Non Gaussian Minkowski functionals and extrema counts for CMB maps

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Abstract. In the conference presentation we have reviewed the theory of non-Gaussian geometrical measures for 3D Cosmic Web of the matter distribution in the Universe and 2D sky data, such as Cosmic Microwave Background (CMB) maps that was developed in a series of our papers. The theory leverages symmetry of isotropic statistics such as Minkowski functionals and extrema counts to develop post Gaussian expansion of the statistics in orthogonal polynomials of invariant descriptors of the field, its first and second derivatives. The application of the approach to 2D fields defined on a spherical sky was suggested, but never rigorously developed. In this paper we present such development treating the effects of the curvature and finiteness of the spherical space S_2 exactly, without relying on flat-sky approximation. We present Minkowski functionals, including Euler characteristic and extrema counts to the first non-Gaussian correction, suitable for weakly non-Gaussian fields on a sphere, of which CMB is the prime example.

Random fields are ubiquitous phenomena in physics appearing in areas ranging from turbulence to the landscape of string theories. In cosmology, the sky-maps of the polarized Cosmic Microwave Background (CMB) radiation – a focal topic of current research – is a prime example of such 2D random fields, specified on S_2 spherical space. Modern view of the cosmos, developed primarily through statistical analysis of these fields, points to a Universe that is statistically homogeneous and isotropic with a hierarchy of structures arising from small Gaussian fluctuations of quantum origin. While the Gaussian limit provides the fundamental starting point in the study of random fields, non-Gaussian features of the CMB fields are of great interest. Indeed, CMB inherits a high level of Gaussianity from initial fluctuations, but small non-Gaussian deviations may provide a unique window into the details of processes in the early Universe. The search for the best methods to analyze non-Gaussian random fields is ongoing.

In the paper Pogosyan, Gay & Pichon (2009) the general invariant based formalism for computing topological and geometrical characteristics of non Gaussian fields was presented. The general formulae for the Euler characteristic to all orders has been derived, which encompasses the well known first correction Matsubara (2003) and which was later confirmed to the next order by Matsubara (2010). This work was followed by the detailed exposition of the theory in 2D and 3D flat (Cartesian) space in Pogosyan, Pichon & Gay (2011) and Gay, Pichon & Pogosyan (2012), and generalized to the 3D redshift space where isotropy is broken in Codis *et al.* (2013).

The goal of this paper is to extend these results to the fields defined on a finite curved spherical space S_2 without reliance on the flat field (small angle) approximation. While these proceedings were being prepared, similar work has been done for statistics of peaks in the Gaussian case within Marcos-Caballero *et al.* (2016). Here our focus is on non-Gaussian corrections. We discuss how to compute exact Minkowski functionals for the

excursion sets of a scalar field on a S_2 sphere to all orders in non-Gaussian expansion and provide an explicit expression for the Euler characteristic to the first order. Expressions for the total extrema counts to the first non-Gaussian order are also given, while analytical formulas for differential extrema counts to the same order will be published elsewhere due to their length. These results have a direct relevance to CMB data analysis.

1. Joint distribution function of the field and its derivatives on S_2 sphere.

The statistics of Minkowski functionals, including the Euler number, as well as extrema counts requires the knowledge of the one-point joint probability distribution function (JPDF) $P(x, x_i, x_{ij})$ of the field x, its first, x_i, and second, x_{ij}, derivatives of the field x. Let us consider random field x defined on a 2D sphere S_2 of radius R represented as the expansion in spherical harmonics

$$x(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi) \tag{1.1}$$

where for the Gaussian statistically homogeneous and isotropic field random coefficients a_{lm} are uncorrelated with m-independent variances C_l of each harmonic

$$\langle a_{lm} a_{l'm'}^* \rangle = C_l \delta_{ll'} \delta_{mm'} \tag{1.2}$$

The variance of the field is then given by

$$\sigma^2 \equiv \langle x^2 \rangle = \frac{1}{4\pi} \sum_l C_l(2l + 1) \tag{1.3}$$

When considering derivatives in the curved space, we use covariant derivatives $x_{;\theta}$, $x_{;\phi}$, $x^{;\theta}{}_{;\theta}$, $x^{;\phi}{}_{;\phi}$, $x^{;\theta}{}_{;\phi}$ where it will be seen immediately that mixed version for the second derivatives is the most appropriate choice. The 2D rotation-invariant combinations of derivatives are

$$q^2 = x_{;\phi} x^{;\phi} + x_{;\theta} x^{;\theta} \;,\; J_1 = \left(x^{;\theta}{}_{;\theta} + x^{;\phi}{}_{;\phi} \right)^2 \;,\; J_2 = \left(x^{;\theta}{}_{;\theta} - x^{;\phi}{}_{;\phi} \right)^2 + 4 x^{;\theta}{}_{;\phi} x^{;\phi}{}_{;\theta} \tag{1.4}$$

where J_1 is linear in the field and q^2 and J_2 are quadratic, always positive, quantities. The derivatives are also random Gaussian variables, which variances are easily computed

$$\sigma_1^2 \equiv \langle q^2 \rangle = \frac{1}{4\pi R^2} \sum_l C_l l(l+1)(2l+1) \tag{1.5}$$

$$\sigma_2^2 \equiv \langle J_1^2 \rangle = \frac{1}{4\pi R^4} \sum_l C_l l^2 (l+1)^2 (2l+1) \tag{1.6}$$

$$\sigma_2'^2 \equiv \langle J_2 \rangle = \frac{1}{4\pi R^4} \sum_l C_l (l-1)l(l+1)(l+2)(2l+1) \tag{1.7}$$

where the fundamental difference between a sphere and the 2D Cartesian space is in the fact that $\sigma_2' \neq \sigma_2$. Among the cross-correlations the only non-zero one is between the the field and its Laplacian $\left\langle x \left(x^{;\theta}{}_{;\theta} + x^{;\phi}{}_{;\phi} \right) \right\rangle = -\sigma_1^2$.

From now on we rescale all random quantities by their variances, so that rescaled variables have $\langle x^2 \rangle = \langle J_1^2 \rangle = \langle q^2 \rangle = \langle J_2 \rangle = 1$. Introducing $\zeta = (x + \gamma J_1)/\sqrt{1 - \gamma^2}$ (where the spectral parameter $\gamma = -\langle x J_1 \rangle = \sigma_1^2/(\sigma \sigma_2)$) leads to the following JPDF for

the Gaussian 2D field

$$G_{2D} = \frac{1}{2\pi} \exp\left[-\frac{1}{2}\zeta^2 - q^2 - \frac{1}{2}J_1^2 - J_2\right].$$ (1.8)

In Pogosyan, Gay & Pichon (2009) we have observed that for non-Gaussian JPDF the invariant approach immediately suggests a Gram-Charlier expansion in terms of the orthogonal polynomials defined by the kernel G_{2D}. Since ζ, q^2, J_1 and J_2 are uncorrelated variables in the Gaussian limit, the resulting expansion is

$$P_{2D}(\zeta, q^2, J_1, J_2) = G_{2D}\left[1+\right.$$

$$\left.\sum_{n=3}^{\infty} \sum_{i,j,k,l=0}^{i+2j+k+2l=n} \frac{(-1)^{j+l}}{i!\,j!\,k!\,l!} \left\langle \zeta^i q^{2j} J_1{}^k J_2{}^l \right\rangle_{GC} H_i(\zeta) L_j(q^2) H_k(J_1) L_l(J_2)\right], \quad (1.9)$$

where terms are sorted in the order of the field power n and $\sum_{i,j,k,l=0}^{i+2j+k+2l=n}$ stands for summation over all combinations of non-negative i, j, k, l such that $i + 2j + k + 2l$ adds to the order of the expansion term n. H_i are (*probabilists'*) Hermite and L_j are Laguerre polynomials. The coefficients of expansion

$$\left\langle \zeta^i q^{2j} J_1^k J_2^l \right\rangle_{GC} = \frac{j!\,l!}{(-1)^{j+l}} \left\langle H_i(\zeta) L_j(q^2) H_k(J_1) L_l(J_2) \right\rangle.$$ (1.10)

are related (and for the first non-Gaussian order n=3 are equal) to the moments of the field and its derivatives (see Gay, Pichon & Pogosyan (2012) for details).

Up to now our considerations are practically identical to the theory in the Cartesian space, which facilitates using many of the Cartesian calculations. We stress again the only, but important difference being $\sigma_2' \neq \sigma_2$. We shall see in the next sections how this difference plays out. Here we introduce the spectral parameter β that describes this difference

$$\beta \equiv 1 - \frac{\sigma_2'^2}{\sigma_2^2} = 2\frac{\sum_l C_l l(l+1)(2l+1)}{\sum_l C_l l^2(l+1)^2(2l+1)}$$ (1.11)

Let us review the scales and parameters that the theory has. As in the flat space, we have two scales $R_0 = \sigma/\sigma_1$ and $R_* = \sigma_1/\sigma_2$ and the spectral parameter $\gamma = R_*/R_0$ (which also describes correlation between the field and its second derivatives). On a sphere we have a third scale, the curvature radius R. The meaning of β becomes clear if we notice that $\sigma_2^2 - \sigma_2'^2 = 2\sigma_1^2/R^2$, thus $\beta = 2R_*^2/R^2$, i.e describes the ratio of the correlation scale R_* to the curvature of the sphere. As with the γ, β varies from 0 to 1, with $\beta = 0$ corresponding to the flat space limit. From Eq. (1.11) we find that $\beta = 1$ is achieved when the field has only the monopole and the dipole in its spectral decomposition.

2. Minkowski functionals on S_2 beyond the Gaussian limit

There are three Minkowski functionals that are defined for the excursion set above threshold ν of a 2D field, namely the filling factor, $f_V(\nu)$, i.e the volume fraction occupied by the region above the threshold ν, the length (per unit volume) of isofield contours, $\mathcal{L}(\nu)$ and Euler characteristic $\chi(\nu)$. Statistics of the first two do not depend on second derivatives of the field, and thus are identical on S_2 and the 2D Cartesian space. Here, for completeness, we reproduce the non-Gaussian expansions for these quantities from

Gay, Pichon & Pogosyan (2012)

$$f(\nu) = \frac{1}{2}\text{Erfc}\left(\frac{\nu}{\sqrt{2}}\right) + \frac{1}{\sqrt{2\pi}}e^{-\frac{\nu^2}{2}}\sum_{n=3}^{\infty}\frac{\langle x^n\rangle_{\text{GC}}}{n!}H_{n-1}(\nu). \tag{2.1}$$

$$\mathcal{L}(\nu) = \frac{1}{2\sqrt{2}R_0}e^{-\frac{\nu^2}{2}}\left(1 + \frac{1}{2\sqrt{\pi}}\sum_{n=3}^{\infty}\sum_{i,j}^{i+2j=n}\frac{(-1)^{j+1}}{i!j!}\frac{\Gamma(j-\frac{1}{2})}{\Gamma(j+1)}\langle x^i q^{2j}\rangle_{\text{GC}}H_i(\nu)\right). \tag{2.2}$$

Euler characteristic density of the region above a threshold $x = \nu$ is a more interesting case. It is given by the average of the determinant of the Hessian matrix of the second derivatives of the field at the points where the first derivatives vanish Adler (1981), Longuet-Higgins (1957)

$$\chi(\nu) = \int_{\nu}^{\infty}\text{d}x\int\text{d}^3x_{ij}P(x, x_i = 0, x_{ij})\det(x_{ij}). \tag{2.3}$$

It has been argued in Pogosyan, Gay & Pichon (2009) that on S_2 the determinant should be that of the Hessian of the mixed covariant derivatives $\det(x^{:i}_{;j})$. It is this choice that provides the density relative to the invariant volume element $R^2\sin^2\theta\text{d}\theta\text{d}\phi$ and has a scalar trace equal to the Laplacian of the field. Using scaled invariant variables

$$\det(x^{:i}_{;j}) = \frac{\sigma_2^2}{4}\left(J_1^2 - (1-\beta)J_2\right) \equiv \sigma_2^2 I_2, \tag{2.4}$$

where we have introduced another scaled quadratic invariant I_2. In terms of the eigenvalues of the Hessian, $\sigma_2^2 I_2 = \lambda_1\lambda_2$, while $\sigma_2^2 J_1 = \lambda_1 + \lambda_2$ and $\sigma_2^2(1-\beta)J_2 = (\lambda_1 - \lambda_2)^2$.

In the Gaussian limit the Euler characteristic density becomes

$$\chi(\nu) = \frac{\sigma_2^2}{8\pi^2\sigma_1^2}\int_{-\infty}^{\infty}\text{d}J_1\int_0^{\infty}\text{d}J_2\int_{\frac{\nu+\gamma J_1}{\sqrt{1-\gamma^2}}}^{\infty}\text{d}\zeta\exp\left[-\frac{1}{2}\zeta^2 - \frac{1}{2}J_1^2 - J_2\right]\left(J_1^2 - (1-\beta)J_2\right) \tag{2.5}$$

It evaluates to

$$\chi(\nu) = \frac{\gamma^2}{4\pi\sqrt{2\pi}R_*^2}\nu e^{-\frac{\nu^2}{2}} + \frac{\beta}{8\pi R_*^2}\text{erfc}\left(\frac{\nu}{\sqrt{2}}\right) \tag{2.6}$$

which differ from the well known Cartesian result by the $\beta \neq 0$ term. On a sphere which has a finite volume $4\pi R^2$ it is appropriate to quote the total Euler characteristic in the whole volume, which, recalling the relation between γ, β, R and R_* becomes

$$4\pi R^2\chi(\nu) = \frac{R^2}{\sqrt{2\pi}R_0^2}\nu e^{-\frac{\nu^2}{2}} + \text{Erfc}\left(\frac{\nu}{\sqrt{2}}\right) \tag{2.7}$$

which explicitly demonstrates that if $\nu = -\infty$. i.e the whole space is included in the excursion set, the total Euler characteristic is equal to that of a sphere, $4\pi R^2\chi(-\infty) = 2$, as expected.

Evaluation of the non-Gaussian expansion for $\chi(\nu)$ entails integration Eq. (2.3) with distribution function P_{2D} given by the Eq. (1.9). The procedure is similar to that in Cartesian space as elaborated in detail in Gay, Pichon & Pogosyan (2012) and which led to the complete expression for the Euler characteristic to all orders first reported in Pogosyan, Gay & Pichon (2009). Indeed, the quantity I_2 that we average over all the range of J_2 can be rewritten as $H_2(J_1) + \beta + (1-\beta)L_1(J_2)$. Thus only $l = 0, 1$ terms of the expansion, i.e containing $L_0(J_2)$ or $L_1(J_2)$, do not vanish after integration. Here we should limit ourselves with presenting only the result of the most practical use - up to

the first, cubic in the field, non-Gaussian correction

$$\chi(\nu) = \frac{\beta}{8\pi R_*^2} \text{Erfc}\left(\frac{\nu}{\sqrt{2}}\right) + \frac{1}{4\pi\sqrt{2\pi}R_*^2} \exp\left(-\frac{\nu^2}{2}\right)$$

$$\times \left[\gamma^2 H_1(\nu) + 2\gamma \langle q^2 J_1 \rangle + 4\langle xI_2 \rangle - (\gamma^2 \langle xq^2 \rangle + \gamma \langle x^2 J_1 \rangle) H_2(\nu) + \frac{\gamma^2}{6} \langle x^3 \rangle H_4(\nu)\right.$$

$$\left. + \beta\left(-\langle xq^2 \rangle H_0(\nu) + \frac{1}{6}\langle x^3 \rangle H_2(\nu)\right)\right], \tag{2.8}$$

where the Gram-Charlier moments of the non-primary variables x and I_2 are understood as correspondent combinations of Gram-Charlier moments of the expansion variables ζ, J_1 and J_2. The first term of Eq. (2.8) is the Gaussian result on the sphere that is responsible for the total Euler number of the excursion set to be that of the total sphere when $\nu = -\infty$. The last $\propto \beta$ terms is a correction to non-Gaussian result due to the curvature of the sphere. In conclusion we as well write explicitly the result for the total Euler number above threshold ν

$$4\pi R^2 \chi(\nu) = \text{Erfc}\left(\frac{\nu}{\sqrt{2}}\right) + \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{\nu^2}{2}\right)\left(\frac{1}{6}\langle x^3 \rangle H_2(\nu) - \langle xq^2 \rangle H_0(\nu)\right)$$

$$+ \frac{R^2}{\sqrt{2\pi}R_0^2} \exp\left(-\frac{\nu^2}{2}\right) \tag{2.9}$$

$$\times \left[H_1(\nu) + \frac{2}{\gamma}\langle q^2 J_1 \rangle + \frac{4}{\gamma^2}\langle xI_2 \rangle - \left(\langle xq^2 \rangle + \frac{1}{\gamma}\langle x^2 J_1 \rangle\right) H_2(\nu) + \frac{1}{6}\langle x^3 \rangle H_4(\nu)\right].$$

3. Extrema counts on S_2 beyond the Gaussian limit

The number density of extrema above a threshold ν is given by an integral very similar to the Euler characteristic Adler (1981), Longuet-Higgins (1957)

$$n_{\text{ext}}(\nu) = \int_\nu^\infty dx \int d^3 x_{ij} P(x, x_i = 0, x_{ij})|x_{ij}|\Theta_{ext}(\lambda_m). \tag{3.1}$$

where the theta function $\Theta_{ext}(\lambda_m)$ chooses the regions of integration in the space of second derivatives with appropriate to the particular extremum type signs of the Hessian eigenvalues. In 2D, assuming $\lambda_1 \geqslant \lambda_2$, $\Theta_{ext}(\lambda_m) = \Theta(-\lambda_1)$ for maxima, $= \theta(\lambda_2)$ for minima, and $= \theta(\lambda_1)\theta(-\lambda_2)$ for saddle points. Indeed, we have a well-known topological relation

$$\chi(\nu) = n_{\max}(\nu) - n_{\text{sad}}(\nu) + n_{\min}(\nu) \tag{3.2}$$

The integral Eq. (3.1) has a very transparent form when the Hessian is described in invariant variables. It is equivalent to Eq. (2.5), except that the limits of integration over J_1 are partitioned into the regions of the fixed sign of the determinant I_2. Namely, maxima correspond to the range $J_1 \in (-\infty, -\sqrt{(1-\beta)J_2})$, minima to $J_1 \in (\sqrt{(1-\beta)J_2}, \infty)$ and saddle points to $J_1 \in (-\sqrt{(1-\beta)J_2}, \sqrt{(1-\beta)J_2})$.

Calculations for differential density of extremal points, $\partial n_{\text{ext}}/\partial(\nu)$ can be carried out analytically even for the general expression Eq, (1.9) (see discussion in Gay, Pichon & Pogosyan (2012) for the flat case). The resulting expressions are cumbersome, and here we limit ourselves to presenting results for the total density of extrema in the first non-Gaussian order only. The total number density of maxima is given by

$$n_{\max} = \frac{\sigma_2^2}{4\sigma_1^2} \int_0^\infty dJ_2 \int_{-\infty}^{-\sqrt{(1-\beta)J_2}} dJ_1 P_{2D}(q^2 = 0, J_1, J_2) \left|J_1^2 - (1-\beta)J_2\right| \tag{3.3}$$

and, similarly, for the minima and the saddle points. The result is

$$n_{\mathrm{max/min}} = \frac{(1-\beta)^{3/2} + \beta\sqrt{3-\beta}}{8\pi\sqrt{3-\beta}R_*{}^2} \pm \frac{6(3-\beta)\left\langle q^2 J_1\right\rangle - (5-3\beta)\left\langle J_1^3\right\rangle + 6(1-\beta)\left\langle J_1 J_2\right\rangle}{6\pi\sqrt{2\pi}R_*{}^2(3-\beta)^2},$$

$$n_{\mathrm{sad}} = \frac{(1-\beta)^{3/2}}{4\pi\sqrt{3-\beta}R_*{}^2}, \tag{3.4}$$

where we immediately see that $n_{\mathrm{max}} + n_{\mathrm{min}} - n_{\mathrm{sad}} = \beta/(4\pi R_*^2) = 2/(4\pi R^2)$ as expected. The total number of saddles, as well as of all the extremal points, $n_{\mathrm{max}} + n_{\mathrm{min}} + n_{\mathrm{sad}}$, are preserved in the first order (the latter following from the former), but the symmetry between the minima and the maxima is broken.

It is instructive to look how the Gaussian extrema counts are modified by the properties of spherical space when the curvature radius is large relative to the typical extrema separation scale R_*, i.e when β is small. Up to the first order in β

$$n_{\mathrm{max/min}} \sim \frac{1}{8\sqrt{3}\pi R_*^2} + \frac{9-4\sqrt{3}}{36\pi R^2} \approx \frac{1}{8\sqrt{3}\pi R_*^2}\left(1 + 0.8R_*^2/R^2\right) \tag{3.5}$$

$$n_{\mathrm{sad}} \sim \frac{1}{4\sqrt{3}\pi R_*^2} - \frac{2}{3\sqrt{3}\pi R^2} \approx \frac{1}{4\sqrt{3}\pi R_*^2}\left(1 - 2.7R_*^2/R^2\right) \tag{3.6}$$

This shows that being on a sphere, increases the number density of maxima and minima, but decrease (and in more significant way) the number of saddles. Incidently, assuming large-angle CMB power spectrum, truncated at $l = 30$ gives $\beta \approx 1/170$, i.e 1% correction to the count of extrema relative to the flat-sky approximation.

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