-1}f(x_{j-1}))$ are the same for $f \in D_1(\tau, p, I)$ and for $f \in D_1(\tau, p, I_j)$. A change in this vector amounts to changing

$$\tau_0 f(x) = \sum_{\nu=0}^{n-1}\tau_\nu f(x_{j-1})\frac{(x-x_{j-1})^\nu}{\nu!}+\int_{x_{j-1}}^{x}[K_1^{(n)}(x,\xi)\tau_n f(\xi)+K_2^{(n)}(x,\xi)f(\xi)]d\xi$$

by the addition of a polynomial $R(x)$ of degree $n-1$. As this addition could be modified by a C^n function outside of I_j, the admissible class of polynomials in both cases consists of those for which $R(x)/p_0(x)$ belongs to $L_p(I)$. Thus the admissible values of the vector are the same in both cases and we have succeeded in proving that the adjoint of $\hat{T}_{1j}(\tau, p, I)$ is $T_0(\tau^*, q, I)$. This completes the proof.

COROLLARY 3.3. *If $p \neq \infty$ the closure of $\hat{T}_{1j}(\tau, p, I)$ is $T_1(\tau, p, I_j)$.*

For a more general τ-operator T the domain $D(T)$ is determined by a finite set of boundary conditions $\alpha_1, \alpha_2, \ldots, \alpha_r$. Even if we assume that these are separated in the sense that each one depends only on values of a function near one point of $\mathfrak{B}$, it does not follow that any of them can be considered as boundary conditions on any I_j. This is due to the fact that a boundary condition at a point x_j of $\mathfrak{B}$, which is interior to I, will usually depend on values of a function on both sides of x_j. If none of the boundary conditions determining $D(T)$ can be considered as boundary conditions on I_j then it is clear that the second adjoint of $\hat{T}_j$ will again be $T_1(\tau, p, I_j)$.

One can give a more definitive answer to the question of permutability with the projections P_j.

THEOREM 3.4. *If $T_0(\tau, p, I) \subset T \subset T_1(\tau, p, I)$ then $P_jT = TP_j$ if and only if $P_jD(T) = D_0(\tau, p, I_j)$ for $j = 2, 3, \ldots, m-1$ and $F(P_jf) = 0$ for any $f \in D(T)$ and boundary condition F on I_1 or I_m, which depends only on the values of a function near x_1 or x_{m-1} respectively.*

Proof. If P_j is to leave $D(T)$ invariant it is clear that $f \in D(T)$ implies $\chi_j f \in D(T)$ so $\tau_k f(x_j) = 0$ for $k = 0, 1, \ldots, n-1$; $j = 1, 2, \ldots, m-1$. Noting that $T_1(\tau^*, q, I_j)$ is the closure of τ^* on $C^{n-k}j(I_j) \cap L_q(I_j)$ we see that for $g \in C^{n-k}j(I_j) \cap L_q(I_j)$, $\langle f, g\rangle_{I_j}$ depends only on values of f and g

near x_0 or x_m if either belongs to I_j, and is zero otherwise. Thus clearly $\chi_j f \in D_0(\tau, p, I_j)$ for $j = 2, 3, \ldots, m-1$, and $F(P_j f) = 0$ for any boundary condition F on I_1 or I_m which depends only on the values of a function near x_1 or x_{m-1} respectively. The converse implication is trivial.

Thus if T is to commute with P_j the boundary conditions must be chosen with great care, and even this will be impossible unless $P_j D_0(\tau, p, I) = D_0(\tau, p, I_j)$ for $j = 2, 3, \ldots, m$.

4. Formally self-adjoint operators on $L_2(I)$. In dealing with the L_2 case we shall conform to the standard Hilbert space notation and τ^* will now denote the conjugate of (1.2). As the q_k's are now conjugates of those in (1.2) the expressions for $\tau_k^* f$ in (1.3) must have conjugates on the p_k's. When we speak of adjoints of operators we now mean the usual Hilbert space adjoint.

We shall assume henceforward that $\tau = \tau^*$. Thus $T_0(\tau, I) = T_0(\tau, I, 2)$ is a symmetric operator and $T_0(\tau, I) \subset T_1(\tau, I) = T_0^*(\tau, I)$. As usual we are interested in discussing the spectral resolution of operators T such that $T_0(\tau, I) \subset T \subset T_1(\tau, I)$, and particularly in those which are maximal symmetric or self-adjoint. The general theory of such problems has been considered by many authors, so we shall only consider a particular aspect. Coddington **(2, 3, 4, 5)** has shown how the resolution of the identity of a self-adjoint extension, or the generalized resolution of the identity for a maximal symmetric extension, can be expressed as integral operators in such a way as to yield an expansion theorem and Parseval equality; provided that the resolvent, or generalized resolvent, is an integral operator of Carleman type. This is also related to the work of Mautner **(6)**, Bade and Schwartz **(1)**, and Nelson **(7)** on eigenfunction expansions.

We shall prove only the following theorem, which allows one to apply these results to obtain an expansion theorem.

THEOREM 4.1. *If $\mathfrak{B}$ is finite, $T_0(\tau, I) \subset T \subset T_1(\tau, I)$ where T is maximal symmetric or self-adjoint, then the generalized resolvent or resolvent of T is an integral operator of Carleman type.*

Proof. On I_j τ gives rise to a problem of the type considered by Coddington **(3, 4)**, who showed that there are either maximal symmetric extensions possessing generalized resolvents, which are integral operators of Carleman type, or self-adjoint extensions with resolvents having the same property. Let $G_j(x, \xi, \lambda)$ be the kernel of such a resolvent or generalized resolvent.

Now the essential spectrum of $T_0(\tau, I)$, and thus of $T_1(\tau, I)$, is contained in the real axis, so we can apply a result of Rota **(9)** to construct for $\mathcal{I}m\lambda > 0$ an orthonormal basis

$$\phi_1^{(j)}(x, \lambda), \ldots, \phi_{\omega_j+}^{(j)}(x, \lambda)$$

of the null-space of $T_1(\tau, I_j) - \lambda$, and a similar basis

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$$\phi_1^{(j)}(x, \lambda), \ldots, \phi_{\omega_j^-}^{(j)}(x, \lambda)$$

for $\mathcal{I}m\lambda < 0$. These bases will be analytic in λ.

Now if $g \in D_1(\tau, I)$ and $(T_1(\tau, I) - \lambda)g = f$ we see that on I_j g can differ from

$$\int_{I_j} G_j(x, \xi, \lambda)f(\xi)d\xi$$

only by a linear combination of $\phi_k^{(j)}(x, \lambda)$ ($k = 1, 2, \ldots, \omega_j(\lambda)$; $\omega_j(\lambda) = \omega_{j+}$ for $\mathcal{I}m\lambda > 0$ and $= \omega_{j-}$ for $\mathcal{I}m\lambda < 0$). Thus

$$g(x) = \sum_{k=1}^{\omega_j(\lambda)} \alpha_{jk}\phi_k^{(j)}(x, \lambda) + \int_{I_j} G_j(x, \xi, \lambda)f(\xi)d\xi, \ x \in I_j. \tag{4.1}$$

Since $g \in D_1(\tau, I)$ we must also have

$$\tau_e \int_{I_j} G_j(x_j - 0, \xi, \lambda)f(\xi)d\xi + \sum_{k=1}^{\omega_j(\lambda)} \alpha_{jk}\tau_e\phi_k^{(j)}(x_j - 0, \lambda)$$
$$= \tau_e \int_{I_{j+1}} G_{j+1}(x_j + 0, \xi, \lambda)f(\xi)d\xi + \sum_{k=1}^{\omega_{j+1}(\lambda)} \alpha_{j+1k}\,\tau_e\phi_k^{(j+1)}(x_j + 0, \lambda)$$

for $j = 1, 2, \ldots, m - 1$; $e = 0, 1, \ldots, n - 1$; where

$$\tau_e\phi_k^{(j)}(x_j - 0, \lambda)$$

and

$$\tau_e\phi_k^{(j+1)}(x_j + 0, \lambda)$$

clearly exist as the x_j's involved are finite. Similarly $\tau_e G_j(x_j - 0, \xi, \lambda)$ and $\tau_e G_{j+1}(x_j + 0, \xi, \lambda)$ exist and are in $L_2(I_j)$ and $L_2(I_{j+1})$ respectively as functions of ξ. These identities determine certain of the α_{jk}'s in terms of certain others and in terms of such expressions as

$$\int_{I_j} \tau_e G_j(x_j - 0, \xi, \lambda)f(\xi)d\xi.$$

Thus we may rewrite (4.1) in the form

$$g(x) = \sum_{k=1}^{\omega(\lambda)} \alpha_k\phi_k(x, \lambda) + \int_I G(x, \xi, \lambda)f(\xi)d\xi, \tag{4.2}$$

where the α_k's are the α_{jk}'s which remained undetermined above, and $G(x, \xi, \lambda) \in L_2(I)$ as function of ξ. Clearly the functions $\phi_k(x, \lambda)$ belong to the null-space of $T_1(\tau, I)$, and we can assume that they form a basis for this space which is orthonormal and analytic in λ.

Now if $\mathcal{I}m\lambda > 0$, $\omega(\lambda) = \omega+$, and if $\mathcal{I}m\lambda < 0$, $\omega(\lambda) = \omega-$. We may assume $\omega+ \leqslant \omega-$ so that a maximal symmetric extension of $T_0(\tau, I)$ is determined by an isometric V from $\mathfrak{N}(T_1(\tau, I) - i)$ into $\mathfrak{N}(T_1(\tau, I) + i)$ in the following manner

$$D(T_v) = \{f \in D_1(\tau, I) | f = f_0 + (I - V)f_+, f_0 \in D_0(\tau, I), f_+ \in \mathfrak{N}(T_1(\tau, I) - i)\}$$

then

$$D(T_v^*) = \{f \in D_1(\tau, I) | f = f_0 + (I - V^*)f_-, f_0 \in D_0(\tau, I), f_- \in \mathfrak{N}(T_1(\tau, I) + i)\}.$$

This clearly amounts to imposing $\omega +$ boundary conditions, and allows us to determine from (4.2) a Carleman kernel $G_v(x, \xi, \lambda)$, which is a kernel for the generalized resolvent of the maximal symmetric operator T_v.

5. Examples. Here we shall discuss briefly three examples of operators which are formally self-adjoint.

(a) If $\tau y = (xy')'$ on $I = [-1, 1]$ one finds that $\mathfrak{B} = \{-1, 0, 1\}$ and

$$\langle f, g, \rangle = \alpha_1(g)\alpha_2(f) - \alpha_2(g)\alpha_1(f) + \alpha_3(g)\alpha_4(f) - \alpha_4(g)\alpha_3(f) + \alpha_5(g)\alpha_6(f) - \alpha_6(g)\alpha_5(f),$$

where

$$\alpha_1(f) = f(1), \alpha_2(f) = f'(1), \alpha_3(f) = \lim_{x \to 0} xf'(x) = \tau_1 f(0),$$
$$\alpha_4(f) = \lim_{x \to 0+} [f(x) - f(-x)], \alpha_5(f) = f(-1), \text{and } \alpha_6(f) = f'(-1).$$

Thus this operator of the second order requires three boundary conditions to determine a self-adjoint extension on $L_2(I)$.

With the boundary conditions $\alpha_1(f) = \alpha_3(f) = \alpha_5(f) = 0$ we obtain a self-adjoint extension on $L_2(I)$ which commutes with the projections P_1 and P_2.

On the other hand, with the boundary conditions $\alpha_1(f) = \alpha_4(f) = \alpha_5(f) = 0$ we obtain a self-adjoint extension which does not commute with P_1 and P_2.

We might also note that this differential expression has particularly simple properties as the equation $(T_1(\tau, p, I) - \lambda)f = 0$ has three linearly independent solutions, which are entire functions of λ for any $p < \infty$. These are

$$u_1(x, \lambda) = \begin{cases} J_0(2(-\lambda x)^{\frac{1}{2}}) & x > 0 \\ 0 & x < 0 \end{cases}$$

$$u_2(x, \lambda) = \begin{cases} 0 & x > 0 \\ J_0(2(-\lambda x)^{\frac{1}{2}}) & x < 0 \end{cases}$$

$$u_3(x, \lambda) = Y_0(2(-\lambda x)^{\frac{1}{2}}) - \frac{2}{\pi} J_0(2(-\lambda x)^{\frac{1}{2}})(\log(-\lambda)^{\frac{1}{2}} + \gamma)$$
$$= \frac{1}{\pi} \log |x| \, J_0(2(-\lambda x)^{\frac{1}{2}}) - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(\lambda x)^n}{(n!)^2} \phi(n),$$

where

$$\phi(n) = \sum_{k=1}^{n} \frac{1}{k}.$$

For $p = \infty$ only $u_1(x, \lambda)$ and $u_2(x, \lambda)$ belong to $L_\infty(I)$. However, in this case $\alpha_3(f) = \alpha_4(f) = 0$ automatically.

(b) If $\tau y = (x^2y')' + \frac{1}{4}y$ on $I = [-1, 1]$, one finds that $\mathfrak{B} = \{-1, 0, 1\}$ again, but

$$\langle f, g, \rangle = \alpha_1(g)\alpha_2(f) - \alpha_2(g)\alpha_1(f) - \alpha_3(g)\alpha_4(f) + \alpha_4(g)\alpha_3(f),$$

with the same notation as in (a). Here any extension defined by separated boundary conditions commutes with the projections P_1 and P_2, but extensions given by non-separated boundary conditions will not have this property of course. The spectrum is purely continuous ($\lambda \geqslant -\frac{1}{4}$) for separated boundary conditions, but there may also be point spectrum in the general situation.

(c) If

$$\tau y = \begin{cases} (x^3y')' + iy' & x \geqslant 0 \\ iy' & x \leqslant 0 \end{cases} \qquad \text{on } I = [-1, 1]$$

one again finds that $\mathfrak{B} = \{-1, 0, 1\}$ but here

$$\langle f, \bar{g} \rangle = \int_{-1}^{1} (\tau f\bar{g} - f\overline{\tau g})dx = f'(1)\overline{g(1)} - f(1)\overline{g'(1)} + if(1)\overline{g(1)} - if(-1)\overline{g(-1)}.$$

Thus $T_0(\tau, I)$ has no self-adjoint extensions on $L_2(I)$, but a maximal symmetric extension will commute with P_1 and P_2 if its domain is defined by separated boundary conditions.

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