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ON ODD-DIMENSIONAL COMPLEX ANALYTIC KLEINIAN GROUPS

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Abstract

We shall explain here an idea to generalize classical complex analytic Kleinian group theory to any odd-dimensional cases. For a certain class of discrete subgroups of $\mathrm{PGL}_{2n+1}(\mathbf{C})$ acting on $\mathbf{P}^{2n+1}$, we can define their domains of discontinuity in a canonical manner, regarding an n-dimensional projective linear subspace in $\mathbf{P}^{2n+1}$ as a point, like a point in the classical one-dimensional case. Many interesting (compact) non-Kähler manifolds appear systematically as the canonical quotients of the domains. In the last section, we shall give some examples.

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Notation

- $M(p \times q, \mathbf{C})$: the set of matrices of size $p \times q$ with coefficients in $\mathbf{C}$.
- $M_p(\mathbf{C})$: the set of matrices of size $p \times p$ with coefficients in $\mathbf{C}$.

1. Introduction

The theory of discrete subgroups of $\mathrm{PGL}_2(\mathbf{C})$ has a long history. Let Γ be a discrete subgroup of $\mathrm{PGL}_2(\mathbf{C})$. We say that the action of Γ at a point $z \in \mathbf{P}^1$ is *discontinuous* if there is a neighborhood W of z such that $\gamma(W) \cap W = \emptyset$ for all but finitely many $\gamma \in \Gamma$. Following Maskit [11], we call a subgroup $\Gamma \subset \mathrm{PGL}_2(\mathbf{C})$ whose action is discontinuous at some point $z \in \mathbf{P}^1$ a *Kleinian group*.

Let $\Gamma \subset \mathrm{PGL}_2(\mathbf{C})$ be a Kleinian group. The set $\Omega(\Gamma)$ of points $z \in \mathbf{P}^1$ at which Γ acts discontinuously is called the *set of discontinuity* of Γ. The set $\Omega(\Gamma)$ is a Γ-invariant open subset in $\mathbf{P}^1$ on which Γ acts properly discontinuously. The geometry of the quotient space $\Omega(\Gamma)/\Gamma$ is one of the main themes in the classical Kleinian group theory.

If we seek a higher-dimensional version of the Kleinian group theory, we must first define the set of discontinuity for a given discrete subgroup. Let $n \geq 2$. Take a discrete

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subgroup $\Gamma \subset \mathrm{PGL}_{n+1}(\mathbf{C})$ acting of $\mathbf{P}^n$. Consider, as above, the set $\Omega(\Gamma)$ of points $z \in \mathbf{P}^n$ at which Γ acts discontinuously. Then it is true that Γ acts on $\Omega(\Gamma)$, but the action is not properly discontinuous in general. Therefore, we must find another definition of the set of discontinuity to get a good quotient space.

In this paper, we consider a class of discrete subgroups in $\mathrm{PGL}_{2n+2}(\mathbf{C})$ $(n \geq 1)$, that is, the class of *type* **L** *groups* (Definition 4.5). A type **L** group Γ has the nonempty set of discontinuity $\Omega(\Gamma)$, which is defined in a canonical manner. The set $\Omega(\Gamma)$ contains a subdomain $W \subset \mathbf{P}^{2n+1}$, which is biholomorphic to

$$\{z \in \mathbf{P}^{2n+1} : |z_0|^2 + \cdots + |z_n|^2 < |z_{n+1}|^2 + \cdots + |z_{2n+1}|^2\} \tag{1.1}$$

and satisfies

$$\gamma(W) \cap W = \emptyset \quad \text{for any } \gamma \in \Gamma \setminus \{1\},$$

where $z = [z_0 : \cdots : z_n : z_{n+1} : \cdots : z_{2n+1}]$.

For type **L** groups, n-dimensional projective linear subspaces in $\mathbf{P}^{2n+1}$ play the same role as points do in one-dimensional Kleinian group theory. In the following, an n-dimensional projective linear subspace is called an *n-plane* for short. The paper is organized as follows.

In Section 2, we shall make some preparations on the Grassmannian $\mathrm{Gr}(m, 2m)$ of m-dimensional subspaces in $\mathbf{C}^{2m}$. As is well known, $\mathrm{Gr}(m, 2m)$ can be embedded into the projective space $\mathbf{P}^N$, $N = {}_{2m}C_m - 1$, by Plücker coordinates. We remark that the embedded $\mathrm{Gr}(m, 2m)$ is contained in a $\mathrm{PGL}_m(\mathbf{C})$-invariant hyperquadric (Proposition 2.3). This fact plays an important role in studying limit sets of type **L** groups. In Section 3, we study some convergence properties of infinite sequences of projective transformations. In Section 4, we define the set of discontinuity $\Omega(\Gamma)$ for a type **L** group Γ (Definitions 4.3, 4.5), and show that the action of Γ on $\Omega(\Gamma)$ is properly discontinuous (Theorem 4.11). Hence the quotient $\Omega(\Gamma)/\Gamma$ becomes a good space. A domain $\Omega \subset \mathbf{P}^{2n+1}$ is said to be *large* if Ω contains an n-plane. The definition of the term 'large' is different from [10], where a domain $\Omega \subset \mathbf{P}^{2n+1}$ is said to be large if the $4n$-dimensional Hausdorff measure of its complement vanishes. Any holomorphic automorphism of a large domain extends to an element of $\mathrm{PGL}_{2n+2}(\mathbf{C})$ (Ivashkovich [4]). Using this, we show in Section 5, that a large domain which covers a compact manifold is a connected component of $\Omega(\Gamma)$ of some type **L** group Γ (Theorem 5.5). This may justify our definition of $\Omega(\Gamma)$. There are many groups of type **L**. As an example, we explain briefly an analogue of Klein combinations and handle attachments in Section 6. See also [6] on this topic. In Section 7, an analogue of the Ford region is defined. We prove that this region gives a fundamental set of a type **L** group under some additional conditions. In Section 8, we shall give examples of type **L** groups and their quotient spaces $\Omega(\Gamma)/\Gamma$.

2. The Grassmannian Gr(*m*, 2*m*)

Let $\mathrm{Gr}(m, 2m)$, $m \geq 2$, be the Grassmannian of the m-dimensional subspaces in $\mathbf{C}^{2m}$. The aim of the section is to show that $\mathrm{Gr}(m, 2m)$ is embedded in a quadric hypersurface

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in a big projective space by Plücker coordinates. This is a well-known fact. But since this is important for the later argument, we will explain it here for the reader's convenience.

Let $\{e_1, \dots, e_{2m}\}$ be a basis of $\mathbf{C}^{2m}$ and let $\mathcal{I}$ be the set of multiindices

$$I = \{i_1, \dots, i_m\} \subset \{1, \dots, 2m\}, \quad i_1 < \cdots < i_m,$$

of cardinality m. In the set of multiindices $\mathcal{I} = \{I\}$, we introduce the lexicographic order. Namely, for multiindices $I = \{i_1, \dots, i_m\}$, $J = \{j_1, \dots, j_m\}$, $I \neq J$, we write $I < J$, if $i_\mu < j_\mu$ for $\mu = \min\{\lambda : i_\lambda \neq j_\lambda\}$. We put

$$\delta_{JK} = \delta^{KJ} = \begin{cases} (-1)^\nu & \text{if } J \cap K = \emptyset, \\ 0 & \text{if } J \cap K \neq \emptyset, \end{cases}$$

where $\nu = \#\{(p, q) \in J \times K : p > q\}$, and

$$\delta^I_J = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases}$$

Then we have $\delta_{IJ} = (-1)^m \delta_{JI}$, $\delta^{IJ} = (-1)^m \delta^{JI}$, and

$$\delta_{IJ}\delta^{JK} = \begin{cases} 1 & \text{if } I = K, \\ 0 & \text{if } I \neq K. \end{cases} \qquad \text{(Einstein's convention)}$$

As a basis of $\Lambda^m(\mathbf{C}^{2m})$ of the space of m-vectors, we use $\{e_I\}_I$, where

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_m}, \quad I = \{i_1, \dots, i_m\} \in \mathcal{I}.$$

Then

$$e_J \wedge e_K = \delta_{JK} e_1 \wedge \cdots \wedge e_{2m}. \tag{2.1}$$

Any $w \in \Lambda^m(\mathbf{C}^{2m})$ is written uniquely as a linear combination over $\mathbf{C}$,

$$w = w^I e_I, \quad w^I \in \mathbf{C}, \quad \text{(Einstein's convention)}.$$

If $w \neq 0$, it determines a point $[w^I] \in \mathbf{P}^N$, regarding $\{w^I\}_I$ as a homogeneous coordinates, where $N = {}_{2m}C_m - 1$.

Let X be an m-dimensional subspace in $\mathbf{C}^{2m}$ spanned by $2m$-vectors $\{x_1, \dots, x_m\}$. Then X corresponds to the m-vector $\hat{X} = x_1 \wedge \cdots \wedge x_m \in \mathrm{Gr}(m, 2m)$. Letting $x_j = x^k_j e_k$,

$$\hat{X} = x^{k_1}_1 e_{k_1} \wedge \cdots \wedge x^{k_m}_m e_{k_m} = X^K e_K, \tag{2.2}$$

where

$$X^K = \det \begin{pmatrix} x^{k_1}_1 & \cdots & x^{k_1}_m \\ \vdots & & \vdots \\ x^{k_m}_1 & \cdots & x^{k_m}_m \end{pmatrix}.$$

The set of numbers $\{X^K\}_{K \in \mathcal{I}}$ determines the point $[x^K] \in \mathbf{P}^N$, which are the Plücker coordinates of the vector subspace X.

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Let $A \in M_{2m}(\mathbf{C})$ be any element. Put $Ae_j = a_j^k e_k$. Then

$$Ae_J = Ae_{j_1} \wedge \cdots \wedge Ae_{j_m} = a_{j_1}^{k_1} e_{k_1} \wedge \cdots \wedge a_{j_m}^{k_m} e_{k_m} = A_J^K e_K,$$

where

$$A_J^K = \det \begin{pmatrix} a_{j_1}^{k_1} & \cdots & a_{j_m}^{k_1} \\ \vdots & & \vdots \\ a_{j_1}^{k_m} & \cdots & a_{j_m}^{k_m} \end{pmatrix}.$$

Hence, for $J, K \in \mathcal{I}$ with $J \cap K = \emptyset$,

$$A(e_J \wedge e_K) = A_J^L e_L \wedge A_K^M e_M = \delta_{LM} A_J^L A_K^M e_1 \wedge \cdots \wedge e_{2m}.$$

On the other hand,

$$A(e_J \wedge e_K) = \delta_{JK} \det(A) e_1 \wedge \cdots \wedge e_{2m}.$$

Thus

$$\delta_{LM} A_J^L A_K^M = \delta_{JK} \det A. \tag{2.3}$$

Define a bilinear form $Q(z, w)$ on $\mathbf{C}^{N+1}$ by

$$Q(z, w) = \delta_{JK} z^J w^K, \quad z = (z^J), \quad w = (w^K).$$

Put

$$\hat{A}z = (A_I^K z^I), \quad z = (z^I).$$

Then, by (2.3),

$$Q(\hat{A}z, \hat{A}w) = (\det A) Q(z, w).$$

For $\hat{A}$, we define $\hat{A}^* \in M_{N+1}(\mathbf{C})$ by

$$(\hat{A}^*)_J^I = \delta^{IK} \delta_{LJ} A_K^L. \tag{2.4}$$

Then

$$Q(\hat{A}z, w) = Q(z, \hat{A}^* w), \tag{2.5}$$

$$\hat{A}^* \hat{A} = (\det A) I_{N+1}. \tag{2.6}$$

Proposition 2.1. *Let $X, Y \subset \mathbf{C}^{2m}$ be m-dimensional vector subspaces. Put $X = X^K e_K$ and $Y = Y^K e_K$. Then $\dim(X \cap Y) \geq 1$ holds if and only if*

$$Q((X^K), (Y^K)) = 0.$$

In particular, the equation

$$Q((X^K), (X^K)) = 0$$

holds for any m-dimensional subspace X of $\mathbf{C}^{2m}$.

Proof. This is clear from (2.2) and (2.1). □

We apply the above argument to the case $\mathrm{Gr}(n+1, 2n+2)$. Set $N = {}_{2n+2}C_{n+1} - 1$. Then, by Proposition 2.1, we have easily the following proposition.

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PROPOSITION 2.2. *Let $Q(z, w)$ be the quadratic form defined by*

$$Q(z, w) = \delta_{JK} z^J w^K$$

defined on $\mathbf{C}^{N+1} \times \mathbf{C}^{N+1}$. Let ℓ_1, ℓ_2 be n-planes in $\mathbf{P}^{2n+1}$ and let $[x^I]$, $[y^I]$ be their Plücker coordinates. Then ℓ_1 and ℓ_2 intersect if and only if the equality

$$Q([x^I], [y^I]) = 0$$

holds. In particular, for an n-plane with the Plücker coordinates $[x^I]$,

$$Q([x^I], [x^I]) = 0.$$

We have also the following proposition.

PROPOSITION 2.3. *The quadric hypersurface $Q = \{Q(z, z) = 0\}$ in $\mathbf{P}^N$ is invariant by the image group of the group representation*

$$\rho : \mathrm{PGL}_{2n+2}(\mathbf{C}) \to \mathrm{PGL}_{N+1}(\mathbf{C}), \quad \rho(A) = \hat{A},$$

and the Grassmannian $\mathrm{Gr}(n + 1, 2n + 2)$ is contained in Q.

3. Limit of projective transformations

Let $N \geq 1$ and let Γ be a discrete infinite subgroup of $\mathrm{PGL}_{N+1}(\mathbf{C})$ that acts on the projective space $\mathbf{P}^N$. Consider an infinite sequence $\{\sigma_\nu\}$ of elements of Γ. Let $\tilde{\sigma}_\nu \in \mathrm{GL}_{N+1}(\mathbf{C})$ be a representative of σ_ν such that $|\tilde{\sigma}_\nu| = 1$, where, for a matrix $A = (a_{jk})$ of size $N + 1$, we put $|A| = \max_{0\leq j,k\leq N} |a_{jk}|$. We say that $\{\sigma_\nu\}$ is a *normal sequence* if the following conditions are satisfied.

(1) The sequence $\{\sigma_\nu\}$ consists of distinct elements of Γ.
(2) The sequence of matrices $\{\tilde{\sigma}_\nu\}$ can be chosen to be convergent to a matrix $\tilde{\sigma} \in M_{N+1}(\mathbf{C})$.

The projective linear subspace defined by the image of the linear map $\tilde{\sigma} : \mathbf{C}^{N+1} \to \mathbf{C}^{N+1}$ is called the *limit image* of the normal sequence $\{\sigma_\nu\}$ and denoted by $I(\{\sigma_\nu\})$. Similarly, the projective linear subspace defined by the kernel of $\tilde{\sigma}$ is called the *limit kernel* of $\{\sigma_\nu\}$ and is denoted by $K(\{\sigma_\nu\})$. Here $r = \mathrm{rank}\ \tilde{\sigma}$ is called the rank of the normal sequence. Note that $I(\{\sigma_\nu\})$, $K(\{\sigma_\nu\})$ and r are determined independently of the choice of representatives $\tilde{\sigma}_\nu$. Obviously, $\dim I(\{\sigma_\nu\}) = r - 1$ and $\dim K(\{\sigma_\nu\}) = N - r$.

THEOREM 3.1 [12, Satz 2]. *Let $\{\sigma_\nu\} \subset \Gamma$ be a normal sequence. Suppose that the sequence of representatives $\{\tilde{\sigma}_\nu\}$ converges to $\tilde{\sigma} : \mathbf{C}^{N+1} \to \mathbf{C}^{N+1}$. Let I be its limit image and let K be the limit kernel. Then the sequence $\{\sigma_\nu\}$ converges uniformly on compacts in $\mathbf{P}^N \setminus K$ to the projection $\sigma : \mathbf{P}^N \setminus K \to I$ defined by $\tilde{\sigma}$.*

THEOREM 3.2. *Let $\{\sigma_\nu\}_\nu \subset \Gamma$ be a normal sequence such that the sequence $\{\hat{\sigma}_\nu\}_\nu$ is also normal. Then the limit image of $\{\hat{\sigma}_\nu\}_\nu$ is contained in Q, and the limit kernel coincides with the orthogonal subspace (with respect to $Q(z, z)$) of the limit image of $\{\hat{\sigma}_\nu^{-1}\}_\nu$.*

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PROOF. Let $S_\nu \in \mathrm{GL}_{N+1}(\mathbf{C})$ be a representative of $\hat{\sigma}_\nu$. We can assume that $|S_\nu| = 1$ and that the sequence $\{S_\nu\}$ converges to $S \in M_{N+1}(\mathbf{C})$. Since Γ is discrete, $\det S = \lim_\nu \det S_\nu = 0$. Therefore

$$Q(S_\nu z, S_\nu z) = (\det S_\nu) Q(z,z) \quad \text{and} \quad Q(Sz, Sz) = 0,$$

by Proposition 2.3. Hence the limit image of $\{\hat{\sigma}_\nu\}$ is contained in Q. Since

$$\begin{aligned}(\mathrm{Im}\, S^*)^\perp &= \{z \in \mathbf{C}^{N+1} : Q(z, S^* w) = 0\ \forall w \in \mathbf{C}^{N+1}\}\\ &= \{z \in \mathbf{C}^{N+1} : Q(Sz, w) = 0\ \forall w \in \mathbf{C}^{N+1}\}\\ &= \mathrm{Ker}\ S,\end{aligned}$$

we have

$$\mathrm{Ker}\ S = (\mathrm{Im}\ S^*)^\perp.$$

By (2.6), we see that the projection $\mathbf{P}^N \cdots \to \mathbf{P}^N$ defined by S^* is the limit of the normal sequence $\{\hat{\sigma}_\nu^{-1}\}_\nu$. Thus we have the theorem. □

4. Discontinuous groups in the projective $(2n + 1)$-space

Let $\Gamma \subset \mathrm{PGL}_{2n+2}(\mathbf{C})$ be a discrete subgroup. Put $N = {}_{2n+2}C_{n+1} - 1$ and $\mathcal{G} = \mathrm{Gr}(n+1, 2n+2)$. We shall say, from now on, that a sequence $\{\sigma_\nu\} \subset \Gamma$ is *normal* if not only the original sequence $\{\sigma_\nu\}$ is normal but also is the corresponding sequence $\{\hat{\sigma}_\nu\}$, $\hat{\sigma}_\nu = \rho(\sigma_\nu)$, of $\mathrm{PGL}_{N+1}(\mathbf{C})$. Thus a normal sequence $\{\sigma_\nu\} \subset \Gamma$ defines also $I(\{\hat{\sigma}_\nu\})$ and $K(\{\hat{\sigma}_\nu\})$ in $\mathbf{P}^N$. Note that any normal sequence in the old sense contains a subsequence that is normal in the new one.

DEFINITION 4.1. An n-plane ℓ in $\mathbf{P}^{2n+1}$ is called a *limit n-plane* of Γ if there is a normal sequence $\{\sigma_\nu\}$ of Γ with $\hat{\ell} \in \mathcal{G} \cap I(\{\hat{\sigma}_\nu\})$.

Let $\mathcal{L}(\Gamma) \subset \mathcal{G}$ denote the set of points that correspond to limit n-planes of Γ.

DEFINITION 4.2. The union

$$\Lambda(\Gamma) = \bigcup_{\hat{\ell} \in \mathcal{L}(\Gamma)} |\ell|$$

of the support of limit n-planes of Γ is called *the limit set* of Γ.

Here we indicate by $|\ell|$ the support of an n-plane ℓ in $\mathbf{P}^{2n+1}$ in order to express explicitly the set of points on the n-plane.

DEFINITION 4.3. The set

$$\Omega(\Gamma) = \mathbf{P}^{2n+1} \setminus \Lambda(\Gamma)$$

is called the set of discontinuity of the group Γ.

DEFINITION 4.4. A domain Ω in $\mathbf{P}^{2n+1}$ is said to be *large* if Ω contains an n-plane.

There are examples of Γ with nonempty $\Omega(\Gamma)$, but which contain no n-planes. For example, in the case $\mathbf{P}^3$, let Γ be the infinite cyclic group generated by

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$\sigma = \left(\begin{smallmatrix} I & 0 \\ A & I \end{smallmatrix}\right) \in \mathrm{PGL}_4(\mathbf{C})$, $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then $\Omega(\Gamma) = \mathbf{P}^3 \setminus \{z_0 = 0\}$. Thus we define type **L** groups as follows.

Definition 4.5. A discrete subgroup in $\mathrm{PGL}_{2n+2}(\mathbf{C})$ is said to be of type **L** if $\Omega(\Gamma)$ contains a large domain.

From now on, *we assume that* $\Gamma \subset \mathrm{PGL}_{2n+2}(\mathbf{C})$ *is of type* **L**, if not stated otherwise explicitly. This is our higher-dimensional *complex analytic* analogue of Kleinian groups.

Lemma 4.6. *$\mathcal{G} \cap I(\{\hat{\sigma}_\nu\})$ consists of a single point for any normal sequence $\{\sigma_\nu\}$ in Γ.*

Proof. Let $\{\sigma_\nu\}_\nu$ be any normal sequence in $\Gamma \subset \mathrm{PGL}_{2n+2}(\mathbf{C})$ and let $\{S_\nu\}_\nu \subset \mathrm{GL}_{N+1}(\mathbf{C})$ be any convergent sequence of representatives of $\{\hat{\sigma}_\nu\}_\nu$ with $|S_\nu| = 1$. Put $S = \lim S_\nu$. Let $I = [\mathrm{Im}\, S] = I(\{\hat{\sigma}_\nu\})$, $K = [\mathrm{Ker}\, S] = K(\{\hat{\sigma}_\nu\}) \subset \mathbf{P}^N$ be the limit image and the limit kernel of $\{S_\nu\}$, respectively. If the algebraic set $I \cap \mathcal{G}$ is of positive dimension, then $B = \bigcup_{\hat{\ell} \in I \cap \mathcal{G}} |\ell|$ is an algebraic manifold contained in $\Lambda(\Gamma)$ with dimension more than n. This is absurd since $\Omega(\Gamma)$ contains an n-plane which does not intersect B. Hence $I \cap \mathcal{G}$ is a finite set. Consequently, $I \cap \mathcal{G}$ consists of a single point since it is the set of limit points of $\mathcal{G} \setminus K$, which is connected. □

Proposition 4.7. *The limit image $I(\{\hat{\sigma}_\nu\})$ consists of a single point in $\mathcal{G}$ for any normal sequence $\{\sigma_\nu\}$ in Γ.*

Proof. We use the notation in the proof of the lemma above. By the lemma, we have $I \cap \mathcal{G} = \{\hat{\ell}\}$ for some point $\hat{\ell} \in \mathcal{G}$. Suppose that $\dim I > 0$. The linear map S defines the projection $S : \mathbf{P}^N - K \to I$. Since $\mathcal{G}$ is not contained in any proper linear subspace in $\mathbf{P}^N$, there is a point $w \in I \setminus \mathcal{G}$. The fiber $S^{-1}(w)$ does not intersects $\mathcal{G}$ outside K since, otherwise, for $x \in \mathcal{G} \setminus K$, we have $w = S(x) = \lim_\nu S_\nu(x) \in \mathcal{G}$. This is absurd. Thus $\mathcal{G} \subset K \cup S^{-1}(\hat{\ell})$. This contradicts again the fact that the manifold $\mathcal{G}$ is not contained in any proper linear subspace in $\mathbf{P}^N$. Hence we have $I = I \cap \mathcal{G} = \{\hat{\ell}\}$. Thus we have the proposition. □

Theorem 4.8. *Let $\{\sigma_\nu\}$ be a sequence of distinct elements of Γ. Then there are limit n-planes ℓ_I, ℓ_K and a subsequence $\{\tau_\nu\}$ of $\{\sigma_\nu\}$ such that $\{\tau_\nu\}$ is uniformly convergent to ℓ_I on $\mathbf{P}^{2n+1} \setminus \ell_K$ in the sense that, for any compact subset $M \subset \mathbf{P}^{2n+1} \setminus \ell_K$ and for any neighborhood V of ℓ_I, there is an integer m_0 such that $\tau_\nu(M) \subset V$ for any $m > m_0$.*

Proof. Choose a normal subsequence $\{\tau_\nu\}$ of $\{\sigma_\nu\}$ such that $\{\tau_\nu^{-1}\}$ also has a convergent sequence of representatives. Let $\{T_\nu\} \subset \mathrm{GL}_{N+1}(\mathbf{C})$ be the convergent sequence corresponding to $\{\hat{\tau}_\nu\}$. Put $T = \lim_\nu T_\nu$. Note that $\{T'_\nu\}$, $T'_\nu = |T^*_\nu|^{-1} T^*_\nu$, is a convergent sequence of representatives of $\{\hat{\tau}_\nu^{-1}\}$ by (2.4) and (2.6). Hence $\{\tau_\nu^{-1}\}$ is also a normal sequence. Put $T' = \lim_\nu T'_\nu$. By Proposition 4.7, $[\mathrm{Im}\, T]$ is a single point in $\mathcal{G}$, which corresponds to a limit n-plane, denoted by ℓ_I, in $\mathbf{P}^{2n+1}$. On the other hand, since $[\mathrm{Im}\, T']$ is the limit image of the normal sequence $\{\hat{\tau}_\nu^{-1}\}$, $[\mathrm{Im}\, T']$ consists of a single point corresponding to a limit n-plane in $\mathbf{P}^{2n+1}$ by Proposition 4.7, which we denote by ℓ_K. Note that $[\mathrm{Im}\, T']^\perp$ is the set of points parameterizing n-planes intersecting ℓ_K by Proposition 2.2. Since $\mathrm{Ker}\, T = (\mathrm{Im}\, T')^\perp$ by Theorem 3.2, and $\{\hat{\tau}_\nu\}$ converges

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uniformly on compact sets in $\mathbf{P}^N \setminus [\mathrm{Ker}\, T]$ to $[\mathrm{Im}\, T]$ by Theorem 3.1, we see that $\{\tau_\nu\}$ converges uniformly compact on sets on $\mathbf{P}^{2n+1} \setminus \ell_K$ to ℓ_I. This proves the theorem. □

In the course of the proof, we have shown the following proposition.

Proposition 4.9. *Let ℓ_0 be a limit n-plane of Γ. Then there is a limit n-plane ℓ_∞ and a normal sequence $\{\sigma_\nu\} \subset \Gamma$ such that $\{\sigma_\nu\}$ is uniformly convergent to ℓ_0 on any compact set in $\mathbf{P}^{2n+1} \setminus \ell_\infty$ and that $\{\sigma_\nu^{-1}\}$ is uniformly convergent to ℓ_∞ on any compact set in $\mathbf{P}^{2n+1} \setminus \ell_0$.*

Next we shall show the following theorem.

Theorem 4.10. *For a type* **L** *group Γ, $\Lambda(\Gamma)$ is a closed, nowhere dense Γ-invariant subset in $\mathbf{P}^{2n+1}$.*

Proof. To show that $\Lambda(\Gamma)$ is Γ-invariant, we take any point $x \in \Lambda(\Gamma)$. Since x is on a limit n-plane, say, ℓ_0, there is a normal sequence $\{\sigma_\nu\}$ of Γ with $I(\{\hat{\sigma}_\nu\}) = \hat{\ell}_0$ by Proposition 4.9. Then $\{\sigma \circ \sigma_\nu\}$ is a normal sequence with $I(\{\hat{\sigma} \circ \hat{\sigma}_\nu\}) = \hat{\sigma}(\hat{\ell}_0)$. Since the limit n-plane $\sigma(\ell_0)$ passes through the point $\sigma(x)$, $\Lambda(\Gamma)$ is Γ-invariant.

To show that $\Lambda(\Gamma)$ is closed, let $\{x_\nu\}$ be a sequence of points of $\Lambda(\Gamma)$ such that $\lim_\nu x_\nu = x$ for some point $x \in \mathbf{P}^{2n+1}$. Let ℓ_ν be a limit n-plane through x_ν. By Proposition 4.9, for each ν, we can find a limit n-plane $\ell_{\nu,\infty}$ and a normal sequence $\{\sigma_{\nu,k}\}_k$ such that $I(\{\hat{\sigma}_{\nu,k}\}_k) = \hat{\ell}_\nu$ and that the sequence $\{\sigma_{\nu,k}\}_k$ is uniformly convergent to ℓ_ν on compact sets in $\mathbf{P}^{2n+1} \setminus \ell_{\nu,\infty}$. Taking a subsequence of $\{\ell_\nu\}$, we can assume that the ℓ_ν are all distinct and that $\{\hat{\ell}_\nu\}$ and $\{\hat{\ell}_{\nu,\infty}\}_\nu$ are convergent in $\mathcal{G}$.

Since $\{\hat{\ell}_{\nu,\infty}\}_\nu$ is convergent, there is an n-plane ℓ_a which is disjoint from the closure of $\bigcup_\nu |\ell_{\nu,\infty}|$. Take a small tubular neighborhood W of ℓ_a, which is biholomorphic to the domain (1.1), such that the closure $[W]$ is still disjoint from the closure of $\bigcup_\nu |\ell_{\nu,\infty}|$.

Fix a metric on $\mathcal{G}$ and consider the distance of points on $\mathcal{G}$. Let δ_ν be the minimal distance from $\hat{\ell}_\nu$ to any other $\hat{\ell}_\mu$ in $\mathcal{G}$. Obviously, $\lim_\nu \delta_\nu = 0$. Set

$$N_{\delta_\nu}(\hat{\ell}_\nu) = \{z \in \mathcal{G} : \mathrm{distance}(z, \hat{\ell}_\nu) < \delta_\nu\}.$$

Choose $k(\nu)$ such that

$$\hat{\sigma}_{\nu,k(\nu)}([\hat{W}]) \subset N_{\delta_\nu}(\hat{\ell}_\nu)$$

and that the $\sigma_{\nu,k(\nu)}$, $\nu = 1, 2, 3, \ldots$, are all distinct, where $\hat{W} = \{\hat{\ell} \in \mathcal{G} : \ell \subset \mathcal{W}\}$. Put $\hat{\ell} = \lim_\nu \hat{\ell}_\nu$. Take any $\delta > 0$. Then there is ν_0 such that $N_{\delta_\nu}(\hat{\ell}_\nu) \subset N_\delta(\hat{\ell})$ holds for any $\nu > \nu_0$. Thus, for $\nu > \nu_0$, we have $\hat{\sigma}_{\nu,k(\nu)}([\hat{W}]) \subset N_\delta(\hat{\ell})$. This implies that $\{\sigma_{\nu,k(\nu)}\}$ converges to ℓ uniformly on W. Thus ℓ is a limit n-plane passing through x. Hence $\Lambda(\Gamma)$ is closed.

Lastly, we shall show that $\Lambda(\Gamma)$ is nowhere dense. Let x be any point in $\Lambda(\Gamma)$. By Proposition 4.9, there are n-planes ℓ_0, ℓ_∞ in $\mathbf{P}^{2n+1}$ and a normal sequence $\{\sigma_\nu\}$ such that $x \in \ell_0$ and that $\lim_\nu \hat{\sigma}_\nu(\hat{K}) = \hat{\ell}_0$ for any compact set $K \subset \mathbf{P}^{2n+1} \setminus \ell_\infty$. By the property **L**, we can set K as a single n-plane ℓ contained in $\Omega(\Gamma)$. Then, for every neighborhood W of x, there is an integer ν_0 such that $W \cap \sigma_\nu(\ell) \neq \emptyset$ for $\nu \geq \nu_0$. Thus W contains a point in $\Omega(\Gamma)$. Hence $\Lambda(\Gamma)$ is nowhere dense. □

Theorem 4.11. *For a type* **L** *group Γ, the action of Γ on $\Omega(\Gamma)$ is properly discontinuous.*

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PROOF. Take any compact set M in $\Omega(\Gamma)$. Suppose that there is an infinite sequence $\{\sigma_\nu\}_\nu$ of distinct elements of Γ such that $M \cap \sigma_\nu(M) \neq \emptyset$ for any ν. By Proposition 4.9, replacing $\{\sigma_\nu\}$ with its normal subsequence, we can assume that there are limit n-planes ℓ_K and ℓ_I such that $\{\sigma_\nu\}$ converges uniformly on $\mathbf{P}^{2n+1} \setminus \ell_K$ to ℓ_I. Since $\Omega(\Gamma)$ has no intersection with limit n-planes, we see that $M \cap (\ell_I \cup \ell_K) = \emptyset$. Therefore $\{\sigma_\nu(M)\}$ converges to a subset on ℓ_I. This contradicts the assumption that $M \cap \sigma_\nu(M) \neq \emptyset$ for any ν. □

By Theorem 4.11, we can define canonically the quotient space $\Omega(\Gamma)/\Gamma$, which we denote by $X(\Gamma)$,

$$X(\Gamma) = \Omega(\Gamma)/\Gamma.$$

REMARK 4.12. There are examples of Γ for which $X(\Gamma)$ is not connected. Such an example can be constructed easily in the case $n = 1$ by considering a flat twistor space over a conformally flat four-manifold [7], where every connected component of $\Omega(\Gamma)$ is large. We do not know, however, whether this is the case for all type **L** groups or not.

5. Discontinuous group actions on large domains

In this section, we shall show that a large domain that covers a compact manifold is a connected component of $\Omega(\Gamma)$ of some Γ of type **L**.

PROPOSITION 5.1. *Let Γ be a group of holomorphic automorphisms of a large domain Ω in $\mathbf{P}^{2n+1}$. Suppose that Γ is torsion free and that the action of Γ on Ω is properly discontinuous. Then Γ is of type* **L**.

PROOF. First we shall prove that Γ is a subgroup of $\mathrm{PGL}_{2n+2}(\mathbf{C})$. By a *line* we shall mean a one-dimensional projective linear subspace of a projective space. Since every line in $\mathbf{P}^{2n+1}$ has a tubular neighborhood with a smooth convex–concave boundary, the following lemma follows immediately from a theorem of Ivashkovich [4].

LEMMA 5.2. *Let L_ν, $\nu = 1, 2$, be lines in $\mathbf{P}^m$ $(m \geq 2)$ and let U_ν be a tubular neighborhood of L_ν. Suppose that $\gamma : U_1 \to U_2$ is a biholomorphic mapping. Then γ extends to an element of* $\mathrm{PGL}_{m+1}(\mathbf{C})$.

LEMMA 5.3. *Let $\sigma \in \Gamma \setminus \{1\}$ be any element and let $\tilde{\sigma} \in \mathrm{GL}_{2n+2}(\mathbf{C})$ be a representative of σ. Then the inequality*

$$\operatorname{rank}(\tilde{\sigma} - \alpha I) \geq n + 1 \tag{5.1}$$

holds for any $\alpha \in \mathbf{C}$.

PROOF. Consider the subspace

$$V = \{z \in \mathbf{C}^{2n+2} : (\tilde{\sigma} - \alpha I)z = 0\}.$$

Each point of the projectivized linear subspace $[V] \subset \mathbf{P}^{2n+1}$ is fixed by σ. Suppose that $\operatorname{rank}(\tilde{\sigma} - \alpha I) \leq n$. Then $\dim V \geq n + 2$. Therefore any n-plane in Ω intersects $[V]$ and every point on the intersection is fixed by σ. This is absurd, since Γ is torsion free and properly discontinuous on Ω. Thus we have the lemma. □

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LEMMA 5.4.[1] *If (5.1) holds for any $\alpha \in \mathbf{C}$, then there is an n-plane ℓ such that $\sigma(\ell) \cap \ell = \emptyset$.*

PROOF. We have to choose a subspace $L \subset \mathbf{C}^{2n+2}$ of dimension $n+1$ such that $\tilde{\sigma}(L) \cap L = \{0\}$. Put

$$\rho = \min_{\alpha \in \mathbf{C}} \operatorname{rank}(\tilde{\sigma} - \alpha I).$$

We can assume that ρ is attained at $\alpha = 1$ without loss of generality. We put $N = \tilde{\sigma} - I$ and then $\rho = \operatorname{rank} N$. Define $\varphi : C^{2n+2} \to \mathbf{C}^{2n+2}$ by $\varphi(z) = Nz$. Since $\rho \geq n+1$ by the assumption, there is an $(n+1)$-dimensional subspace $L_1 \subset \operatorname{Im} \varphi$. Put $\tilde{L}_1 = \varphi^{-1}(L_1)$. Then, since $\dim \operatorname{Ker} \varphi = 2n + 2 - \rho$, we have $\dim \tilde{L}_1 = 3n + 3 - \rho$. Since $\dim L_1 = n + 1$ and $\dim \operatorname{Ker} \varphi = 2n + 2 - \rho$, we can choose a subspace $L \subset \tilde{L}_1$ such that $\dim L = n + 1$, $L \cap L_1 = \{0\}$ and $L \cap \operatorname{Ker} \varphi = \{0\}$. We claim that L is the desired linear subspace in $\mathbf{C}^{2n+2}$. To verify the claim, we choose $X \in M((2n+2) \times (n+1), \mathbf{C})$ with $\operatorname{rank} X = n + 1$ such that

$$L = \{z \in \mathbf{C}^{2n+2} : z = Xu, u \in \mathbf{C}^{n+1}\}.$$

Then $L \cap \tilde{\sigma}(L) = \{0\}$ holds if and only if

$$\det(\tilde{\sigma}X, X) \neq 0.$$

This is equivalent to

$$\det(NX, X) \neq 0.$$

That $L \cap \operatorname{Ker} \varphi = \{0\}$ implies that NX is of maximal rank, and that $L \cap L_1 = \{0\}$ implies that the vectors in NX and X span $\mathbf{C}^{2n+2}$. Thus the claim is verified. □

Now we go back to the proof of Proposition 5.1. By the assumption that Ω is large, there is a relatively compact subdomain $W \subset \Omega$ which is biholomorphic to U. The n-planes in W are parametrized by $\hat{W} \subset \mathcal{G} \subset \mathbf{P}^N$. Since the action of Γ on Ω is properly discontinuous, the set

$$S = \{\sigma \in \Gamma \setminus \{1\} : \hat{\sigma}(\hat{W}) \cap \hat{W} \neq \emptyset\}$$

is finite. Let ℓ be an n-plane in W. For $\sigma \in S$, we have $Q(\hat{\ell}, \hat{\sigma}(\hat{\ell})) = 0$ when ℓ intersects $\sigma(\ell)$. By Lemmas 5.3 and 5.4, we see that the set

$$Y_\sigma = \{\zeta \in \mathcal{G} : Q(\zeta, \hat{\sigma}(\zeta)) = 0\}$$

is a proper analytic subset of $\mathcal{G}$. Hence the set

$$V = \hat{W} \Big\backslash \bigcup_{\sigma \in S} Y_\sigma$$

is not empty. Take a point $\hat{\ell}' \in V$. Then we can choose a neighborhood W' of ℓ' which is biholomorphic to U and satisfies $\sigma(W') \cap W' = \emptyset$ for all σ in S and hence in Γ. □

[1] Compare with [10, Lemma 1.6], which is for 1-planes in $\mathbf{P}^m$.

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THEOREM 5.5. *Let $\Omega \subset \mathbf{P}^{2n+1}$ be a large domain which is an unramified cover of a compact complex manifold. Then there is a type* **L** *group Γ such that $\Omega(\Gamma)$ contains Ω as a connected component.*

PROOF. By Lemma 5.2, there is a group $\Gamma \subset \mathrm{PGL}_{2n+2}(\mathbf{C})$ of holomorphic automorphisms of Ω such that Ω/Γ is compact. Since Γ is finitely generated, we can assume that Γ is torsion free by Selberg's lemma. Hence, by Proposition 5.1, Γ is of type **L**.

We claim that $\Omega \subset \Omega(\Gamma)$. To verify this, suppose, on the contrary, that there is a point $x \in \Omega \cap \Lambda(\Gamma)$. Then there are limit n-planes ℓ_I, ℓ_K such that $x \in \ell_I$ and a sequence $\{\sigma_m\}$ of distinct elements of Γ such that $\{\sigma_m\}$ converges uniformly on $\mathbf{P}^{2n+1} \setminus \ell_K$ to ℓ_I. Let ℓ be an n-plane contained in Ω. Displacing ℓ a little, if necessary, we can assume that $\ell \cap \ell_K = \emptyset$. Let K_x be a compact neighborhood of x contained in Ω. Put $K = K_x \cup \ell$, which is a compact set contained in Ω. Since $\{\sigma_m(\ell)\}$ converges to ℓ_I, we see that $\sigma_m(K) \cap K \neq \emptyset$ for infinitely many m. This contradicts the assumption that Γ is properly discontinuous on Ω. Thus the claim is verified.

Now Ω is contained in a connected component, say, Ω_0, of $\Omega(\Gamma)$. Since Ω is Γ-invariant, so is Ω_0. Therefore, by Theorem 4.11, Ω_0/Γ is a connected complex space that contains Ω/Γ. Since Ω/Γ is compact, we infer that $\Omega/\Gamma = \Omega_0/\Gamma$. Hence $\Omega = \Omega_0$. □

PROPOSITION 5.6. *Let X be a compact Kähler manifold that contains a domain W biholomorphic to*

$$U = \{[z_0 : \cdots : z_{2n+1}] \in \mathbf{P}^{2n+1} : |z_0|^2 + \cdots + |z_n|^2 < |z_{n+1}|^2 + \cdots + |z_{2n+1}|^2\}.$$

Then X is unirational. In particular, X is simply connected.

PROOF. The proof of [6, Corollary 3.1] works also in this case. Take any n-plane $\ell \subset W$. Let B be the irreducible component of the Barlet space which contains the point $\hat{\ell}$ corresponding to ℓ. Since X is Kähler, B is compact. Consider the graph

$$Z = \{(x, b) \in X \times B : x \in b\}.$$

Let $p_X : Z \to X$ and $p_B : Z \to B$ the natural projections. Fix a point $o \in \ell$. Since B is compact, we can apply a theorem of Campana [1, Corollaire 1], which says that $p_X^{-1}(o)$ is a compact *algebraic* variety. Hence $B_o := p_B(p_X^{-1}(o))$ is also compact and algebraic. Put $M := p_B^{-1}(B_o)$ and $f = P_B|_M$. Then $f : M \to B_o$ is a $\mathbf{P}^n$-fiber space over a compact algebraic variety B_o. By the choice of o, B_o is nonsingular at o, and there is a small open neighborhood $N \subset B_o$ centered at $\hat{\ell}$ and a biholomorphic map $\tau : f^{-1}(N) \to N \times \mathbf{P}^n$ such that $f = p \circ \tau$, where $p : N \times \mathbf{P}^n \to N$ is the projection. Let $\mu : M^* \to M$ be a desingularization of M that is a succession of blowing-ups. Here we can assume that μ is biholomorphic on $f^{-1}(N)$. Thus we have a fiber space $g := f \circ \mu : M^* \to B_o$ whose general fiber is $\mathbf{P}^n$.

LEMMA 5.7. *M^* is an algebraic variety.*

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Proof. As in [13, Section 12], we consider the direct image sheaf $g_*\mathcal{O}(-K_{M^*})$, and the associated projective fiber space $\mathbf{P}(g_*\mathcal{O}(-K_{M^*}))$ over B_o. Note that $\mathbf{P}(g_*\mathcal{O}(-K_{M^*}))$ is an algebraic space. Since $g_*\mathcal{O}(-K_{M^*})$ is a locally free sheaf of rank $= {}_{2n+1}C_{n+1}$ on a nonempty Zariski open subset of B_o, we have a commutative diagram

$$\begin{array}{ccc} M^* & \xrightarrow{h} & \mathbf{P}(g_*\mathcal{O}(-K_{M^*})) \\ g \searrow & & \swarrow \pi \\ & B_o, & \end{array}$$

where h is a meromorphic map whose restriction

$$h|_{g^{-1}(b)} : g^{-1}(b) \to \mathbf{P}(g_*\mathcal{O}(-K_{M^*}))_b$$

to a general fiber $g^{-1}(b)$ is the map defined by the linear system $|\mathcal{O}_{\mathbf{P}^n}(n+1)|$. Hence we infer that $\dim h(M^*) = \dim M^*$. Since $\mathbf{P}(g_*\mathcal{O}(-K_{M^*}))$ is algebraic, so is $h(M^*)$. Hence M^* is algebraic. □

Since $X = p_X(M) = p_X(\mu(M^*))$, we see that X is algebraic by Lemma 5.7. Let $\nu : Y \to X$ be a succession of blowing-ups such that Y is projective algebraic. Let $j : U \to W$ be a biholomorphic map. Since any meromorphic function on U extends to a meromorphic function on $\mathbf{P}^{2n+1}$, $\nu^{-1} \circ j : U \to Y$ extends to a meromorphic map $\mathbf{P}^{2n+1} \cdots> Y$. This implies that Y is unirational. Hence X is unirational. □

Theorem 5.8. *A compact complex manifold that is covered by a large domain in $\mathbf{P}^{2n+1}$ is non-Kähler, except for $\mathbf{P}^{2n+1}$ itself.*

Proof. This follows from Propositions 5.1, 5.6 and Theorem 5.5. □

Note that, for a large domain in Theorem 5.8, we assume nothing on its fundamental group nor on its complement in $\mathbf{P}^{2n+1}$. Thus our result gives a slight generalization of [10, Proposition 1.9] for odd-dimensional projective spaces.

6. Klein combinations

Let $\Omega_\nu \subset \mathbf{P}^{2n+1}$, $\nu = 1, 2$, be large domains and let $\Gamma_\nu \subset \mathrm{Aut}(\Omega_\nu)$ be free and properly discontinuous groups. Put

$$U(\varepsilon) = \{|z_0|^2 + \cdots + |z_n|^2 < \varepsilon(|z_{n+1}|^2 + \cdots + |z_{2n+1}|^2)\} \subset \mathbf{P}^{2n+1}, \quad \varepsilon > 1,$$

$$N(\varepsilon) = [U(\varepsilon)] \setminus U(\varepsilon^{-1}).$$

Then

$$\sigma : N(\varepsilon) \to N(\varepsilon), \quad \sigma([z_0 : \cdots : z_n : z_{n+1} : \cdots : z_{2n+1}]) = [z_{n+1} : \cdots : z_{2n+1} : z_0 : \cdots : z_n]$$

is a biholomorphic map. Let $j_\nu : U(\varepsilon) \to X_\nu = \Omega_\nu/\Gamma_\nu$ be holomorphic open embeddings. Then we can consider the gluing

$$X_1 \# X_2 = (X_1 \setminus j_1(U(\varepsilon^{-1}))) \bigcup (X_2 \setminus j_2(U(\varepsilon^{-1})))$$

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by $j_2 \circ \sigma \circ j_1{}^{-1} : j_1(N(\varepsilon)) \to j_2(N(\varepsilon))$ to obtain a new complex manifold. Then

$$X_1 \# X_2 = \Omega/\Gamma$$

for some large domain $\Omega \subset \mathbf{P}^{2n+1}$ and Γ [6]. Here we have $\Gamma \simeq \Gamma_1 * \Gamma_2$. $X_1 \# X_2$ is called the *Klein combination* of X_1 and X_2. If the Γ_ν are cocompact then so is Γ on Ω.

The *handle attachments* can also be defined. In those cases, we have $\Gamma \simeq \Gamma_1 * \mathbf{Z}$. Thus we can get many examples of Γ and $X(\Gamma)$.

7. An analogue of the Ford region

Fix a system of homogeneous coordinates $[z^0 : z^1 : \cdots : z^n : z^{n+1} : \cdots : z^{2n+1}]$ on $\mathbf{P}^{2n+1}$. Put $z' = (z^0, \ldots, z^n), z'' = (z^{n+1}, \ldots, z^{2n+1})$, and write $[z' : z'']$ instead of $[z^0 : z^1 : \cdots : z^n : z^{n+1} : \cdots : z^{2n+1}]$ for brevity. Let ℓ'' be the n-plane defined by $z'' = 0$. Put

$$E = \mathbf{P}^{2n+1} \setminus \ell'',$$

and define the projection by

$$\pi : E \to \mathbf{P}^n, \quad \pi([z' : z'']) = z''.$$

Then E is isomorphic to $\mathcal{O}_{\mathbf{P}^n}(1)^{\oplus(n+1)}$ as a vector bundle over $\mathbf{P}^n$.

7.1. Volume form on *E*. Take the open covering of $E = \bigcup_{\alpha=1}^{n+1} U_\alpha$, where

$$U_\alpha = \{[z' : z''] \in \mathbf{P}^{2n+1} : z^{n+\alpha} \neq 0\} \quad \text{for all } 1 \leq \alpha \leq n+1.$$

On each U_α, we define a system of coordinates by

$$\begin{cases} \zeta_\alpha^j = \dfrac{z^j}{z^{n+\alpha}} & \text{for } 0 \leq j \leq n, \\ x_\alpha^k = \dfrac{z^{n+k}}{z^{n+\alpha}} & \text{for } 1 \leq k < \alpha, \\ x_\alpha^{k-1} = \dfrac{z^{n+k}}{z^{n+\alpha}} & \text{for } \alpha < k \leq n+1. \end{cases}$$

Then $\pi|U_\alpha$ is given by

$$\pi(\zeta_\alpha^0, \ldots, \zeta_\alpha^n, x_\alpha^1, \ldots, x_\alpha^n) = (x_\alpha^1, \ldots, x_\alpha^n).$$

On U_α, we define

$$\begin{aligned} d\zeta_\alpha &= d\zeta_\alpha^0 \wedge \cdots \wedge d\zeta_\alpha^n \\ dx_\alpha &= dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n \end{aligned}$$

and put

$$dV_\alpha = \sqrt{-1}(1 + \|x_\alpha\|^2)^{-2(n+1)} d\zeta_\alpha \wedge \overline{d\zeta_\alpha} \wedge dx_\alpha \wedge \overline{dx_\alpha},$$

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where

$$\|x_\alpha\|^2 = \sum_{k=1}^{n} |x_\alpha^k|^2.$$

It is easy to check that the $(2n+1, 2n+1)$-forms dV_α patch together to give a global volume form

$$dV = dV_\alpha \quad \text{on } U_\alpha$$

on $E = \mathbf{P}^{2n+1} \setminus \ell''$.

LEMMA 7.1. *Consider the projective transformation of* $\mathbf{P}^m$ *defined by*

$$y^\lambda = \frac{c_\mu^\lambda x^\mu + c_0^\lambda}{c_\mu x^\mu + c_0} \quad \textit{for all } 1 \le \lambda, \mu \le m.$$

Then

$$dy^1 \wedge \cdots \wedge dy^m = \frac{\det C}{(c_\mu x^\mu + c_0)^{m+1}} dx^1 \wedge \cdots \wedge dx^m,$$

where

$$C = \begin{pmatrix} c_0 & c_1 & \ldots & c_m \\ c_0^1 & c_1^1 & \ldots & c_m^1 \\ \vdots & & \vdots & \\ c_0^m & c_1^m & \ldots & c_m^m \end{pmatrix}.$$

PROOF. Put

$$P = c_\mu x^\mu + c_0, \quad Q = c_\mu x^\mu, \quad p^\lambda = c_\mu^\lambda x^\mu + c_0^\lambda, \quad q^\lambda = c_\mu^\lambda x^\mu,$$

where μ is summed for $\mu = 1, \ldots, m$. Then

$$\begin{aligned} dy^1 \wedge \cdots \wedge dy^m &= \bigwedge_{\lambda=1}^{m} (P^{-1} dq^\lambda - p^\lambda P^{-2} dQ) \\ &= P^{-2m} \bigwedge_{\lambda=1}^{m} (P dq^\lambda - p^\lambda dQ) \\ &= P^{-(m+1)} \Big(P \bigwedge_{\lambda=1}^{m} dq^\lambda + \sum_{k=1}^{m} (-1)^k p^k dQ \wedge dq^1 \wedge \cdots \wedge dq^{k-1} \wedge dq^{k+1} \wedge \cdots \wedge dq^m \Big). \end{aligned}$$

Define A and A_k by

$$\begin{aligned} A dx^1 \wedge \cdots \wedge dx^m &= dq^1 \wedge \cdots \wedge dq^\lambda, \\ A_k dx^1 \wedge \cdots \wedge dx^m &= dQ \wedge dq^1 \wedge \cdots \wedge dq^{k-1} \wedge dq^{k+1} \wedge \cdots \wedge dq^m. \end{aligned}$$

Then

$$dy^1 \wedge \cdots \wedge dy^m = P^{-(m+1)} \Big((c_\mu x^\mu + c_0) A + \sum_{k=1}^{m} (-1)^k (c_\mu^k x^\mu + c_0^k) A_k \Big) dx^1 \wedge \cdots \wedge dx^m. \tag{7.1}$$

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Note that

$$A = \det\begin{pmatrix} c_1^1 & \dots & c_m^1 \\ \vdots & & \vdots \\ c_1^m & \dots & c_m^m \end{pmatrix} \quad \text{and} \quad A_k = \det\begin{pmatrix} c_1 & \dots & c_m \\ c_1^1 & \dots & c_m^1 \\ \vdots & & \vdots \\ c_1^{k-1} & & c_m^{k-1} \\ c_1^{k+1} & & c_m^{k+1} \\ \vdots & & \vdots \\ c_1^m & \dots & c_m^m \end{pmatrix}.$$

Thus

$$c_\mu A + \sum_{k=1}^m (-1)^k c_\mu^k A_k = \det\begin{pmatrix} c_\mu & c_1 & \dots & c_m \\ c_\mu^1 & c_1^1 & \dots & c_m^1 \\ \vdots & & \vdots & \\ c_\mu^m & c_1^m & \dots & c_m^m \end{pmatrix} = 0$$

for $\mu = 1, \dots, m$, and

$$c_0 A + \sum_{k=1}^m (-1)^k c_0^k A_k = \det\begin{pmatrix} c_0 & c_1 & \dots & c_m \\ c_0^1 & c_1^1 & \dots & c_m^1 \\ \vdots & & \vdots & \\ c_0^m & c_1^m & \dots & c_m^m \end{pmatrix} = \det C.$$

Hence, it follows from (7.1) that

$$dy^1 \wedge \cdots dy^m = P^{-(m+1)} \det C \, dx^1 \wedge \cdots \wedge dx^m. \qquad \square$$

LEMMA 7.2.[1] *For*

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_{2n+2}(\mathbf{C}), \quad A, B, C, D \in M_{n+1}(\mathbf{C})$$

with $\det C \neq 0$, *the pull-back of* dV *is given by*

$$g^* dV = \mu_g^{4(n+1)} dV,$$

where

$$\mu_g(z) = \frac{\|z''\|}{\|Cz' + Dz''\|}.$$

[1]This is the corrected version of [8, Lemma 3.2]. There was a mistake in the calculation there. The results [8, Proposition 3.1, Lemma 3.3] hold true. Calculations in the proofs there should be corrected accordingly, but need no essential changes. Sublemmas 3.1, 3.2 in [8] and their proofs are correct.

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PROOF. We write a square matrix M of size $(n+1)$ as

$$M = \begin{pmatrix} m_0^0 & \dots & m_n^0 \\ \vdots & & \vdots \\ m_0^n & \dots & m_n^n \end{pmatrix}.$$

Set $\alpha = n+1$ and consider the projective transformation g on $U_\alpha = U_{n+1}$. We omit the subscript $n+1$, for simplicity, and write the local coordinates by $(\zeta^0, \dots, \zeta^n, x^1, \dots, x^n)$ instead of $(\zeta^0_{n+1}, \dots, \zeta^n_{n+1}, x^1_{n+1}, \dots, x^n_{n+1})$. Then g sends (ζ^j, x^k) to $(\zeta^{j\prime}, x^{k\prime})$, where

$$\zeta^{j\prime} = \frac{\sum_{\lambda=0}^n a_\lambda^j \zeta^\lambda + \sum_{\mu=0}^{n-1} b_\mu^j x^{\mu+1} + b_n^j}{\sum_{\lambda=0}^n c_\lambda^n \zeta^\lambda + \sum_{\mu=0}^{n-1} d_\mu^n x^{\mu+1} + d_n^n} \quad \text{for all } j = 0, \dots, n,$$

$$x^{k\prime} = \frac{\sum_{\lambda=0}^n c_\lambda^k \zeta^\lambda + \sum_{\mu=0}^{n-1} d_\mu^k x^{\mu+1} + d_n^k}{\sum_{\lambda=0}^n c_\lambda^n \zeta^\lambda + \sum_{\mu=0}^{n-1} d_\mu^n x^{\mu+1} + d_n^n} \quad \text{for all } k = 1, \dots, n.$$

Then, by Lemma 7.1,

$$d\zeta' \wedge \overline{d\zeta'} \wedge dx' \wedge \overline{dx'} = \left| \sum_{\lambda=0}^n c_\lambda^n \zeta^\lambda + \sum_{\mu=0}^{n-1} d_\mu^n x^{\mu+1} + d_n^n \right|^{-4(n+1)} d\zeta \wedge \overline{d\zeta} \wedge dx \wedge \overline{dx}.$$

Hence

$$\begin{aligned} g^* dV &= \sqrt{-1}\left(1 + \sum_{k=1}^n \left| \frac{\sum_{\lambda=0}^n c_\lambda^k \zeta^\lambda + \sum_{\mu=0}^{n-1} d_\mu^k x^{\mu+1} + d_n^k}{\sum_{\lambda=0}^n c_\lambda^n \zeta^\lambda + \sum_{\mu=0}^{n-1} d_\mu^n x^{\mu+1} + d_n^n} \right|^2 \right)^{-2(n+1)} \\ &\quad \times \left| \sum_{\lambda=0}^n c_\lambda^n \zeta^\lambda + \sum_{\mu=0}^{n-1} d_\mu^n x^{\mu+1} + d_n^n \right|^{-4(n+1)} d\zeta \wedge \overline{d\zeta} \wedge dx \wedge \overline{dx} \\ &= \sqrt{-1}\left(\sum_{k=0}^n \left| \sum_{\lambda=0}^n c_\lambda^k \zeta^\lambda + \sum_{\mu=0}^{n-1} d_\mu^k x^{\mu+1} + d_n^k \right|^2 \right)^{-2(n+1)} d\zeta \wedge \overline{d\zeta} \wedge dx \wedge \overline{dx} \\ &= \sqrt{-1} \|C\zeta + D\tilde{x}\|^{-4(n+1)} d\zeta \wedge \overline{d\zeta} \wedge dx \wedge \overline{dx}, \end{aligned}$$

where $\tilde{x} = (x^1, \dots, x^n, 1)$. Thus

$$g^* dV = \left(\frac{\|\tilde{x}\|}{\|C\zeta + D\tilde{x}\|} \right)^{4(n+1)} dV = \left(\frac{\|z''\|}{\|Cz' + Dz''\|} \right)^{4(n+1)} dV.$$

This proves the lemma. □

7.2. *F*-region. Recall that the norm of $u = (u_1, \dots, u_m) \in \mathbf{C}^m$ is defined by

$$\|u\| = (|u_1|^2 + \dots + |u_m|^2)^{1/2}.$$

The norm of a matrix $A = (a_{ij}) \in \mathrm{M}_m(\mathbf{C})$ is defined by the operator norm

$$\|A\| = \sup_{z \neq 0, z \in \mathbf{C}^m} \frac{\|Az\|}{\|z\|}.$$

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Let $\Gamma \subset \mathrm{PGL}_{2n+2}(\mathbf{C})$ be a type $\mathbf{L}$ group, and set $\Omega = \Omega(\Gamma)$, $\Lambda = \Lambda(\Gamma)$. Put $\Gamma^* = \Gamma \setminus \{1\}$. Recall the proof of Proposition 5.1, where it is shown that Y_σ is a proper analytic subset of $\mathcal{G}$. That proof shows that, moving the n-plane $l'' = \{z'' = 0\}$ slightly, if necessary, we can choose a positive number R such that the set

$$V_R = \{[z' : z''] \in \mathbf{P}^{2n+1} : \|z'\| > R\|z''\|\}$$

is contained in Ω and that

$$g(V_R) \cap V_R = \emptyset \tag{7.2}$$

holds for any $g \in \Gamma^*$. Every $g \in \Gamma$ has a representative $\tilde{g} \in \mathrm{SL}_{2n+2}(\mathbf{C})$, which we write as

$$\tilde{g} = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}, \quad A_g, B_g, C_g, D_g \in \mathrm{M}_{n+1}(\mathbf{C}).$$

Lemma 7.3. *There is a constant $R_0 > 0$ such that, for any $g \in \Gamma^*$,*
(i) $\det C_g \neq 0$, (ii) $\|A_g C_g^{-1}\| \leq R_0$, (iii) $\|C_g^{-1} D_g\| \leq R_0$.

Proof. We fix an $R_0 = R$ that satisfies (7.2). (i) Suppose that $\det C_g = 0$. Then there is a point z on l'' such that $g(z) \in l''$. Thus $g(l'') \cap l'' \neq \emptyset$. Since $g \neq 1$, by assumption, this contradicts (7.2). (ii) The n-plane $g(l'')$ is given by $z' = A_g C_g^{-1} z''$. Since $g(l'') \cap V_{R_0} = \emptyset$ by (7.2), we have $\|A_g C_g^{-1}\| \leq R_0$. (iii) The equation of the n plane $g^{-1}(l'')$ is given by $z' = -C_g^{-1} D_g z''$. We have $\|C_g^{-1} D_g\| \leq R_0$ by the argument above. □

Put

$$\begin{aligned}
\Delta_g &= \{z = [z' : z''] \in \mathbf{P}^{2n+1} : \|z''\| < \|C_g z' + D_g z''\|\},\\
\bar{\Delta}_g &= \{z = [z' : z''] \in \mathbf{P}^{2n+1} : \|z''\| \leq \|C_g z' + D_g z''\|\},\\
\Sigma_g &= \{z = [z' : z''] \in \mathbf{P}^{2n+1} : \|z''\| = \|C_g z' + D_g z''\|\},\\
\Delta_g^c &= \{z = [z' : z''] \in \mathbf{P}^{2n+1} : \|z''\| \geq \|C_g z' + D_g z''\|\}
\end{aligned}$$

and

$$\bar{\Delta} = \bigcap_{g \in \Gamma^*} \bar{\Delta}_g.$$

Definition 7.4. Consider the set of interior points

$$F = \mathrm{Int}\bar{\Delta}$$

of $\bar{\Delta}$, which we call the F-region of the type $\mathbf{L}$ groups.

This is an analogue of the Ford region in the Kleinian group theory. Indeed, for some type $\mathbf{L}$ groups that satisfy an additional condition (see (♣), (♠) below), F will give a fundamental set of the action of Γ on Ω. Now we put

$$\begin{aligned}
\bar{F} &= \text{the closure of } F \text{ in } \mathbf{P}^{2n+1},\\
\partial\bar{F} &= \bar{F} \setminus F.
\end{aligned} \tag{7.3}$$

We consider the set of positive real numbers,

$$\mathcal{R} = \{\|C_g^{-1}\| : g \in \Gamma^*\},$$

and consider the conditions on $\mathcal{R}$:

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(♣) $\mathcal{R}$ is bounded in $\mathbf{R}$; and

(♠) $\mathcal{R}$ has no accumulation points other than 0 in $\mathbf{R}$.

Remark 7.5. The number $\|C_g^{-1}\|$ is something like the radius of the isometric circle of g in Kleinian group theory. The conditions (♣) and (♠) may depend on the choice of homogeneous coordinates on $\mathbf{P}^{2n+1}$. But they are preserved under the coordinate change $w = \tau(z)$ of the form $\tau = \left(\begin{smallmatrix} P & Q \\ 0 & S \end{smallmatrix}\right) \in \mathrm{PGL}_{2n+2}(\mathbf{C})$.

Proposition 7.6. *The condition (♣) is satisfied if and only if $\bar{F}$ contains V_R for some $R > 0$.*

Proof. To prove that (♣) is sufficient, let $\rho > 0$ be an upper bound of $\mathcal{R}$. Set $R = R_0 + \rho$, where R_0 is the constant in Lemma 7.3(iii). Then, for any point $z = [z' : z''] \in V_R$ and any $g \in \Gamma^*$,

$$\|C_g z' + D_g z''\| \geq \frac{\|z' + C_g^{-1} D_g z''\|}{\|C_g^{-1}\|} \geq \frac{\|z'\| - R_0\|z''\|}{\|C_g^{-1}\|} \geq \frac{\rho\|z''\|}{\|C_g^{-1}\|} \geq \|z''\|.$$

To prove that (♣) is necessary, take any n-plane

$$\ell_Y : z'' = Yz', \quad Y \in \mathrm{M}_{n+1}(\mathbf{C}), \quad \|Y\| < R^{-1}$$

in V_R. Since $\ell_Y \subset \bar{\Delta}_g$ for any $g \in \Gamma^*$,

$$\|Yz'\| \leq \|C_g z' + D_g Y z'\| \tag{7.4}$$

for any $z' \in \mathbf{C}^2$. Put

$$G = (C_g + D_g Y)^*(C_g + D_g Y) - Y^* Y = (I + C_g^{-1} D_g Y)^* C_g^* C_g (I + C_g^{-1} D_g Y) - Y^* Y,$$

where $M^* = {}^t\bar{M}$.

For Hermitian matrices A, B, we write $A \geq B$ if $A - B$ is positive semidefinite, and we write $A > B$ if $A - B$ is positive definite.

Note that $G \geq 0$ by (7.4) and that $\det(I + C_g^{-1} D_g Y) \neq 0$ holds for any Y with $\|Y\| < \min\{R^{-1}, R_0^{-1}\}$ and any $g \in \Gamma^*$ by Lemma 7.3(iii). Therefore, the inequality

$$C_g^* C_g \geq (I + C_g^{-1} D_g Y)^{*^{-1}} Y^* Y (I + C_g^{-1} D_g Y)^{-1}$$

holds for $\|Y\| < \min\{R^{-1}, R_0^{-1}\}$ and $g \in \Gamma^*$. Set $Y = tI$, $t = \frac{1}{2}\min\{1, R^{-1}, R_0^{-1}\}$. Then

$$C_g^* C_g \geq t^2 (I + tC_g^{-1} D_g)^{*^{-1}} (I + tC_g^{-1} D_g)^{-1} > \frac{t^2}{4} I.$$

Thus

$$\|C_g^{-1}\| \leq \frac{2}{t}.$$

□

Lemma 7.7. *Suppose that Γ satisfies (♣). Then, for any normal sequence $\{g_n\} \subset \Gamma$, the sequence $\{\Sigma_{g_n}\}$ converges as sets to a single limit n-plane if and only if Γ satisfies (♠).*

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Proof. Set $\tilde{g}_n = \left(\begin{smallmatrix} A_n & B_n \\ C_n & D_n \end{smallmatrix}\right) \in \mathrm{SL}_{2n+2}(\mathbf{C})$. By the defining equation $\|C_n z' + D_n z''\| = \|z''\|$,

$$\Sigma_{g_n} = \{[z' : z''] \in \mathbf{P}^{2n+1} : z' = (C_n^{-1}U - C_n^{-1}D_n)\eta, z'' = \eta, \eta \in S^{2n+1}, U \in \mathrm{U}_{n+1}\},$$

where S^{2n+1} is the unit sphere in $\mathbf{C}^{n+1}$ and U_{n+1} is the group of unitary matrices of size $n + 1$. The sequence $\{C_n^{-1}D_n\}$ is bounded by Lemma 7.3(iii) and so is $\{C_n^{-1}\}$ by assumption (♣). Now consider any subsequence of $\{g_n\}$ such that $\{C_n^{-1}\}$ converges. Then take again a subsequence such that $\{C_n^{-1}D_n\}$ also converges. Put $L = -\lim_{n\to\infty} C_n^{-1}D_n$ and $G = \lim_{n\to\infty} C_n^{-1}$. Then we see that the set Σ_{g_n} converges to the set

$$\Sigma := \{[z' : z''] \in \mathbf{P}^{2n+1} : z' = (GU + L)\eta, z'' = \eta, \eta \in S^{2n+1}, U \in \mathrm{U}_{n+1}\}.$$

Thus Σ consists of a single n-plane if and only if $G = 0$. Here $z' = Lz''$ is the limit n-plane of $\{g_n^{-1}(\ell'')\}$. This implies the lemma. □

Lemma 7.8. *Suppose that Γ satisfies (♣) and (♠). Then, for $a \in \Omega$, there can be at most a finite number of Σ_g that contain a.*

Proof. Suppose that there is an infinite number of $g_n \in \Gamma$, $n = 1, 2, \ldots,$ such that $a \in \Sigma_{g_n}$. Then taking a normal subsequence of $\{g_n\}$, we see that a is on a limit n-plane by Lemma 7.7, since Γ satisfies (♠). This contradicts $a \in \Omega$. □

Now recall the definition of μ_g for $g \in \Gamma^*$. We also define

$$\mu_1(z) \equiv 1 \quad \text{for } g = 1.$$

Lemma 7.9. *For $g, h \in \Gamma$,*

$$\mu_{h\circ g}(z) = \mu_h(g(z))\mu_g(z), \quad z \in \mathbf{P}^{2n+1} \setminus \{g^{-1}(\ell'') \cup (h \circ g)^{-1}(\ell'')\}.$$

Proof. This is easy by Lemma 7.2. □

Lemma 7.10. *For any $g \in \Gamma$, we have $g(\bar{\Delta}) \subset \Delta^c_{g^{-1}}$.*

Proof. By Lemma 7.9, $\mu_{g^{-1}}(g(z)) = \mu_g(z)^{-1}$. Since $\mu_g(z) \le 1$ for $z \in \bar{\Delta}$, we have the lemma. □

Theorem 7.11. *Let $\Gamma \subset \mathrm{PGL}_{2n+2}(\mathbf{C})$ be a type $\mathbf{L}$ group. Assume that Γ is torsion free and satisfies both (♣) and (♠). Then F has the following properties.*

(1) *For $g \in \Gamma$, $g(F) \subset F$ holds if and only if $g = 1$.*
(2) *For $g \in \Gamma^*$, $g(F) \cap F = \emptyset$ holds.*
(3) *For every $z \in \Omega$, there is an element $g \in \Gamma$ such that $g(z) \in \Omega \cap \bar{F}$.*
(4) *Suppose that the equality $w = g(z)$ holds for some $z, w \in \Omega \cap \bar{F}$ and $g \in \Gamma^*$. Then both z and w are on $\Omega \cap \partial\bar{F}$.*

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Proof. The following proof is an analogue of [11, pages 33–34].

(1) By Proposition 7.6, F contains a tubular neighborhood W of ℓ''. Suppose that $g(F) \subset F$, Then, by Lemma 7.9,

$$\mu_{h\circ g}(z) = \mu_h(g(z))\mu_g(z) \leq \mu_g(z) \leq 1$$

on W for any h. Letting $h = g^{-1}$, we see that $\mu_g(z) = 1$ on W. This implies that $C_g = 0$. Hence $g(\ell'') = \ell''$. Since ℓ'' is not a limit n-plane, we see that g is of finite order. Since Γ is torsion free, by assumption, we see that $g = 1$. The converse is obvious.

(2) By Lemma 7.10, we have $g(\bar{\Delta}) \subset \Delta^c_{g^{-1}}$. This implies that $g(\bar{F}) \cap \bar{\Delta}_{g^{-1}} = \emptyset$, since F is open. Hence $g(F) \cap F \subset g(F) \cap \bar{\Delta} \subset g(F) \cap \bar{\Delta}_{g^{-1}} = \emptyset$.

(3) Take a point z in Ω. If $g(z) \in \ell''$ for some $g \in \Gamma$, then $g(z) \in \Omega \cap \bar{F}$ by the assumption (♣) and Proposition 7.6. Therefore we can assume that $g(z) \notin \ell''$ for any g. Then $\mu_g(z)$ is defined and hence has a finite-value for any g. By the assumptions (♣) and (♠), $\mu_g(z) < 1$ holds for all except for finitely many $g \in \Gamma$. Therefore we can choose g such that $\mu_g(z)$ is maximal among all g. Then, by Lemma 7.9, we have $\mu_h(g(z)) \leq 1$ for any $h \in \Gamma$. This implies that $g(z) \in \bar{\Delta}$. Thus $\Omega \subset \bigcup_{g\in\Gamma} g(\bar{\Delta})$ and hence

$$\Omega = \bigcup_{g\in\Gamma} g(\Omega \cap \bar{\Delta}). \tag{7.5}$$

We claim that the set $\Omega \cap (\bar{\Delta} \setminus \bar{F})$ is empty. To verify this, we suppose, on the contrary, that a point $w \in \Omega \cap (\bar{\Delta} \setminus \bar{F})$ exists. Since $\bar{\Delta} \setminus \bar{F}$ is thin in $\mathbf{P}^{2n+1}$, so is $\bigcup_{g\in\Gamma} g(\Omega \cap (\bar{\Delta} \setminus \bar{F}))$. Hence, by

$$\Omega \Big\backslash \bigcup_{g\in\Gamma} g(\bar{F}) = \bigcup_{g\in\Gamma} g(\Omega \cap \bar{\Delta}) \Big\backslash \bigcup_{g\in\Gamma} g(\Omega \cap \bar{F}) \subset \bigcup_{g\in\Gamma} g(\Omega \cap (\bar{\Delta} \setminus \bar{F})),$$

we see that the set $\Omega \setminus \bigcup_{g\in\Gamma} g(\bar{F})$ is thin in Ω. Therefore we can find sequences $\{w_n\} \subset \Omega \cap \bar{F} \subset \bar{\Delta}$ and $\{g_n\} \subset \Gamma$ such that $\lim_{n\to\infty} g_n(w_n) = w$. Since $w \notin \partial\bar{F}$, $\{g_n\}$ can be chosen to be a sequence of distinct elements. By Lemma 7.3 and the assumptions (♣) and (♠), we can choose a subsequence of $\{g_n\}$ such that the n-plane $g_n(\ell'') = \{z' + C^{-1}_{g_n^{-1}} D_{g_n^{-1}} z'' = 0\}$ converges to a limit n-plane $\ell_L = \{z' = Lz''\}$, $L = -\lim_{n\to\infty} C^{-1}_{g_n^{-1}} D_{g_n^{-1}}$, and such that $\lim_{n\to\infty} \|C^{-1}_{g_n^{-1}}\| = 0$ holds. The set $\Delta^c_{g_n^{-1}}$ is the image of the map

$$\varphi_n : [0, 1] \times U_{n+1} \times S^{2n+1} \to \mathbf{P}^{2n+1}$$

defined by

$$\varphi_n : (t, U, \eta) \mapsto [tC^{-1}_{g_n^{-1}} U\eta - C^{-1}_{g_n^{-1}} D_{g_n^{-1}} \eta : \eta].$$

Therefore the sequence $\{\Delta^c_{g_n^{-1}}\}$ of sets converges to the image of the limit map

$$\varphi : [0, 1] \times U_{n+1} \times S^{2n+1} \to \mathbf{P}^{2n+1}, \quad (t, U, \eta) \mapsto [L\eta : \eta],$$

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which is ℓ_L. Since $g_n(\ell'') \subset g_n(\bar{\Delta}) \subset \Delta^c_{g_n^{-1}}$ holds by Lemma 7.10, $\{g_n(\bar{\Delta})\}$ also converges to ℓ_L. Hence w is on the limit n-plane ℓ_L. Since $w \in \Omega$, this is absurd. Thus our claim is verified. Now, by (7.5),

$$\Omega = \bigcup_{g \in \Gamma} g(\Omega \cap \bar{F}).$$

This proves (3).

(4) By (2), either $z \in \partial\bar{F}$ or $w \in \partial\bar{F}$. Replacing g with g^{-1}, if necessary, we can assume that $z \in \partial\bar{F}$. Since $z, w \in \bar{F}$, $\mu_f(z) \leq 1$ and $\mu_f(w) \leq 1$ hold for any $f \in \Gamma^*$. Hence, by the equality

$$1 = \mu_1(z) = \mu_{g^{-1}g}(z) = \mu_{g^{-1}}(g(z))\mu_g(z) = \mu_{g^{-1}}(w)\mu_g(z),$$

we have $\mu_{g^{-1}}(w) = \mu_g(z) = 1$. Hence, in particular, we have $w \in \Sigma_{g^{-1}}$. On the other hand, there can be at most a finite number of Σ_f with $w \in \Sigma_f$ by Lemma 7.8. This implies that $w \in \partial\bar{F}$. □

Remark 7.12. For type **L** groups, both conditions (♣) and (♠) are automatically satisfied if the series

$$\sum_{g \in \Gamma^*} \|C_g^{-1}\|^{\delta}$$

is convergent for some constant $\delta > 0$.

Theorem 7.11 is useful when we check whether the quotient space $\Omega(\Gamma)/\Gamma$ becomes compact or not. See examples in Section 8.2.

8. Examples

8.1. Type L groups in general dimension. Suppose that we are given two groups $\Gamma_\nu \subset \mathrm{PGL}_{2n+1}(\mathbf{C})$, $\nu = 1, 2$, of type **L** which satisfy (♣) and (♠). Applying a Klein combination, we can construct another group Γ of type **L** which is isomorphic to the free product $\Gamma_1 * \Gamma_2$. In this subsection, we show that, replacing Γ_ν with their suitable conjugate subgroups in $\mathrm{PGL}_{2n+1}(\mathbf{C})$, we can make Γ also satisfy both (♣) and (♠).

Let F_ν the F-region of Γ_ν with respect to $[z] = [z' : z'']$. Let $\rho_\nu > 0$ be the numbers such that $\|C_g^{-1}\| \leq \rho_\nu$ for all $g \in \Gamma_\nu^*$.

Lemma 8.1. *Let $a \in \mathbf{R}$ be a positive constant, and consider the new system of coordinates $[\zeta' : \zeta'']$ on $\mathbf{P}^{2n+1}$ defined by*

$$\zeta' = az', \quad \zeta'' = a^{-1}z''.$$

Let $\alpha = \begin{pmatrix} aI & 0 \\ 0 & a^{-1}I \end{pmatrix} \in \mathrm{PGL}_{2n+2}(\mathbf{C})$. Then, for a given number $r > 0$, we can choose $a > 0$ so that the following are satisfied simultaneously.

(i) *$\alpha(F_1 \cap F_2)$ contains the set $V = \{\|\zeta'\| \geq r\|\zeta''\|\}$.*

(ii) *$\|C_g^{-1}\| \leq 1$ for all $g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix} \in \alpha\Gamma_\nu^*\alpha^{-1}$, $\nu = 1, 2$.*

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Proof. Since F_ν contains a tubular neighborhood of $z'' = 0$, there is $r_1 > 0$ such that

$$\{\|z'\| \geq r_1 \|z''\|\} \subset F_1 \cap F_2.$$

Choose $a > 0$ satisfying

$$a^2 \leq r_1^{-1} r. \tag{8.1}$$

Take any $[\zeta' : \zeta''] \in V$, and set $[z' : z''] = \alpha^{-1}([\zeta' : \zeta''])$. Then $z' = a^{-1}\zeta'$ and $z'' = a\zeta''$, and

$$\|z'\| = a^{-1}\|\zeta'\| \geq a^{-1} r \|\zeta''\| = a^{-2} r \|z''\| \geq r_1 \|z''\|.$$

Hence $[z' : z''] \in F_1 \cap F_2$. This shows that $[\zeta' : \zeta''] \in \alpha(F_1 \cap F_2)$. Thus (i) is satisfied for $a > 0$ with (8.1).

Let $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_\nu^*$. Then by $g = \alpha\gamma\alpha^{-1} = \left(\begin{smallmatrix} A & a^2 B \\ a^{-2} C & D \end{smallmatrix}\right)$, we have $\|C_g^{-1}\| = a^2 \|C^{-1}\|$. Therefore the number $a > 0$ with

$$a^2 \leq \rho_\nu^{-1}, \quad \nu = 1, 2 \tag{8.2}$$

satisfies (ii). Thus it is enough to choose $a > 0$ which satisfies (8.1) and (8.2). □

Fix $a > 0$ such that (i) and (ii) in the lemma above hold and replace the original coordinates $[z' : z'']$ with $[\zeta' : \zeta'']$, Γ_ν with $\alpha\Gamma_\nu\alpha^{-1}$ and F_ν with $\alpha(F_\nu)$. We use the original notation such as $[z' : z'']$, Γ_ν and F_ν to avoid abuse of notation. Let $U_r = \{\|z'\| \leq r\|z''\|\}$. Then $\gamma(F_\nu) \subset U_r$ for $\gamma \in \Gamma_\nu^*$, $(\nu = 1, 2)$.

Put $\sigma = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) \in \mathrm{PGL}_{2n+2}(\mathbf{C})$, and consider the sets F_1 and $\sigma(F_2)$ as two subsets in the same projective space $\mathbf{P}^{2n+1}$. Note that the set $\{r\|z''\| \leq \|z'\| \leq r^{-1}\|z''\|\}$, $0 < r < 1$, is contained in $F_1 \cap \sigma(F_2)$ and that $\gamma(\sigma(F_2)) \subset \sigma(U_r)$ for $\gamma \in \sigma\Gamma_2\sigma^{-1}$.

Put $\tau = \left(\begin{smallmatrix} I & I \\ -I & I \end{smallmatrix}\right) \in \mathrm{PGL}_{2n+2}(\mathbf{C})$. Introduce a new coordinate system $[w] = [w' : w'']$ by $w = \tau(z)$. Put $\Gamma_1' = \tau\Gamma_1\tau^{-1}$ and $\Gamma_2' = \tau(\sigma\Gamma_2\sigma^{-1})\tau^{-1}$. Let Γ be the group generated by Γ_1' and Γ_2'. This subsection is devoted to proving the following theorem.

Theorem 8.2. *Γ is a group of type* $\mathbf{L}$ *which satisfies* (♣) *and* (♠) *with respect to* $[w]$.

Proof. By the construction, we see that $\tau(F_1 \cap \sigma(F_2))$ is a fundamental set of Γ. The n-plane $\{w'' = 0\}$ has a tubular neighborhood contained in $\tau(F_1 \cap \sigma(F_2))$. Thus, by Proposition 7.6, it is enough to show that (♠) is satisfied. □

Lemma 8.3. *Let $\ell_P : w' = Pw''$ and $\ell_Q : w' = Qw''$ be n-planes in $\tau(U_r)$ and $\tau\sigma(U_r)$, respectively. Then there is a positive constant K_r such that*

$$\|(P - Q)^{-1}\| \leq K_r,$$

where

$$\lim_{r \to 0} K_r = 2^{-1}.$$

Proof. Take the n-planes $\ell_X : z' = Xz''$ in U_r and $\ell_Y : z'' = Yz'$ in $\sigma(U_r)$ such that $\tau(\ell_X) = \ell_P$ and $\tau(\ell_Y) = \ell_Q$, respectively. Then

$$P = (I + X)(I - X)^{-1}, \quad Q = -(I + Y)(I - Y)^{-1}.$$

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Since $\ell_P \cap \ell_Q = \emptyset$, $\det(P - Q) \neq 0$ holds. Hence

$$\|(P - Q)^{-1}\| = \|((I + X)(I - X)^{-1} + (I + Y)(I - Y)^{-1})^{-1}\|.$$

Set

$$K_r = \sup_{\{\|X\| \le r, \|Y\| \le r\}} \|((I + X)(I - X)^{-1} + (I + Y)(I - Y)^{-1})^{-1}\|.$$

It is clear that K_r is finite for $0 \le r < 1$ and that $\lim_{r\to 0} K_r = 2^{-1}$. This implies the lemma. □

Any element $f \in \Gamma^*$ can be written in the *normal* form [11, page 136].

$$f = g_m \cdots g_1.$$

Here either $g_{2j+1} \in \Gamma'^*_1$, $g_{2j} \in \Gamma'^*_2$, or $g_{2j+1} \in \Gamma'^*_2$, $g_{2j} \in \Gamma'^*_1$. The number m is called the *length* of f, which is denoted by $|f|$.

Lemma 8.4. *Take any element $f \in \Gamma^*$, and write f in the normal form as*

$$f = g_m \cdot g_{m-1} \cdots g_1.$$

Then

$$\|C_f^{-1}\| \le K_r^{m-1} \prod_{j=1}^{m} \|C_{g_j}^{-1}\|,$$

where

$$f = \begin{pmatrix} A_f & B_f \\ C_f & D_f \end{pmatrix}, \quad g_j = \begin{pmatrix} A_{g_j} & B_{g_j} \\ C_{g_j} & D_{g_j} \end{pmatrix}.$$

Proof. Set $g = g_m$ and $h = g_{m-1} \cdots g_1$. Comparing the components of $f = gh$ gives

$$C_f = C_g A_h + D_g C_h = C_g (A_h C_h^{-1} + C_g^{-1} D_g) C_h. \tag{8.3}$$

First, assume that $g_1 \in \Gamma'^*_1$. If $g \in \Gamma'^*_1$, then $|f| = m$ is odd. Since $g^{-1} \in \Gamma^*_1$, we have

$$g^{-1}(\{w'' = 0\}) = \{w' = -C_g^{-1} D_g w''\} \subset \tau(U_r).$$

Since $|h|$ is even, we see that $h(\{w'' = 0\}) = \{w' = A_h C_h^{-1} w''\} \subset \tau\sigma(U_r)$. Since $\tau(U_r) \cap \tau\sigma(U_r) = \emptyset$, we see that $\det(A_h C_h^{-1} + C_g^{-1} D_g) \neq 0$. Hence C_f is also nonsingular and, by (8.3),

$$C_f^{-1} = C_h^{-1} (A_h C_h^{-1} + C_g^{-1} D_g)^{-1} C_g^{-1}. \tag{8.4}$$

By Lemma 8.3, it follows that

$$\|(A_h C_h^{-1} + C_g^{-1} D_g)^{-1}\| \le K_r.$$

Hence, by (8.4),

$$\|C_f^{-1}\| \le K_r \|C_g^{-1}\| \cdot \|C_h^{-1}\|. \tag{8.5}$$

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If $g \in \Gamma'^*_2$, then $|f| = m$ is even. Since $g^{-1} \in \Gamma^*_2$, $g^{-1}(\{w'' = 0\}) = \{w' = -C_g^{-1} D_g w''\} \subset \tau\sigma(U_r)$. Since $|h|$ is odd, we see that $h(\{w'' = 0\}) = \{w' = A_h C_h^{-1} w''\} \subset \tau(U_r)$. Then, by the same argument as the case $g = g_m \in \Gamma'^*_1$, we obtain (8.5).

Next, assume that $g_1 \in \Gamma'^*_2$. If $g \in \Gamma'^*_1$, then $|f| = m$ is even. Since $g^{-1} \in \Gamma^*_1$, $g^{-1}(\{w'' = 0\}) = \{w' = -C_g^{-1} D_g w''\} \subset \tau(U_r)$. Since $|h|$ is odd, we see that $h(\{w'' = 0\}) = \{w' = A_h C_h^{-1} w''\} \subset \tau\sigma(U_r)$. Then the rest of the argument is the same as above, and we obtain (8.5).

If $g \in \Gamma'^*_2$, then $|f| = m$ is odd. Since $g^{-1} \in \Gamma^*_2$, $g^{-1}(\{w'' = 0\}) = \{w' = -C_g^{-1} D_g w''\} \subset \tau\sigma(U_r)$. Since $|h|$ is even, we see that $h(\{w'' = 0\}) = \{w' = A_h C_h^{-1} w''\} \subset \tau(U_r)$. Then the rest of the argument is the same as above, and we obtain (8.5).

The lemma follows from (8.5) by induction on m. □

Proof of Theorem 8.2 (continued). It remains to show that Γ satisfies (♠). By Lemma 8.1, we can assume that $\rho_\nu \leq 1$, $\nu = 1, 2$. By Lemma 8.3, we fix small r, $0 < r < 1$, such that $K_r < 1$ holds. Now we shall show that Γ satisfies (♠).

Suppose that (♠) does not hold. Then there is a sequence $\{f_m\}_m \subset \Gamma^*$ such that

$$\lim_{m\to\infty} \|C_{f_m}^{-1}\| = \varepsilon > 0. \tag{8.6}$$

If there is a subsequence $\{h_m\}$ of $\{f_m\}$ such that $\lim_{m\to\infty} |h_m| = \infty$, then $\lim_{m\to\infty} \|C_{h_m}^{-1}\| = 0$ follows from $K_r < 1$ and $\rho_\nu \leq 1$, by Lemma 8.4. This contradicts (8.6). Therefore the sequence $\{|f_m|\}_m$ of lengths is bounded. Let b be a bound of $\{|f_m|\}_m$, that is, $|f_m| \leq b$ for all m. Write f_m in the 'extended' normal form,

$$f_m = g_{m,b} g_{m,b-1} \cdots g_{m,1},$$

where $g_{m,|f_m|} \cdots g_{m,1}$ is the normal form of f_m and $g_{m,j} = 1$ for $|f_m| < j \leq b$. Since both Γ^*_1 and Γ^*_2 satisfy (♣) and (♠), we can find some k, $1 \leq k \leq b$, such that $\{\|C_{g_{m,k}}^{-1}\|\}_m$ contains a subsequence which converges to zero. This implies that the corresponding subsequence of $\{\|C_{f_m}^{-1}\|\}_m$ also converges to zero. This again contradicts (8.6). □

Remark 8.5. A typical higher-dimensional example treated in this subsection is a Schottky group. Let Γ be the infinite cyclic group generated by $g = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathrm{PGL}_{2n+2}(\mathbf{C})$. Let α_j be the eigenvalues of A and let β_k be the eigenvalues of B. Assume that $|\alpha_j| < |\beta_k|$ holds for any pairs (α_j, β_k). Then Γ is a type **L** group, where $\Omega(\Gamma) = \mathbf{P}^{2n+1} \setminus (\{z' = 0\} \cup \{z'' = 0\})$. Introduce a new coordinate $[w' : w'']$ by $w' = z' + z''$ and $w'' = -z' + z''$. Then Γ satisfies (♣) and (♠) with respect to $[w' : w'']$. By successive Klein combinations, we can get type **L** groups with (♣) and (♠) with respect to some coordinate system.

8.2. Type L groups in dimension three. In this subsection, we shall give three examples of type **L** groups. If a finitely generated discrete infinite subgroup $\Gamma \subset \mathrm{PGL}_4(\mathbf{C})$ admits an invariant surface S in $\mathbf{P}^3$ and never admits invariant planes, then S is necessarily one of the following: (i) the tangential surface of a twisted cubic curve; (ii) a nonsingular quadric surface; or (iii) a cone over a nonsingular conic [9].

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Each case has examples of type **L** groups with (♣) and (♠). In the cases (i) and (ii), there are examples with compact connected canonical quotients. The example for the case (ii) is due to Fujiki [2]. For the case (iii), we have only one example at present, whose canonical quotient is connected and noncompact, but it has an invariant plane.

8.2.1. Kleinian groups acting on a twisted cubic curve. Fix a twisted cubic curve $C \subset \mathbf{P}^3$, which is defined to be the image of the map

$$\tau : \mathbf{P}^1 \to \mathbf{P}^3, \quad \tau([s:1]) = [s^3 : s^2 : s : 1].$$

Then τ determines a group representation

$$\tau_* : \mathrm{PSL}_2(\mathbf{C}) \to \mathrm{PSL}_4(\mathbf{C})$$

such that $\tau \circ g = \tau_*(g) \circ \tau$. Explicitly, for $g = \pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbf{C})$, $\tau_*(g)$ is given by $\tau_*(g) = \pm\tilde{\tau}_*(g) \in \mathrm{PSL}_4(\mathbf{C})$, where

$$\tilde{\tau}_*(g) = \begin{pmatrix} a^3 & 3a^2b & 3ab^2 & b^3 \\ a^2c & a^2d + 2abc & 2abd + b^2c & b^2d \\ ac^2 & 2acd + bc^2 & ad^2 + 2bcd & bd^2 \\ c^3 & 3c^2d & 3cd^2 & d^3 \end{pmatrix} \in \mathrm{SL}_4(\mathbf{C}).$$

The group $\mathrm{PGL}_4(\mathbf{C})$ acts on the set of lines in $\mathbf{P}^3$, that is, on $\mathrm{Gr}(2,4)$. Using Plücker coordinates, we can embed $\mathrm{Gr}(2,4)$ into $\mathbf{P}^5 = \mathbf{P}(\wedge^2\mathbf{C}^4)$. Since any $A \in \mathrm{GL}_4(\mathbf{C})$ defines a linear automorphism on $\wedge^2\mathbf{C}^4 \simeq \mathbf{C}^6$, we have the group homomorphism

$$\rho : \mathrm{GL}_4(\mathbf{C}) \to \mathrm{GL}_6(\mathbf{C}).$$

Let $e_0 = {}^t(1,0,0,0), e_1 = {}^t(0,1,0,0), e_2 = {}^t(0,0,1,0), e_3 = {}^t(0,0,0,1)$ and $e_j \wedge e_k$ the linear two-space spanned by $\{e_j, e_k\}$, where $e_j \wedge e_k = -e_k \wedge e_j$. In this subsection, in the following, we write $\tilde{g} = \rho \circ \tilde{\tau}_*(g) \in \mathrm{GL}_6(\mathbf{C})$, which is well defined for $g = \pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbf{C})$. Then, with respect to the basis

$$\{e_0 \wedge e_1, e_0 \wedge e_2, e_0 \wedge e_3, e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\},$$

in $\mathbf{C}^6$, $\tilde{g} = \rho \circ \tilde{\tau}_*(g) \in \mathrm{GL}_6(\mathbf{C})$ is given by

$$\tilde{g} = \begin{pmatrix} a^4 & 2a^3b & a^2b^2 & 3a^2b^2 & 2ab^3 & b^4 \\ 2a^3c & a^2(ad+3bc) & ab(ad+bc) & 3ab(ad+bc) & b^2(3ad+bc) & 2b^3d \\ 3a^2c^2 & 3ac(ad+bc) & a^2d^2+abcd+b^2c^2 & 9abcd & 3bd(ad+bc) & 3b^2d^2 \\ a^2c^2 & ac(ad+bc) & abcd & a^2d^2+abcd+b^2c^2 & bd(ad+bc) & b^2d^2 \\ 2ac^3 & c^2(3ad+bc) & cd(ad+bc) & 3cd(ad+bc) & d^2(ad+3bc) & 2bd^3 \\ c^4 & 2c^3d & c^2d^2 & 3c^2d^2 & 2cd^3 & d^4 \end{pmatrix}.$$

Limit sets. In the following, in this subsection, we let $\Gamma \subset \mathrm{PSL}_2(\mathbf{C})$ be a Kleinian group whose set of discontinuity $\Omega_{\mathbf{P}^1}$ contains $[1:0] \in \mathbf{P}^1$. Put $\Lambda_{\mathbf{P}^1} = \mathbf{P}^1 \setminus \Omega_{\mathbf{P}^1}$. We consider the group $\tilde{\Gamma} = \tau_*(\Gamma)$, which we regard as a subgroup of $\mathrm{PGL}_4(\mathbf{C})$.

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Let $\{\gamma_n\} \subset \Gamma$ be a normal sequence. Let

$$g_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \mathrm{SL}_2(\mathbf{C}) \quad \text{for all } n = 1, 2, \ldots$$

be a sequence of representatives of $\{\gamma_n\}$ such that $\{c_n^{-1} g_n\}$ converges to a matrix of the form $h = \left(\begin{smallmatrix} \mu & -\lambda\mu \\ 1 & -\lambda \end{smallmatrix}\right) \in M_2(\mathbf{C})$, $\lambda, \mu \in \mathbf{C}$, since $\{a_n c_n^{-1}\}$ and $\{c_n^{-1} d_n\}$ are bounded (compare with Lemma 7.3). Put

$$G_n = c_n^{-4} \tilde{\tau}_*(g_n) \in \mathrm{GL}_6(\mathbf{C}).$$

Then

$$G := \lim_n G_n = \begin{pmatrix} \mu^4 & -2\lambda\mu^4 & \lambda^2\mu^4 & 3\lambda^2\mu^4 & -2\lambda^3\mu^4 & \lambda^4\mu^4 \\ 2\mu^3 & -4\lambda\mu^3 & 2\lambda^2\mu^3 & 6\lambda^2\mu^3 & -4\lambda^3\mu^3 & 2\lambda^4\mu^3 \\ 3\mu^2 & -6\lambda\mu^2 & 3\lambda^2\mu^2 & 9\lambda^2\mu^2 & -6\lambda^3\mu^2 & 3\lambda^4\mu^2 \\ \mu^2 & -2\lambda\mu^2 & \lambda^2\mu^2 & 3\lambda^2\mu^2 & -2\lambda^3\mu^2 & \lambda^4\mu^2 \\ 2\mu & -4\lambda\mu & 2\lambda^2\mu & 6\lambda^2\mu & -4\lambda^3\mu & 2\lambda^4\mu \\ 1 & -2\lambda & \lambda^2 & 3\lambda^2 & -2\lambda^3 & \lambda^4 \end{pmatrix}.$$

The limit G defines a projection to the limit image

$$\mathbf{P}^5 \setminus H \to I := \{[\mu^4 : 2\mu^3 : 3\mu^2 : \mu^2 : 2\mu : 1]\}.$$

The limit kernel H is the four-plane defined by

$$\{\zeta = [\zeta_j] \in \mathbf{P}^5 : \zeta_0 - 2\lambda\zeta_1 + \lambda^2\zeta_2 + 3\lambda^2\zeta_3 - 2\lambda^3\zeta_4 + \lambda^4\zeta_5 = 0\}.$$

Let ℓ_μ be the tangent line to the curve C at $[\mu : 1]$. Then $\hat{\ell}_\mu \in \mathrm{Gr}(2,4) \subset \mathbf{P}^5$ is given by

$$\begin{aligned} &(3\mu^2 e_0 + 2\mu e_1 + e_2) \wedge (\mu^3 e_0 + \mu^2 e_1 + \mu e_2 + e_3) \\ &\quad = \mu^4 e_0 \wedge e_1 + 2\mu^3 e_0 \wedge e_2 + 3\mu^2 e_0 \wedge e_3 + \mu^2 e_1 \wedge e_2 + 2\mu e_1 \wedge e_3 + e_2 \wedge e_3, \end{aligned}$$

which is nothing but the limit image $I = [\mu^4 : 2\mu^3 : 3\mu^2 : \mu^2 : 2\mu : 1] \in \mathbf{P}^5$. Hence ℓ_μ is the limit image of the sequence $\{\tau_*(\gamma_n)\}$. Here the limit kernel $H \cap \mathrm{Gr}(2,4)$ is the set of lines in $\mathbf{P}^3$ that intersect the tangent line to C at the limit point $\tau([\lambda : 1])$. Thus we have the following theorem.

Theorem 8.6. *Let $\Gamma \subset \mathrm{PSL}_2(\mathbf{C})$ be a Kleinian group. Then*

$$\tilde{\Gamma} = \tau_*(\Gamma) \subset \mathrm{PGL}_4(\mathbf{C})$$

is a group of type **L**. *The limit set is given by*

$$\Lambda(\tilde{\Gamma}) = \bigcup_{\lambda \in \Lambda_{\mathbf{P}^1}} |\ell_\lambda|,$$

where $|\ell_\lambda|$ is the support of the tangent line ℓ_λ to the twisted cubic curve at $\tau([\lambda : 1])$.

Proposition 8.7. *Let $\Gamma \subset \mathrm{PSL}_2(\mathbf{C})$ be a Kleinian group whose set of discontinuity contains the point $[1 : 0] \in \mathbf{P}^1$. Then the series*

$$\sum_{\tilde{g} \in \tilde{\Gamma}^*} \|C_g^{-1}\|^\delta$$

is convergent for any $\delta \geq 4$. Thus Γ satisfies (♣) and (♠).

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PROOF. By our choice of coordinates on $\mathbf{P}^1$, we see that $c_g \neq 0$ for $g \neq 1$ and $\{a_g/c_g\}_g, \{b_g/c_g\}_g, \{d_g/c_g\}_g$ are uniformly bounded. Since

$$C_g = \begin{pmatrix} a_g c_g^2 & 2a_g c_g d_g + b_g c_g^2 \\ c_g^3 & 3c_g^2 d_g \end{pmatrix} = c_g^3 \begin{pmatrix} a_g/c_g & 2a_g d_g/c_g^2 + b_g/c_g \\ 1 & 3d_g/c_g \end{pmatrix},$$

we see that

$$\|C_g^{-1}\| \leq M|\det C_g|^{-1}|c_g|^3 = M|c_g|^{-1}$$

holds for some $M > 0$. It is well known that, for Kleinian groups, the series $\sum_{g\in\Gamma^*} |c_g|^{-4}$ is convergent [11, Theorem II.B.5]. Hence we have the proposition. □

Compact quotients. As an application of Theorem 7.11, we obtain the following theorem.

THEOREM 8.8. *If Γ is convex-cocompact*[1]*, then $(\mathbf{P}^3 \setminus \Lambda(\tilde{\Gamma}))/\tilde{\Gamma}$ is compact.*

PROOF. Let $\Gamma \subset \mathrm{SL}_2(\mathbf{C})$ be a convex-cocompact Kleinian group. It is known that every limit point of a convex-cocompact group is a point of approximation [5, Definitions 4.43, 4.71 and 4.76]. We can assume, further, that Γ is torsion free without loss of generality. By Theorem 7.11, it is enough to show that the set $\bar{F}$ defined by (7.3) is a compact subset contained in $\Omega(\tilde{\Gamma})$. If $\bar{\Delta}$ is contained in $\Omega(\tilde{\Gamma})$, then the quotient $\Omega(\tilde{\Gamma})/\tilde{\Gamma}$ becomes compact, since $\bar{F} \subset \bar{\Delta}$ and $\bar{\Delta}$ is compact. Thus it sufficient to show the following proposition. □

PROPOSITION 8.9. *Any limit line does not intersect $\bar{\Delta}$.*

PROOF. Let ℓ_λ be any limit line, which is the tangent line to C at $\tau([\lambda : 1])$, $\lambda = [\lambda : 1] \in \Lambda_{\mathbf{P}^1}$. More explicitly, ℓ_λ is given by $z' = L_\lambda z''$, where $z = [z' : z''] \in \mathbf{P}^3$ and

$$L_\lambda = \begin{pmatrix} 3\lambda^2 & -2\lambda^3 \\ 2\lambda & -\lambda^2 \end{pmatrix}.$$

Recall that every limit point of Γ is a point of approximation. Hence, there is a sequence $\{g_m\}$ of distinct elements of Γ and a constant $\delta > 0$ such that

$$|g_m(\lambda) - g_m(\infty)| \geq \delta \tag{8.7}$$

for any m. Let

$$g_m = \begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix} \in \mathrm{SL}_2(\mathbf{C}).$$

The inequality (8.7) is equivalent to

$$\left|\frac{a_m\lambda + b_m}{c_m\lambda + d_m} - \frac{a_m}{c_m}\right| \geq \delta.$$

This implies that

$$|c_m(c_m\lambda + d_m)| \leq \delta^{-1}. \tag{8.8}$$

[1] Geometrically finite and no parabolic elements [5, page 95].

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Since $\infty = [1:0] \in \Omega_{\mathbf{P}^1}$, we know that $\lim_{m\to\infty} |c_m| = \infty$. Hence, it follows from (8.8) that

$$\lim_{m\to\infty} |c_m\lambda + d_m| = 0, \quad \lim_{m\to\infty} |c_m(c_m\lambda + d_m)^2| = 0. \tag{8.9}$$

Again, since $\infty = [1:0] \in \Omega(\Gamma)$, there is a positive constant M such that

$$\left|\frac{a_m\lambda + b_m}{c_m\lambda + d_m}\right| \le M, \quad \left|\frac{a_m}{c_m}\right| \le M.$$

Hence, also

$$\lim_{m\to\infty} |a_m\lambda + b_m| = 0 \tag{8.10}$$

and

$$\lim_{m\to\infty} |a_m(c_m\lambda + b_m)^2| = 0. \tag{8.11}$$

Put

$$\tilde{\tau}_*(g_m) = \begin{pmatrix} A_m & B_m \\ C_m & D_m \end{pmatrix}.$$

□

LEMMA 8.10. $\lim_{m\to\infty} \|C_m L_\lambda + D_m\| = 0$.

PROOF. We calculate the components of $C_m L_\lambda + D_m$. Put $C_m L_\lambda + D_m = \left(\begin{smallmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{smallmatrix}\right)$. Then

$$\begin{aligned}
\alpha_{11} &= a_m(c_m\lambda + d_m)^2 + 2(a_m\lambda + b_m)c_m(c_m\lambda + d_m)\\
\alpha_{12} &= (a_m\lambda + b_m)(c_m\lambda + d_m)^2 - a_m\lambda(c_m\lambda + d_m)^2 - 2(a_m\lambda + b_m)c_m\lambda(c_m\lambda + d_m)\\
\alpha_{21} &= 3c_m(c_m\lambda + d_m)^2\\
\alpha_{22} &= (c_m\lambda + d_m)^3 - 3c_m\lambda(c_m\lambda + d_m)^2.
\end{aligned}$$

Then

$$\lim_{m\to\infty} \alpha_{ij} = 0$$

follows easily from (8.9), (8.10) and (8.11). □

PROOF OF THE PROPOSITION (CONTINUED). Suppose that $\ell_\lambda \cap \bar{\Delta}$ contains a point $a = [a' : a''] \in \mathbf{P}^3$, where $a' = L_\lambda a''$. Then,

$$\|(C_g L_\lambda + D_g)a''\| \ge \|a''\|$$

for any $g \in \Gamma$. Since $a'' \neq 0$, this contradicts Lemma 8.10. □

REMARK 8.11. The condition that Γ should not contain parabolic elements is indispensable. Indeed, the group $\tilde{\Gamma}$ induced by the rank two abelian group $\Gamma = \{\tau_1, \tau_2\}$, $\tau_1(z) = z + 1$, $\tau_2(z) = z + i$, gives a counter example.

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8.2.2. Kleinian groups acting on a quadric surface. Let $S \in \mathbf{P}^3$ be the quartic surface $S : z_0 z_3 - z_1 z_2 = 0$, and let

$$q : \mathbf{P}^1 \times \mathbf{P}^1 \to S$$

be the Segre map $q([u_0 : u_1], [v_0 : v_1]) = [u_0 v_0 : u_0 v_1 : u_1 v_0 : u_1 v_1]$. We consider the case where a subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbf{C})$ acts trivially on the second component of $\mathbf{P}^1 \times \mathbf{P}^1$. This case was studied by Fujiki [2] and Guillot [3]. Here we shall reprove a theorem of Fujiki, as an application of Theorem 7.11.

Then the Segre map q defines a group representation

$$q_* : \Gamma \to \mathrm{PGL}_4(\mathbf{C}),$$

which is induced by the following commutative diagram.

$$\begin{array}{ccc} \mathbf{P}^1 \times \mathbf{P}^1 & \stackrel{q}{\longrightarrow} & \mathbf{P}^3 \\ g \times 1 \downarrow & & \downarrow q_*(g) \\ \mathbf{P}^1 \times \mathbf{P}^1 & \stackrel{q}{\longrightarrow} & \mathbf{P}^3 \end{array}$$

Explicitly, for $g = \pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbf{C})$, $\tilde{g} = q_*(g)$ is given by

$$\tilde{g} = \pm\begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix} \in \mathrm{PGL}_4(\mathbf{C}), \tag{8.12}$$

where I denotes the identity matrix of size two.

Limit sets. Let $\Gamma \subset \mathrm{PSL}_2(\mathbf{C})$ be a Kleinian group whose set of discontinuity $\Omega_{\mathbf{P}^1}$ contains $[1 : 0] \in \mathbf{P}^1$. Put $\Lambda_{\mathbf{P}^1} = \mathbf{P}^1 \setminus \Omega_{\mathbf{P}^1}$ and $\tilde{\Gamma} = q_*(\Gamma)$.

PROPOSITION 8.12. *The limit set of $\tilde{\Gamma}$ is given by*

$$\Lambda(\tilde{\Gamma}) = q(\Lambda_{\mathbf{P}^1} \times \mathbf{P}^1).$$

Thus $\tilde{\Gamma}$ is of type **L** *and satisfies* (♣) *and* (♠).

PROOF. As in Section 8.2.1, we embed $\mathrm{Gr}(2, 4)$ into $\mathbf{P}^5 = \mathbf{P}(\wedge^2 \mathbf{C}^4)$, and we consider the group homomorphism

$$\bar{\rho} : \mathrm{PGL}_4(\mathbf{C}) \to \mathrm{PGL}_6(\mathbf{C}).$$

Let $g = \pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbf{C})$. With respect to the basis

$$\{e_0 \wedge e_1, e_0 \wedge e_2, e_0 \wedge e_3, e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$$

of $\wedge^2 \mathbf{C}^4 = \mathbf{C}^6$, the matrix

$$G(a, b, c, d) := \begin{pmatrix} a^2 & 0 & ab & -ab & 0 & b^2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ ac & 0 & ad & -bc & 0 & bd \\ -ac & 0 & -bc & ad & 0 & -bd \\ 0 & 0 & 0 & 0 & 1 & 0 \\ c^2 & 0 & cd & -cd & 0 & d^2 \end{pmatrix} \in \mathrm{SL}_4(\mathbf{C})$$

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represents $\bar{\rho}(\tilde{g}) \in \mathrm{PGL}_6(\mathbf{C})$. Let $\{\gamma_n\} \subset \Gamma$ be a normal sequence. Let

$$g_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \mathrm{SL}_2(\mathbf{C}) \quad \text{for all } n = 1, 2, \ldots$$

be a sequence of representatives of $\{\gamma_n\}$. Since $[1 : 0] \in \Omega_{\mathbf{P}^1}$, $\{c_n^{-1} g_n\}$ converges to a matrix of the form $h = \left(\begin{smallmatrix} \mu & -\lambda\mu \\ 1 & -\lambda \end{smallmatrix}\right) \in M_2(\mathbf{C})$, $\lambda, \mu \in \mathbf{C}$. Letting $G_n = c_n^{-2} G(a_n, b_n, c_n, d_n)$, we calculate the limit: that is,

$$G := \lim_n G_n = \begin{pmatrix} \mu^2 & 0 & -\lambda\mu^2 & \lambda\mu^2 & 0 & \lambda^2\mu^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mu & 0 & -\lambda\mu & \lambda\mu & 0 & \lambda^2\mu \\ -\mu & 0 & \lambda\mu & -\lambda\mu & 0 & -\lambda^2\mu \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -\lambda & \lambda & 0 & \lambda^2 \end{pmatrix}.$$

Thus G defines a projection to a single point,

$$\mathbf{P}^5 \setminus H \to I = \{[\mu^2 : 0 : \mu : -\mu : 0 : 1]\},$$

where H is the four-plane defined by

$$H = \{\zeta \in \mathbf{P}^5 : \zeta_0 - \lambda\zeta_2 + \lambda\zeta_3 + \lambda^2\zeta_5 = 0\}.$$

Note that I is contained in $\mathrm{Gr}(2, 4)$ and corresponds to the line

$$(\mu e_0 + e_2) \wedge (\mu e_1 + e_3) = \{z' = \mu z''\} \tag{8.13}$$

in $\mathbf{P}^3$. This line coincides with $q([\mu : 1] \times \mathbf{P}^1)$. That $\tilde{\Gamma}$ satisfies (♣) and (♠) follows from (8.12) and the fact that $\sum_{g \in \Gamma^*} |c_g|^{-4} < +\infty$ in the Kleinian group theory [11, Theorem II.B.5]. □

Compact quotients. As an application of Theorem 7.11, we have the following theorem.

THEOREM 8.13 [2]. *If Γ is convex-cocompact, then $(\mathbf{P}^3 \setminus \Lambda(\tilde{\Gamma}))/\tilde{\Gamma}$ is compact.*

PROOF. The outline of the proof is the same as that of Theorem 8.8. As in that proof, it is sufficient to prove the following proposition. □

PROPOSITION 8.14. *Any limit line does not intersect $\bar{\Delta}$.*

PROOF. A limit line ℓ_λ is given by $z' = \lambda z''$ by (8.13), where $[\lambda : 1] \in \mathbf{P}^1$ is the limit point of Γ. Now suppose that there exits a limit line ℓ_λ such that $\ell_\lambda \cap \bar{\Delta}$ is nonempty. Take a point $[a' : a''] \in \ell_\lambda \cap \bar{\Delta}$. Then, by $a' = \lambda a''$, $\tilde{g} = \left(\begin{smallmatrix} a_g I & b_g I \\ c_g I & d_g I \end{smallmatrix}\right)$ and $\|C_g a' + D_g a''\| \geq \|a''\|$, we have $\|(c_g \lambda + d_g) a''\| \geq \|a''\|$ for any $g \in \Gamma$. Since $a'' \neq 0$, this contradicts (8.9). □

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8.2.3. Kleinian groups acting on a cone over a conic. For the moment, we have only a very simple example of type **L** in this case. Many discrete subgroups acting on the cone can be constructed by the method used in [9, page 278]. It is plausible that some of them are of type **L**, but their canonical quotients will be noncompact.

EXAMPLE 8.15. Let $\Gamma \subset SL_4(\mathbf{C})$ be an infinite cyclic group generated by

$$g = \begin{pmatrix} \alpha^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha^{-2} & 0 \\ p & q & r & 1 \end{pmatrix} \quad \text{for } |\alpha| > 1.$$

With respect to the basis

$$\{e_0 \wedge e_1, e_0 \wedge e_2, e_1 \wedge e_2, e_0 \wedge e_3, e_1 \wedge e_3, e_2 \wedge e_3\},$$

we have

$$\rho(g^n) = \begin{pmatrix} \alpha^{2n} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha^{-2n} & 0 & 0 & 0 \\ n\alpha^{2n}q & \dfrac{1-\alpha^{2n}}{\alpha^{-2}-1}r & 0 & \alpha^{2n} & 0 & 0 \\ -\dfrac{\alpha^{2n}-1}{\alpha^2-1}p & 0 & \dfrac{\alpha^{-2n}-1}{\alpha^{-2}-1}r & 0 & 1 & 0 \\ 0 & -\dfrac{1-\alpha^{-2n}}{\alpha^2-1}p & -n\alpha^{-2n}q & 0 & 0 & \alpha^{-2n} \end{pmatrix} \in PGL_6(\mathbf{C}).$$

This implies that the limit image of the sequence $\{\rho(g^n)\}$, $n \to +\infty/-\infty$, is a point if and only if $q \neq 0$. If $q \neq 0$, there are exactly two limit lines, which are

$$\ell_1 = e_0 \wedge e_3 \quad \text{and} \quad \ell_2 = e_2 \wedge e_3.$$

Thus Γ is of type **L** if and only if $q \neq 0$. The cone $S = \{z_0 z_2 - z_1^2 = 0\}$ contains ℓ_1 and ℓ_2, and they are invariant by Γ. Note that the quotient space $\Omega(\Gamma)/\Gamma = (\mathbf{P}^3 \setminus \{\ell_1 \cup \ell_2\})/\Gamma$ contains a noncompact surface $(S \setminus \{\ell_1 \cup \ell_2\})/\Gamma$ as a closed submanifold, which is a $\mathbf{C}$-bundle over the elliptic curve $\mathbf{C}^*/\langle\alpha\rangle$. Therefore $\Omega(\Gamma)/\Gamma$ is not compact. The group satisfies (♣) and (♠) with respect to a new system $[w]$ of coordinates, such as $w_0 = z_0 + z_1 - z_2 - z_3, w_1 = -z_0 + z_1 + z_2 - z_3, w_2 = z_0 - z_1 + z_2 + z_3, w_3 = z_3$.

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