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ON THE ABSENCE OF ZEROS IN INFINITE ARITHMETIC PROGRESSION FOR CERTAIN ZETA FUNCTIONS

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Abstract

Putnam ['On the non-periodicity of the zeros of the Riemann zeta-function', *Amer. J. Math.* **76** (1954), 97–99] proved that the sequence of consecutive positive zeros of $\zeta(\frac{1}{2}+it)$ does not contain any infinite arithmetic progression. We extend this result to a certain class of zeta functions.

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1. Introduction and statement of results

In 1954 Putnam [3] proved that the set of positive zeros of $\zeta(\frac{1}{2}+it)$ does not contain any infinite arithmetic progression of the form $\{d, 2d, 3d, \ldots\}$ with $d > 0$. Later, Lapidus and van Frankenhuijsen [1] extended Putnam's theorem to a large class of zeta functions and L-series by using a different proof. Recently, Li and Radziwiłł [2] showed that at least one-third of the points in a vertical arithmetic progression are not zeros of the Riemann zeta function. Li and Radziwiłł proved this for 'inhomogeneous' arithmetic progressions of the form $\{\frac{1}{2}+i(a+nd)\}$ by investigating moments of $\zeta(s)$. Putnam's approach does not depend on such detailed information about $\zeta(s)$, but does not seem to extend to more general arithmetic progressions asymmetrically distributed about the real axis.

Since we wish to cover a variety of examples of zeta functions, we introduce an axiomatic setting. Let C be the set of meromorphic functions f in the half-plane $\sigma > 0$ of the complex plane ($z = \sigma + it$) satisfying the following conditions:

(i) f is a meromorphic function in the half-plane $\sigma > 0$ with at most one pole at $z = 1$ of order $m > 0$;
(ii) there exist a complex-valued function $A(z)$ and a real-valued function $B(x)$ such that

$$f(z) = zA(z) - z\int_0^\infty B(x)e^{-(\sigma+it)x}\,dx, \quad 0 < \sigma < 1, \tag{1.1}$$

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and also, for $n = 1, 2, \ldots$ and $d > 0$,

$$\mathrm{Re}(A(\tfrac{1}{2} + idn)) = O(n^{-\delta}) \quad \text{for some } \delta > 1, \tag{1.2}$$

$$B(x) = O(1) \quad \text{for } \log n \le x < \log(n+1) \tag{1.3}$$

and such that the one-sided limits $\lim_{h\to 0^+} B(\log n \pm h)$ exist and

$$\lim_{h\to 0^+} B(\log n + h) - \lim_{h\to 0^-} B(\log n + h) < 0. \tag{1.4}$$

We extend Putnam's theorem to the class C.

Theorem 1.1. *Let $f \in C$. Then f cannot vanish on any infinite arithmetic progression $\{\frac{1}{2} + idn : n \in \mathbb{N}\}$, where d is a positive real number.*

To recover the original case, we note that the Riemann zeta function $\zeta(s)$ belongs to the class C. Write

$$\zeta(s) = \frac{s}{s-1} - s\int_0^\infty (e^x - \lfloor e^x \rfloor)e^{-xs}\,dx, \quad 0 < \mathrm{Re}(s) < 1.$$

For $n = 1, 2, \ldots$, it is clear that $A(z) = 1/(z-1)$ satisfies $\mathrm{Re}(A(\frac{1}{2} + idn)) = O(n^{-2})$ and that $B(x) = e^x - \lfloor e^x \rfloor = O(1)$ for $\log n \le x < \log(n+1)$. Moreover,

$$\lim_{h\to 0^+} (e^{(\log n+h)} - \lfloor e^{(\log n+h)} \rfloor) - \lim_{h\to 0^-} (e^{(\log n+h)} - \lfloor e^{(\log n+h)} \rfloor) < 0.$$

Therefore, Putnam's theorem follows from our Theorem 1.1.

We show that the Hurwitz zeta function, $\zeta(s, \alpha)$, is another example which belongs to the class C.

Corollary 1.2. *Let $0 < \alpha \le 1$. Then the set of zeros of $\zeta(s, \alpha)$ does not contain any infinite arithmetic progression $\{\frac{1}{2} + idn : n \in \mathbb{N}\}$, where d is a positive real number.*

The technique used to prove Putnam's theorem requires a pole of the function under consideration. The Dirichlet L-function $L(s, \chi)$ has no pole if the Dirichlet character χ is a nonprincipal character. Thus, in this case, we cannot apply Theorem 1.1 directly. However, in Section 3, we show that the product $\zeta(s)L(s, \chi)$ satisfies the axioms of class C, from which we may deduce the following theorem.

Theorem 1.3. *For any Dirichlet character χ, the set of zeros of $L(s, \chi)$ does not contain any infinite arithmetic progression $\{\frac{1}{2} + idn : n \in \mathbb{N}\}$, where d is a positive real number.*

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $f \in C$. By hypothesis (1.1), $f(z)$ is given by the expression

$$f(z) = zA(z) - z\int_0^\infty B(x)e^{-(\sigma+it)x}\,dx$$

for $0 < \sigma < 1$. If $z = \sigma + it$ is a zero of $f(z)$, it follows that

$$A(z) = \int_0^\infty B(x)e^{-(\sigma+it)x}\,dx$$

and hence

$$\int_0^\infty B(x)e^{-\sigma x}\cos(tx)\,dx = \mathrm{Re}(A(z)). \tag{2.1}$$

Now we assume that there exists some number $d > 0$ such that, for $n = 1, 2, \ldots$, the numbers $z = \frac{1}{2} + idn$ are zeros of $f(z)$.

We extend the domain of $B(x)$ to all real numbers, by putting $B(-x) = B(x)e^{-x}$, $0 \le x < \infty$, and define $\mathcal{D}(x)$ on $-\infty < x < \infty$ by

$$\mathcal{D}(x) = \sum_{k=-\infty}^{\infty} B\Big(x + \frac{2\pi k}{d}\Big)e^{-(x+2\pi k/d)/2}. \tag{2.2}$$

In view of (1.3), for $-\infty < x < \infty$,

$$\Big|B\Big(x + \frac{2\pi k}{d}\Big)e^{-(x+2\pi k/d)/2}\Big| \le e^{-|x|/2}$$

and so the series $\mathcal{D}(x)$ is uniformly convergent on any finite interval. By (2.2), $\mathcal{D}(x)$ is periodic with period $2\pi/d$, that is, $\mathcal{D}(x + 2\pi/d) = \mathcal{D}(x)$ and

$$\int_0^{2\pi/d} \mathcal{D}(x)e^{idnx}\,dx = \int_{-\infty}^{\infty} B(x)e^{-(x/2)+idnx}\,dx.$$

In view of (2.1),

$$\int_0^{2\pi/d} \mathcal{D}(x)\cos(dnx)\,dx = \mathrm{Re}\Big(A\Big(\frac{1}{2} + idn\Big)\Big)$$

and

$$\int_0^{2\pi/d} \mathcal{D}(x)\sin(dnx)\,dx = 0.$$

From (1.2), the Fourier coefficients of $\mathcal{D}(x)$ are $O(n^{-\delta})$ with $\delta > 1$. Thus, the series for $\mathcal{D}(x)$ is uniformly convergent and the function $\mathcal{D}(x)$ is a continuous function. Let $h > 0$. From (1.4), the one-sided limits $\lim_{h\to0^+} B(\log n \pm h)$ exist for all n. Again, by the uniform convergence, the one-sided limits $\lim_{h\to0^+} \mathcal{D}(\log n \pm h)$ exist. In order to reach a contradiction of the assertion, we will show that

$$\lim_{h\to0^+} \mathcal{D}(x + h) - \lim_{h\to0^-} \mathcal{D}(x + h) < 0$$

for at least one value of x, contrary to the continuity of $\mathcal{D}(x)$.

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To see this, let $x = \log m$, where $m \geq 2$ is an integer to be determined later. Then

$$\begin{aligned}
&\lim_{h\to 0^+} \mathcal{D}(\log m + h) - \lim_{h\to 0^-} \mathcal{D}(\log m + h)\\
&\quad = \lim_{h\to 0^+} B(\log m + h)e^{-(\log m+h)/2} - \lim_{h\to 0^-} B(\log m + h)e^{-(\log m+h)/2}\\
&\qquad + \sum_{k\neq 0}\Big(\lim_{h\to 0^+} B\Big(x + \frac{2\pi k}{d} + h\Big)e^{-(x+(2\pi k/d)+h)/2}\\
&\qquad - \lim_{h\to 0^-} B\Big(x + \frac{2\pi k}{d} + h\Big)e^{-(x+(2\pi k/d)+h)/2}\Big).
\end{aligned}$$

In view of (1.4),

$$\lim_{h\to 0^+} B(\log m + h)e^{-(\log m+h)/2} - \lim_{h\to 0^-} B(\log m + h)e^{-(\log m+h)/2} < 0$$

and, for $n = 2, 3, \ldots$,

$$\lim_{h\to 0^+} B(x+h)e^{-(x+h)/2} - \lim_{h\to 0^-} B(x+h)e^{-(x+h)/2}\begin{cases} >0 & \text{if } x = -\log n,\\ <0 & \text{if } x = \log n.\end{cases}$$

Thus, it is sufficient to show that m can be chosen so that

$$\log m + \frac{2\pi k}{d} \neq -\log n \quad \text{for } k = \pm 1, \pm 2, \ldots \text{ and } n = 2, 3, \ldots.$$

That is, we wish to prove that, for some integer $m \geq 2$,

$$\frac{2\pi k}{d} \neq \log mn \quad \text{holds for } k = 1, 2, \ldots \text{ and } n = 2, 3, \ldots. \tag{2.3}$$

Suppose that for every integer $m \geq 2$, (2.3) is not true. Then, if $m = m_j \geq 2$ $(j = 1, 2)$ denote arbitrary integers, there exist integers $k = k_j \geq 1$ and $n = n_j \geq 2$ such that $2\pi k_j/d = \log m_j n_j$, that is, $(m_2n_2)^{k_1} = (m_1n_1)^{k_2}$. If we choose m_2 relatively prime to both m_1 and n_1, then the last equality is not true.

Hence, (2.3) is true for some integer $m = m' \geq 2$ and therefore

$$\lim_{h\to 0^+} \mathcal{D}(x \log m' + h) - \lim_{h\to 0^-} \mathcal{D}(\log m' + h) < 0.$$

The contradiction follows from this and the proof of the theorem is complete. □

Proof of Corollary 1.2. The Hurwitz zeta function $\zeta(s, \alpha)$ has a simple pole at $s = 1$ and analytical continuation to $0 < \mathrm{Re}(s) < 1$. It can be represented as

$$\zeta(s, \alpha) = \alpha^{-s} + \frac{s}{s-1} - \lfloor 1 - \alpha\rfloor + s\int_1^\infty (\lfloor x - \alpha\rfloor - x)x^{-s-1}\,dx.$$

Here,

$$A(s) = \frac{\alpha^{-s} - \lfloor 1-\alpha\rfloor}{s} + \frac{1}{s-1}$$

and $B(x) = \lfloor e^x - \alpha\rfloor - e^x$ for $\log n \leq x < \log(n+1)$ and $n = 1, 2, \ldots$. For $\mathrm{Re}(s) = \frac{1}{2}$, the Hurwitz zeta function $\zeta(s, \alpha)$ satisfies the conditions (1.2)–(1.4). Thus, $\zeta(s, \alpha) \in \mathcal{C}$. □

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3. Nonperiodicity of the zeros of Dirichlet L-functions

In this section we study the nonperiodicity of the zeros of zeta functions which do not satisfy Condition (i) of the class C. For example, the Dirichlet L-function $L(s,\chi)$ has no pole if the Dirichlet character χ is a nonprincipal character. Thus, in this case, we cannot apply Theorem 1.1 directly.

We consider the product of the Riemann zeta function and the Dirichlet L-function, say, $F(s) = \zeta(s)L(s,\chi)$, where χ is a Dirichlet character modulo q. For $\mathrm{Re}(s) > 1$,

$$F(s) = \zeta(s)L(s,\chi) = \sum_{n=1}^{\infty} f(n)n^{-s},$$

where $f(n) = \sum_{d|n} \chi(d)$. In order to show that the zeros of Dirichlet L-functions do not contain any arithmetic progression, we will show that the zeros of $F(s)$ also do not contain any arithmetic progression. We will do this by showing that $F(s)$ belongs to the class C.

Lemma 3.1 [4, Theorem 12.2]. *Suppose that $x > 1$ and χ is a Dirichlet character modulo $q > 1$. Then*

$$\sum_{n\le x} f(n) = L(1,\chi)x + O(x^{1/3+\epsilon}).$$

Proof of Theorem 1.3. We show first that $F(s) = \zeta(s)L(s,\chi)$ has an analytical continuation to $\frac{1}{3} < \mathrm{Re}(s) < 1$. By partial summation,

$$\begin{aligned}\sum_{n\le M} f(n)n^{-s} &= \sum_{n\le M} f(n)M^{-s} + s\int_1^M \Big(\sum_{n\le u} f(n)\Big)u^{-s-1}\,du\\ &= L(1,\chi)M^{1-s} + O(M^{1/3-s+\epsilon}) + s\int_1^M (L(1,\chi)u + O(u^{1/3+\epsilon}))u^{-s-1}\,du\\ &= \frac{L(1,\chi)}{1-s}M^{1-s} + \frac{sL(1,\chi)}{s-1} + O(M^{1/3-s+\epsilon}) + s\int_1^M O(u^{1/3+\epsilon})u^{-s-1}\,du.\end{aligned}$$

Let $M \to \infty$. For $\mathrm{Re}(s) > 1$,

$$F(s) = \frac{sL(1,\chi)}{s-1} + s\int_1^\infty O(u^{1/3+\epsilon})u^{-s-1}\,du.$$

Hence, $F(s)$ has an analytical continuation for $\mathrm{Re}(s) > \frac{1}{3}$ except for $s = 1$.

Now let $R(u) := \sum_{n\le u} f(n) - L(1,\chi)u$. For $\frac{1}{3} < \sigma < 1$, we can write

$$F(s) = \frac{sL(1,\chi)}{s-1} + s\int_0^\infty R(e^x)e^{-\sigma x}e^{-itx}\,dx.$$

This representation shows that $F(s)$ belongs to the class C provided that we can prove (1.4), that is, we must show that there is no positive integer n such that

$$\lim_{h\to 0^+} R(ne^h)e^{-(\log n+h)/2} - \lim_{h\to 0^-} R(ne^h)e^{-(\log n+h)/2} < 0.$$

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To see this, substitute the definition of $R(ne^h)$ and cancel the terms involving $L(1,\chi)$, giving

$$\begin{aligned}\lim_{h\to 0^+} R(ne^h)e^{-(\log n+h)/2} - \lim_{h\to 0^-} R(ne^h)e^{-(\log n+h)/2}\\ = \lim_{h\to 0^+}\sum_{k\le ne^h} f(k)e^{-(\log n+h)/2} - \lim_{h\to 0^-}\sum_{k\le ne^h} f(k)e^{-(\log n+h)/2}\\ = \frac{1}{\sqrt{n}}f(n) = \frac{1}{\sqrt{n}}\sum_{d\mid n}\chi(d)\ge 0\end{aligned}$$

for all integers n. Then, for $\mathrm{Re}(s) > \frac{1}{3}$, $L(s,\chi)\zeta(s)$ belongs to the class C. Since every zero $s = \frac{1}{2} + it$ of $L(s,\chi)$ is also a zero of $L(s,\chi)\zeta(s)$, it follows that the zeros of $L(\frac{1}{2} + it,\chi)$ cannot contain an infinite arithmetic progression. □

Now we extend this to other Dirichlet series. Let $\mathcal{L}$ be the class of the Dirichlet series $L(s) = \sum_{n=1}^{\infty} l(n)/n^s$, convergent for $\mathrm{Re}(s) > 1$, such that the following conditions hold:

(1) $L(s)$ has no pole in $0 < \mathrm{Re}(s) < 1$;
(2) $\sum_{n\le M} l(n) = L(1)M + O(x^{\alpha})$ as $M\to\infty$ with $\alpha < \frac{1}{2}$;
(3) there is no positive integer n such that $\sum_{d\mid n} l(d) < 0$.

Analogously to Theorem 1.3, we obtain the following result.

Theorem 3.2. *If $f\in\mathcal{L}$, then the set of zeros of $f(s,\chi)$ does not contain any infinite arithmetic progression $\{\frac{1}{2} + idn : n\in\mathbb{N}\}$, where d is a positive real number.*

Example 3.3. Let K be a quadratic field with discriminant d and let χ_d be the Kronecker symbol of d. We can write the Dedekind zeta function as

$$\zeta_K(s) = \zeta(s)L(s,\chi_d),$$

where $L(s,\chi_d)$ is the Dirichlet L-function associated to χ_d. Since $\zeta_K(s)\in\mathcal{L}$, Theorem 3.2 show that the set of zeros of $\zeta_K(\frac{1}{2} + it)$ does not contain an infinite arithmetic progression. This remark also shows that the set of zeros of $L(\frac{1}{2} + it,\chi_d)$ does not contain an infinite arithmetic progression.

Example 3.4. Let $r = f * g$ with f, g both periodic with period q, q', respectively, and consider the associated L-series

$$Z(s) = L(s, f)L(s, g) = \sum_{n=1}^{\infty}\frac{r(n)}{n^s},\quad \mathrm{Re}(s) > 1.$$

If f, g are both even, then $Z(s)$ belongs to the class $\mathcal{L}$. This follows by the same method as in Example 3.3. Thus, the set of zeros of $L(\frac{1}{2} + it, f)$ of the Dirichlet series associated to the arithmetic function f with period $q\ge 1$ does not contain an infinite arithmetic progression.

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