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**Functorial destackification of tame stacks with
abelian stabilisers**

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Functorial destackification of tame stacks with abelian stabilisers

Daniel Bergh

ABSTRACT

We give an algorithm for removing stackiness from smooth, tame Artin stacks with abelian stabilisers by repeatedly applying stacky blow-ups. The construction works over a general base and is functorial with respect to base change and compositions with gerbes and smooth, stabiliser-preserving maps. As applications, we indicate how the result can be used for destackifying general Deligne–Mumford stacks in characteristic 0, and to obtain a weak factorisation theorem for such stacks. Over an arbitrary field, the method can be used to obtain a functorial algorithm for desingularising varieties with simplicial toric quotient singularities, without assuming the presence of a toroidal structure.

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1. Introduction and main theorems

Consider an algebraic stack X which is smooth over a field k . If X has finite inertia, then there is a canonical map $X \rightarrow X_{\text{cs}}$ to a coarse (moduli) space. The algebraic space X_{cs} will, however, in general not be smooth. Given a morphism $f: X' \rightarrow X$ of stacks with finite inertia, we get an induced map $f_{\text{cs}}: X'_{\text{cs}} \rightarrow X_{\text{cs}}$. If f is proper and birational, we call f a *stacky modification*. Our goal is to find nice choices of f and X' such that the map f_{cs} becomes a desingularisation.

The stacky modifications we will work with are usual *blow-ups* with smooth centres and *root stacks*, where we take roots of smooth divisors. Such modifications will collectively be referred to as *stacky blow-ups* with smooth centres, and sequences of such stacky blow-ups will be referred to as *smooth stacky blow-up sequences* (see Definition 2.2).

It is useful to think of the process described above as a process to remove stackiness from a smooth stack. The method described in this paper will produce a roof-shaped diagram

$$\begin{array}{ccc} & X' & \\ \pi \swarrow & & \searrow f \\ X'_{\text{cs}} & & X \end{array}$$

where π is the coarse map. The map f is a composition of a sequence of stacky blow-ups and π is a root stack if we start with an orbifold X and a composition of a gerbe and a root stack otherwise. We will use the term *destackification* (see Definition 2.3) for a process producing such a roof.

In this paper, we will focus on the case when X has diagonalisable stabilisers. This allows us to attack the problem with toric methods. The combinatorial nature of toric methods makes them quite insensitive to assumptions on the base we are working over. Hence, we will assume that the base is an arbitrary scheme rather than a field. In fact, we could just as easily work over an arbitrary algebraic stack if we used the appropriate relative versions of concepts such as coarse space and stabilisers, but we will not work in this generality.

Just as in the classical method for desingularisation by Hironaka [Hir64a, Hir64b], divisors with simple normal crossings will play an important role in the algorithms used in this paper. Typically, the divisors will be exceptional divisors for the various blow-ups during the destackification process. Furthermore, the centre of each stacky blow-up will have normal crossings with the given simple normal crossings divisor. As in Hironaka's method, it will be crucial to keep track of the order in which the divisors have been created in order to achieve functoriality. The main object that we will work with will therefore be a pair (X, \mathbf{E}) , where X is a tame, smooth stack and \mathbf{E} an ordered set of smooth divisors on X which have simple normal crossings. For brevity, we will call such a pair a *standard pair* (see Definition 2.1 for technical details). The elements of \mathbf{E} will be called the *components* of \mathbf{E} .

The first step in the destackification process is to create enough components of the divisor \mathbf{E} to be able to attack the problem with toric methods. We do this by making the pair (X, \mathbf{E}) *divisorial* (see Definition 7.13). The reader should be warned that the term *divisorial* in this context is used in a non-standard way. If X is an orbifold, the pair (X, \mathbf{E}) is divisorial precisely when the stacky locus is contained in the divisor \mathbf{E} . An alternative description is that (X, \mathbf{E}) is divisorial when the \mathbb{G}_m^n -torsor associated to the divisor \mathbf{E} is an algebraic space (see Remark 7.14).

THEOREM 1.1 (Functorial divisorialification). *Let $(X, \mathbf{E})/S$ be a standard pair, as defined in Definition 2.1. If X has diagonalisable stabilisers, then there exists a smooth, ordinary blow-up sequence*

$$\Pi: (X_n, \mathbf{E}_n) \rightarrow \cdots \rightarrow (X_0, \mathbf{E}_0) = (X, \mathbf{E})$$

such that the pair (X_n, \mathbf{E}_n) is divisorial. The construction is functorial (see Definition 2.6) with respect to arbitrary base change $S' \rightarrow S$ and with respect to gerbes and smooth, stabiliser-preserving morphisms $X' \rightarrow X$.

In [KKMS73], a combinatorial method for desingularising toroidal varieties is described. Toroidal varieties naturally occur as coarse spaces of *toroidal stacks* (see Definition 7.4) and the methods in [KKMS73] could quite easily be adapted to handle destackification of toroidal stacks. However, it seems non-trivial to obtain a toroidal structure on a smooth stack with diagonalisable stabilisers if not given one from the start. Note that *toroidality* is a much stronger property than the *divisoriality* described above and, whereas divisorialification may be reached via the naivest possible method using just usual blow-ups (see Algorithm C), toroidalification requires the whole arsenal of stacky blow-ups. In fact, it seems like the easiest way to obtain a toroidal structure is to simultaneously achieve destackification.

The method described in this paper makes use of two different invariants associated to each point of the stack. The *independency index* (see Definition 7.1) measures how far the stack is from being destackified at the point and the *toroidal index* (see Definition 7.4) measures how far the stack is from being toroidal. The destackification process alternates between reducing the toroidal index and the independency index in a controlled way. A complication is that the locus where the toroidal index is maximal is not smooth in general, and therefore cannot be blown up. Instead, other invariants must be used to single out suitable substacks for modification. The result of the process is summarised in the following theorem, which is the main theorem of the article.

THEOREM 1.2 (Functorial destackification). *Let $(X, \mathbf{E})/S$ be a standard pair, as defined in Definition 2.1, over a quasi-compact scheme S . If X has diagonalisable stabilisers, then there exists a smooth, stacky blow-up sequence*

$$\Pi: (X_m, \mathbf{E}_m) \rightarrow \cdots \rightarrow (X_0, \mathbf{E}_0) = (X, \mathbf{E}),$$

which is a destackification as in Definition 2.3. In particular, the coarse space of X_m is smooth, and the coarse map can be factored as a gerbe followed by a root stack. The construction is functorial (see Definition 2.6) with respect to arbitrary base change $S' \rightarrow S$ and with respect to gerbes and smooth, stabiliser-preserving morphisms $X' \rightarrow X$.

The functorial properties of the algorithm have the pleasant side effect that they guarantee that the locus U of the stack X which only has generic stackiness will not be modified by the algorithm. More precisely, each of the stacky blow-ups in the blow-up sequence will have centres lying above the complement $X \setminus U$. In particular, the algorithm will not modify the locus where X is an algebraic space.

Applications

To illustrate how the destackification theorem may be applied, we will study three corollaries. The proofs given here will be sketchy, since a more detailed account will appear later in a joint paper with David Rydh.

The destackification algorithm is useful even if one is not primarily interested in stacks. Let X be a variety over a field k whose singular points are all simplicial, toric singularities. By this we mean that each point $\xi \in X$ has an étale neighbourhood $X' \rightarrow X$ with $X' = U/\Delta$ for some smooth variety U and finite diagonalisable group Δ . In this situation, there exists a canonical stack X_{can} which is smooth and has X as coarse space [Vis89, Sat12]. By applying the functorial destackification algorithm on X_{can} , we obtain a functorial desingularisation algorithm.

COROLLARY 1.3 (Functorial desingularisation of simplicial toric singularities). *Let X be an algebraic space of finite type over an arbitrary field k . Assume that X has simplicial toric singularities only. Then there exists a sequence*

$$\Pi: X_m \rightarrow \cdots \rightarrow X_0 = X$$

of proper birational modifications such that X_m is smooth. The construction is functorial with respect to change of base field and with respect to smooth morphisms $X' \rightarrow X$.

Note that no toroidal structure is needed. This makes the corollary more general than the toroidal methods described in [KKMS73]. On the other hand, the methods described in this article are somewhat less explicit. It recently came to the author's knowledge that a similar result was obtained in the recent preprint [Buo15].

At first sight, the assumption in Theorem 1.2 that the stack X has diagonalisable stabilisers seems to be quite restrictive. But at least if we work over a field of characteristic 0, this can be overcome. By first using functorial embedded desingularisation on the stacky locus of X with the Bierstone–Milman variant of Hironaka's method [BM97], we reduce to the case where the stacky locus is contained in a simple normal crossings divisor. But this implies that the stabilisers are in fact diagonalisable [RY00, Theorem 4.1], so we are in a situation where we can apply Theorem 1.2.

COROLLARY 1.4 (Functorial destackification of Deligne–Mumford stacks in characteristic 0). *Let X be a Deligne–Mumford stack which is smooth and of finite type over a field of characteristic 0. Also assume that X has finite inertia. Then there exists a smooth stacky blow-up sequence Π , as in the functorial destackification theorem, such that (X_m, \mathbf{E}_m) has the same properties as mentioned in that theorem.*

Finally, destackification can be used to obtain a stacky version of the weak factorisation theorem by Włodarczyk [Wł00] for Deligne–Mumford stacks in characteristic 0. The corollary is obtained by applying Włodarczyk's result on the algebraic space obtained after destackifying using Corollary 1.4.

COROLLARY 1.5 (Weak factorisation of orbifolds in characteristic 0). *Consider a proper birational map $f: X \dashrightarrow Y$ of orbifolds over a field of characteristic 0. Then there exists a factorisation of f in stacky blow-ups and blow-downs which is an isomorphism over the non-stacky locus where f is an isomorphism.*

Outline of the paper

Section 2 collects some preliminaries on algebraic stacks and clarifies the terminology used in this paper. We will also make precise definitions of certain terms, such as functoriality and blow-up sequence, used in the main theorems. In §3, we will review some basic facts about toric stacks. These will be used in §4, where we describe two algorithms, Algorithms A and B, which prove

the destackification theorems in the toric case. The algorithms are based on the classical toric desingularisation algorithm, but have an additional twist in order to make the process functorial.

From § 5 and onwards, we leave the realm of toric stacks and work with more general smooth stacks with finite diagonalisable stabilisers. First we show that any such stack is locally toric, which allows us to work with local homogeneous coordinates. Then, in § 6, we introduce an invariant, which we call the *conormal representation*. This invariant captures the local structure of a stack near each point. In characteristic 0, we could have worked with the canonical action of the stabiliser on the tangent space at each point, but, in positive characteristic, the tangent space is not well behaved. Instead, we work with the conormal bundle of the residual gerbe. We will also study a framework for constructing special purpose invariants, called *conormal invariants*, based on the conormal representation. Simple, well-known examples of such invariants are the order of the stabiliser and the multiplicity of the toric singularity of the corresponding point in the coarse space.

In § 7, we give an outline of the general destackification algorithms and introduce all conormal invariants used by these algorithms. Finally, in § 8, we go through the actual destackification algorithms and prove their correctness.

The paper also includes three appendices, collecting results of more general interest. In Appendix A, we prove a structure theorem for smooth, tame stacks in the spirit of the general structure theorem given in [AOV08]. We will also simplify parts of the proof of the general structure theorem given in [AOV08]. In Appendix B, we compute the cotangent complex of a basic toric stack and, in Appendix C, we give an alternative interpretation of the conormal representation in terms of the cotangent complex.

2. Stacky blow-up sequences and functoriality

2.1 Preliminaries and basic terminology

The cyclic (abstract) group of order n will be denoted by C_n .

We will use the definitions of algebraic stack and algebraic space used in the Stacks project [Sta17]. By a *sheaf*, we mean a sheaf on the site of schemes with coverings given by faithfully flat morphisms which are locally of finite presentation, i.e., the *fppf topology*. By a *stack*, we mean a stack in groupoids over the same site. An *atlas* for a stack X is a 1-morphism $f: U \rightarrow X$, where U and f are representable by algebraic spaces and f is faithfully flat and locally of finite presentation. If the morphism f is smooth, we call it a *smooth atlas*. A stack is *algebraic* if it admits an atlas, and it is a theorem that every algebraic stack admits a smooth atlas.

Let X be an algebraic stack. A morphism $\pi: X \rightarrow X_{cs}$ is called a *coarse space* if it is initial among morphisms to algebraic spaces and the induced map $|\pi|: |X| \rightarrow |X_{cs}|$ between topological spaces is a homeomorphism. Usually, this is called a *coarse moduli space*, but we drop the word *moduli* since we are discussing algebraic stacks without having any specific moduli problem in mind. Due to a classical theorem by Keel and Mori [KM97] with generalisations by Rydh [Ryd13], an algebraic stack X has a coarse space if its inertia stack is finite over X .

Let X be an algebraic stack which is quasi-separated and locally of finite presentation over a base scheme S . Following Abramovich *et al.* [AOV08], we say that X is *tame* if it has finite inertia and linearly reductive stabilisers. This property is reviewed in Appendix A. We will be particularly interested in the case when X has diagonalisable stabilisers. We will use the term *orbifold*, in the relative sense, for a tame stack $X \rightarrow S$ which is smooth over the base scheme, and which has fibrewise generically trivial stabilisers.

We recall the definition of a simple normal crossings divisor on a regular scheme X . Let $E = E^1 + \dots + E^r$ be an effective Cartier divisor on X . Given a point $\xi \in X$, we consider the stalk $\mathcal{O}_{X,\xi}$ with maximal ideal \mathfrak{m}_ξ . Each divisor E^i defines a regular element $f_\xi^i \in \mathcal{O}_{X,\xi}$ which is well defined up to multiplication by a unit. We say that E is a *simple normal crossings divisor* if it can be written as a sum $E = E^1 + \dots + E^r$ of Cartier divisors such that at each point $\xi \in X$, the subsequence of regular parameters f_ξ^i satisfying $f_\xi^i \in \mathfrak{m}_\xi$ extends to a regular system of parameters $(t_\xi^1, \dots, t_\xi^n)$. A closed subscheme Z of X is said to have *simple normal crossings* with E if at each point $\xi \in X$, the system $(t_\xi^1, \dots, t_\xi^n)$ described above can be chosen such that the ideal corresponding to Z in $\mathcal{O}_{X,\xi}$ is generated by a subsequence of $(t_\xi^1, \dots, t_\xi^n)$.

The definitions above generalise directly to stacks. Let X be a regular algebraic stack with a smooth atlas $\pi: U \rightarrow X$. An effective Cartier divisor E is a *simple normal crossings divisor* if E can be written as a sum $E^1 + \dots + E^r$ of regular divisors and the pull-back E_U to U is a simple normal crossings divisor. Similarly, a closed substack Z has *simple normal crossings* with E if the same holds for Z_U with respect to E_U .

We will also consider the relative version of the definitions above. Let $X \rightarrow S$ be a smooth stack over a scheme and let $E = E^1 + \dots + E^r$ be an effective Cartier divisor on X , with each E^i smooth over S . Note that E is a relative effective Cartier divisor in the sense of [EGAIV, §21.15]. We say that E is a *(relative) simple normal crossings divisor* if E_ξ is a simple normal crossings divisor on the fibre X_ξ for each geometric point $\xi: \text{Spec } \bar{k} \rightarrow S$. Similarly, we define what is meant for a substack $Z \subset X$ which is smooth over S to have simple normal crossings with E .

DEFINITION 2.1. Let S be a scheme and consider a pair $(X, \mathbf{E})/S$, where:

- (i) X is a tame algebraic stack which is smooth and of finite presentation over S ;
- (ii) $\mathbf{E} = (E^1, \dots, E^r)$ is an ordered set of distinct, effective Cartier divisors on X , called the *components* of \mathbf{E} . Each component E^i is required to be smooth over S and their sum $E = \sum E^i$ is required to be a simple normal crossings divisor.

We call such a pair $(X, \mathbf{E})/S$ a *standard pair*.

Note that the term *component* in this context does not refer to *connected component*; the components of \mathbf{E} , as in the definition above, may well be empty or disconnected.

When referring to the ordering of the components of an ordered simple normal crossings divisor, we will use an age metaphor. The components of such a divisor form a sequence E^1, \dots, E^r . The indices may be thought of as birth dates of the components, and we say that E^i is *older* than E^j , and that E^j is *younger* than E^i , provided that $i < j$.

2.2 Stacky blow-up sequences

Let S be a scheme and let $(X, \mathbf{E})/S$ be a standard pair. By a *smooth blow-up* of $(X, \mathbf{E})/S$, we mean a blow-up $\pi: \text{Bl}_Z X \rightarrow X$ in a centre Z which is smooth over S and has simple normal crossings with E . The *transform* of $(X, \mathbf{E})/S$ along π is the pair $(\text{Bl}_Z X, \mathbf{E}')$, where \mathbf{E}' denotes the ordered set of the strict transforms of the components of \mathbf{E} followed by the exceptional divisor. In other words, the ordering of the components \mathbf{E}' is the one induced by the ordering on \mathbf{E} with the exceptional divisor added as the youngest divisor.

The *root construction* of a stack in an effective Cartier divisor is thoroughly described in for instance [AGV08, Cad07] and [FMN10, §1.3.b]. Let $(X, \mathbf{E})/S$ be a standard pair. We will only consider root stacks with roots taken of components of \mathbf{E} . Such a root stack will be called

a *smooth root stack*. If $E^i \in \mathbf{E}$, we use the notation $X_{d^{-1}E^i} \rightarrow X$ for the d th root of E^i . If \mathbf{d} is a sequence of positive integers indexed by the elements of \mathbf{E} , then $X_{\mathbf{d}^{-1}\mathbf{E}} \rightarrow X$ denotes the fibre product of the stacks $X_{d_i^{-1}E^i}$ over X for all $E^i \in \mathbf{E}$ with d_i as corresponding element in \mathbf{d} . The sum of the components of \mathbf{E} for which the corresponding index in \mathbf{d} is larger than one is called the *centre* of the root stack. If \mathbf{E}' is a subsequence of \mathbf{E} and \mathbf{d} is a sequence of positive integers indexed by \mathbf{E}' , then we sometimes use the notation $X_{\mathbf{d}^{-1}\mathbf{E}'} \rightarrow X$ for the root stack obtained by extending \mathbf{d} to \mathbf{E} by ones. The *transform* of a pair $(X, \mathbf{E})/S$ along a smooth root stack $\pi: X_{\mathbf{d}^{-1}\mathbf{E}} \rightarrow X$ is the pair $(X_{\mathbf{d}^{-1}\mathbf{E}}, \pi^{-1}\mathbf{E})$. Here $\pi^{-1}\mathbf{E}$ denotes the sequence $(d_i^{-1}\pi^*E^i)$ of roots of the components E^i of \mathbf{E} .

A *smooth stacky blow-up* is either a smooth root stack or a smooth blow-up. Note that the transform of a standard pair along a smooth stacky blow-up is again a standard pair.

DEFINITION 2.2. Let $(X_0, \mathbf{E}_0)/S$ be a standard pair. A *smooth, stacky blow-up sequence* of $(X_0, \mathbf{E}_0)/S$ of length n is a sequence

$$\Pi: (X_n, \mathbf{E}_n) \xrightarrow{\pi_n} \dots \xrightarrow{\pi_1} (X_0, \mathbf{E}_0),$$

where each π_i , for $1 \leq i \leq n$, is a smooth stacky blow-up in a centre Z_{i-1} and each (X_i, \mathbf{E}_i) is the transform of $(X_{i-1}, \mathbf{E}_{i-1})$ along π_i . The centres Z_i for $0 \leq i \leq n - 1$, although suppressed from the notation, are considered part of the structure. We require each Z_i to have positive codimension in X_i at each of its points. If all stacky blow-ups are in fact usual blow-ups, we call Π a *smooth, ordinary blow-up sequence*.

Since all blow-up sequences we consider in this article will be smooth, stacky blow-up sequences, we will usually drop the modifiers *smooth* and *stacky* and just say *blow-up sequence*.

DEFINITION 2.3. Let $(X_0, \mathbf{E}_0)/S$ be a standard pair and let

$$\Pi: (Y, \mathbf{F}) = (X_n, \mathbf{E}_n) \xrightarrow{\pi_n} \dots \xrightarrow{\pi_1} (X_0, \mathbf{E}_0)$$

be a smooth, stacky blow-up sequence on $(X_0, \mathbf{E}_0)/S$. Let $\pi: Y \rightarrow Y_{\text{cs}}$ be the coarse space. We call Π a *destackification* if the following conditions hold.

- (i) The space Y_{cs} is smooth over S .
- (ii) Each component of $\mathbf{F}_{\text{cs}} = (F_{\text{cs}}^i \mid F^i \in \mathbf{F})$ is smooth and $\sum_i F^i$ has simple normal crossings.
- (iii) The divisor \mathbf{F} is a \mathbf{d} th root of the pull-back $\pi^*\mathbf{F}_{\text{cs}}$ for some sequence \mathbf{d} of positive integers indexed by the components of \mathbf{F} .
- (iv) The canonical factorisation $Y \rightarrow (Y_{\text{cs}})_{\mathbf{d}^{-1}\mathbf{F}_{\text{cs}}} \rightarrow Y_{\text{cs}}$ through the root stack makes Y a gerbe over $(Y_{\text{cs}})_{\mathbf{d}^{-1}\mathbf{F}_{\text{cs}}}$. In particular, if X_0 is an orbifold, then $Y \rightarrow (Y_{\text{cs}})_{\mathbf{d}^{-1}\mathbf{F}_{\text{cs}}}$ is an isomorphism.

The conditions (i) and (ii) can be summarised by saying that the pair $(Y_{\text{cs}}, \mathbf{F}_{\text{cs}})/S$ is a standard pair.

A stacky blow-up is said to be *empty* if the centre is empty. Although the algorithms used in the constructions mentioned in the main theorems will never produce blow-up sequences containing empty blow-ups, such may occur after pulling back blow-up sequences along morphisms which are not surjective. We will consider such pull-backs when discussing

functoriality below. We regard two blow-up sequences Π and Π' to be *equivalent* if, after pruning them from empty blow-ups, they fit into a 2-commutative ladder

$$\begin{array}{ccc} \Pi: (X_n, \mathbf{E}_n) & \longrightarrow \cdots \longrightarrow & (X_0, \mathbf{E}_0) \\ \downarrow & & \downarrow \\ \Pi': (X'_n, \mathbf{E}'_n) & \longrightarrow \cdots \longrightarrow & (X'_0, \mathbf{E}'_0) \end{array}$$

such that the vertical morphisms are isomorphisms preserving the centres.

2.3 Gerbes

Given a stack X , we denote by $X \rightarrow X_{\text{sh}}$ the canonical morphism to the *coarse sheaf*. This is, by definition, a morphism which is initial among morphisms to sheaves in the fppf topology. An algebraic stack X is called a *gerbe* if its associated coarse sheaf is an algebraic space. In this case, the morphism $\pi: X \rightarrow X_{\text{sh}}$ is called the *structure morphism* of the gerbe. There is also a relative notion of gerbe. A morphism $\pi: X \rightarrow Y$ of algebraic stacks is called a *gerbe* if the base change $X' \rightarrow Y'$ of π along any morphism $Y' \rightarrow Y$, with Y' an algebraic space, is the structure morphism of a gerbe in the sense described above.

This definition is standard and coincides with the one given in, for instance, the Stacks project [Sta17]. It is related to the definition given by Giraud [Gir71, Definition 2.1.1] in the following way. A stack Y has a canonical structure of a site induced by the fppf topology on the site of schemes. A morphism $\pi: X \rightarrow Y$ of algebraic stacks is a gerbe in our sense if and only if X is a gerbe in the sense of Giraud when viewed as a stack in groupoids over the site Y .

Note that in the context of algebraic stacks, the terminology for gerbes sometimes conflicts with algebrogeometric notions. For instance, the structure morphism $\pi: X \rightarrow Y$ of a gerbe in our sense is always smooth (see Proposition A.2). In particular, the morphism π always admits a section étale locally on Y . This does, however, not imply that X is a gerbe as a stack in groupoids over Y considered as a site endowed with the smooth or étale topology. This is a potential source of confusion as it makes terms such as *smooth gerbe* ambiguous. In this article, we try to avoid the confusion by consistently working with the fppf topology.

2.4 Stabiliser-preserving maps

We recall the definition and some basic facts about stabiliser-preserving 1-morphisms of stacks. Let $f: X \rightarrow Y$ be a 1-morphism of stacks. Given a generalised point $\xi: T \rightarrow X$, where T is a scheme, we get an induced map of stabilisers $\text{Stab}_\xi X \rightarrow \text{Stab}_{f \circ \xi} Y$ over T . The map f also induces a pair of 2-commutative diagrams

$$\begin{array}{ccc} I_X & \longrightarrow & I_Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X_{\text{sh}} & \longrightarrow & Y_{\text{sh}} \end{array}$$

Here $I_X \rightarrow X$ denotes the inertia stack of X , and the map $X \rightarrow X_{\text{sh}}$ is the coarse sheaf of X (as defined in the previous section).

DEFINITION 2.4. The 1-morphism $f: X \rightarrow Y$ is called *stabiliser-preserving* if any of the following conditions, which are easily seen to be equivalent, hold.

- (i) The map $\text{Stab}_\xi X \rightarrow \text{Stab}_{f \circ \xi} Y$ is an isomorphism for any generalised point ξ .

- (ii) The left 2-commutative square above is 2-cartesian.
- (iii) The right 2-commutative square above is 2-cartesian.

If the first condition holds for all geometric points, we say that f is *point-wise stabiliser-preserving*.

In particular, monomorphisms between stacks are stabiliser-preserving. Note that the notions of *stabiliser-preserving* and *point-wise stabiliser-preserving* maps are distinct. Both properties are preserved under composition and arbitrary base change. They are also local on the base in the fppf topology.

Example 2.5. Let k be a field and let $X = \text{Spec } k[\varepsilon]$ be the spectrum of the dual numbers over k . Furthermore, let the group μ_2 act on X by giving ε degree 1. Then we get a map $[X/\mu_2] \rightarrow \text{B}\mu_2$ from the quotient stack to the classifying stack of μ_2 , which is point-wise stabiliser-preserving, but not stabiliser-preserving.

A useful fact is that if $f: X \rightarrow Y$ is an étale map between algebraic stacks with finite inertia, then the locus where f is point-wise stabiliser-preserving is open in X , and f is stabiliser-preserving over this locus [Ryd11, Proposition 6.5]. In fact, if the stacks are tame, the corresponding fact for smooth morphisms is also true, but we will not use this here.

2.5 Functoriality

We consider two basic situations when a smooth stacky blow-up sequence can be transferred from one standard pair to another. Fix a standard pair $(X, \mathbf{E})/S$.

The first situation is when we change base scheme. Given a morphism $S' \rightarrow S$, we can form the pull-backs $X_{S'} = X \times_S S'$ and $\mathbf{E}_{S'} = \mathbf{E} \times_S S'$. Then the pair $(X_{S'}, \mathbf{E}_{S'})/S'$ is also standard, and any smooth stacky blow-up sequence on $(X, \mathbf{E})/S$ pulls back to a smooth stacky blow-up sequence on $(X_{S'}, \mathbf{E}_{S'})/S'$.

The second situation is when we have a morphism of stacks $X' \rightarrow X$ which is smooth. Then we can form the pull-back $\mathbf{E}' = \mathbf{E} \times_X X'$, and we get a standard pair $(X', \mathbf{E}')/S$. Again, any smooth stacky blow-up sequence on $(X, \mathbf{E})/S$ pulls back to a smooth stacky blow-up sequence on $(X', \mathbf{E}')/S$.

DEFINITION 2.6. Let $(X, \mathbf{E})/S$ be a standard pair. We say that a construction of a blow-up sequence is *functorial with respect to base change* $S' \rightarrow S$ if the blow-up sequence obtained by applying the construction to $(X_{S'}, \mathbf{E}_{S'})/S'$ is equivalent (see § 2.2) to the pull-back of the blow-up sequence obtained by applying the construction to $(X, \mathbf{E})/S$ as described in the first situation above.

We say that a construction of a blow-up sequence is *functorial with respect to a morphism* $X' \rightarrow X$ if the blow-up sequence obtained by applying the construction to $(X', \mathbf{E}')/S$ is equivalent to the pull-back of the blow-up sequence obtained by applying the construction to $(X, \mathbf{E})/S$ as described in the second situation above. In order for this to make sense, the morphism $X' \rightarrow X$ must be smooth.

The constructions in the main theorems are functorial with respect to arbitrary base change $S' \rightarrow S$ and with respect to morphisms $X' \rightarrow X$ which are either *gerbes* or *smooth and stabiliser-preserving*. It is, however, not reasonable to expect the construction to be functorial with respect to any smooth morphism. Indeed, if $X' \rightarrow X$ is a smooth atlas, any reasonable destackification of X' must be trivial, whereas a destackification of X cannot be trivial in general.

2.6 Distinguished structure

We do not want our algorithms to modify the locus lying over the smooth locus of the coarse space of the original stack. This poses a problem when it comes to root stacks, since they always modify the entire divisor of which the root is taken. Thus, we would like to keep track of divisors which we are allowed to root. We do this by marking certain divisors as *distinguished*.

DEFINITION 2.7. Let $(X, \mathbf{E})/S$ be a standard pair, and let $\mathbf{D} \subset \mathbf{E}$ be a subset such that all divisors in \mathbf{D} are younger than the divisors in the complement $\mathbf{E} \setminus \mathbf{D}$. We say that $(X, \mathbf{E}, \mathbf{D})/S$ is a standard pair with *distinguished structure*, and call the components of \mathbf{E} lying in \mathbf{D} *distinguished*. A *smooth stacky blow-up* of $(X, \mathbf{E}, \mathbf{D})/S$ is simply a smooth stacky blow-up of the underlying standard pair $(X, \mathbf{E})/S$. Such a blow-up is called *admissible* if its centre is contained in the support of the divisors in \mathbf{D} .

The transform $(X', \mathbf{E}')/S$ of an admissible stacky blow-up of a standard pair with distinguished structure $(X, \mathbf{E}, \mathbf{D})/S$ again has a distinguished structure $\mathbf{D}' \subset \mathbf{E}'$. It consists of the strict transforms of all distinguished divisors along with the exceptional divisor.

3. Smooth toric stacks

The theory of toric stacks has been treated by several authors. We mention a few. Borisov *et al.* [BCS05] gave a basic definition of smooth toric Deligne–Mumford stacks via the Cox construction. Iwanari gave a moduli interpretation of toric stacks using logarithmic geometry [Iwa09b]. He also gave a structure theorem, characterising toric orbifolds over a field of characteristic 0 in terms of stacks with torus actions [Iwa09a]. A similar result was obtained independently by Fantechi *et al.* [FMN10], using a bottom-up construction. Geraschenko and Satriano [GS15a, GS15b] extended the theory to non-smooth stacks and stacks with positive-dimensional stabilisers and unified the theory with other notions of toric stacks.

In this section, we summarise some basic results from the theory of toric stacks. Since there are several different suggestions on what a toric stack should be in the literature, we give the following comparison.

(i) For the definition of *toric stack*, we follow [BCS05]. The same definition is used in [Iwa09b], [Iwa09a] and [FMN10]. In the language of [GS15a, GS15b], this is equivalent to a non-strict, smooth, separated toric stack.

(ii) We use the term *toric orbifold* for a toric stack which is also an orbifold. That is, a toric stack with trivial generic stabilisers.

(iii) For the definition of a *stacky fan*, we basically follow [BCS05, §3]. However, since we only need to consider stacky fans which give rise to toric orbifolds, we only consider such fans. We call such fans *orbifold fans*.

(iv) We work with toric stacks over an arbitrary base scheme S . Some of the references restrict the discussion to the situation when S is a field, but at least for our needs considering general S comes with no extra complications.

(v) Note that this definition of stacky fan used here is essentially different to the one given in [GS15a, Definition 2.4]. The definition given in [GS15a, Definition 2.4] is less suitable for our needs, since it makes the relation between the fan and the coarse space of the toric stack less explicit.

Whenever it applies, we try to follow the notation and terminology for classical toric varieties from [CLS11] as closely as possible.

3.1 Basic toric stacks

First we introduce *basic toric stacks*. They play a similar role in the theory of toric stacks as affine toric varieties in the theory of toric varieties. Toric stacks in general are obtained by gluing basic toric stacks together along toric morphisms in the Zariski topology. Note that the term basic toric stack is non-standard.

Fix a scheme S . Consider a finitely generated abelian group A and an A -graded quasi-coherent sheaf of \mathcal{O}_S -algebras \mathcal{R} . The A -grading on \mathcal{R} corresponds to an action of the Cartier dual A^\vee (over S) on the relative spectrum $\text{Spec}_{\mathcal{O}_S} \mathcal{R}$. We get an algebraic stack associated to the pair (\mathcal{R}, A) by taking the stack quotient $[\text{Spec}_{\mathcal{O}_S} \mathcal{R}/A^\vee]$.

Let (\mathcal{R}', A') be another pair as above. A *graded homomorphism* $(f, \varphi): (\mathcal{R}, A) \rightarrow (\mathcal{R}', A')$ is a pair where $\varphi: A \rightarrow A'$ is a group homomorphism and $f: \mathcal{R} \rightarrow \mathcal{R}'$ a morphism of sheaves of \mathcal{O}_S -algebras. The morphism f is required to take homogeneous sections of degree $a \in A$ to homogeneous sections of degree $\varphi(a)$. Such a graded homomorphism induces a morphism $[\text{Spec}_{\mathcal{O}_S} \mathcal{R}'/A'^\vee] \rightarrow [\text{Spec}_{\mathcal{O}_S} \mathcal{R}/A^\vee]$ of quotient stacks.

DEFINITION 3.1. An algebraic stack X associated to a pair (\mathcal{R}, A) , as described above, is called a *basic toric stack* provided that the following conditions hold.

- (a) The sheaf of rings \mathcal{R} is of the form $\mathcal{O}_S[x_1, \dots, x_r][x_{s+1}^{-1}, \dots, x_r^{-1}]$ for some r and s such that $0 \leq s \leq r$.
- (b) Each coordinate function x_i , with $1 \leq i \leq r$, is homogeneous of degree $a_i \in A$.
- (c) The quotient group $A/\langle a_{s+1}, \dots, a_r \rangle$ is finite.

The triple $(\mathcal{R}, A, \mathbf{a})$, where $\mathbf{a} = (a_1, \dots, a_r)$, is called a *homogeneous coordinate ring* for X . The closed substacks of the form $E^i = V(x_i)$, for $1 \leq i \leq s$, are called the *toric divisors* of X . A morphism of basic toric stacks is called *toric* provided that it comes from a morphism $(\mathcal{R}, A) \rightarrow (\mathcal{R}', A')$ such that the underlying \mathcal{O}_S -algebra homomorphism $\mathcal{R} \rightarrow \mathcal{R}'$ takes monomials to monomials. A basic toric stack admitting a homogeneous coordinate ring with $s = r$ is said to be *without torus factors*.

It should be noted that the homogeneous coordinate ring does not determine the isomorphism class of the basic toric stack uniquely.

PROPOSITION 3.2. Let X be a basic toric stack with homogeneous coordinate ring $(\mathcal{R}, A, \mathbf{a})$ with

$$\mathcal{R} = \mathcal{O}_S[x_1, \dots, x_r][x_{s+1}^{-1}, \dots, x_r^{-1}].$$

Let K be the subgroup $\langle a_{s+1}, \dots, a_r \rangle$ and define r' as r minus the rank of K . Consider the A/K -graded sheaf of \mathcal{O}_S -algebras

$$\mathcal{R}' = \mathcal{O}_S[x_1, \dots, x_{r'}][x_{s+1}^{-1}, \dots, x_{r'}^{-1}]$$

with the grading given by the weight vector $\bar{\mathbf{a}} = (\bar{a}_1, \dots, \bar{a}_{r'})$, where \bar{a}_i denotes the image of a_i in A/K . Then the basic toric stack associated to the triple $(\mathcal{R}', A/K, \bar{\mathbf{a}})$ is isomorphic to X .

Proof. The proposition is trivial if all of the weights a_{s+1}, \dots, a_r are zero. Hence, we assume that one of these weights, which we without loss of generality may assume is a_r , is non-zero. By induction, it is enough to prove that the stack X is isomorphic to the stack associated to $(\mathcal{R}', A/K, \bar{\mathbf{a}})$ with $K = \langle a_r \rangle$ and \mathcal{R}' , r' and $\bar{\mathbf{a}}$ being as in the statement of the proposition.

Assume that a_r has infinite order. Then it is straightforward to verify that the graded homomorphism $\mathcal{R} \rightarrow \mathcal{R}'$ defined by

$$x_i \mapsto \begin{cases} 1 & \text{if } i = r, \\ x_i & \text{otherwise} \end{cases}$$

induces an isomorphism of stacks.

Now assume that a_r has finite order m . By a simple group theoretical argument, we may find an element $a'_r \in A$ such that $a_r = na'_r$ for some positive integer n and A splits as $A_0 \oplus \langle a'_r \rangle$. Consider the surjective homomorphism $A' = A_0 \oplus \mathbb{Z} \rightarrow A$ taking $(a, 1)$ to $a + a'_r$. We endow the sheaf of rings $\mathcal{R}[t, t^{-1}]$ with an A -grading, with $|x_r| = (1, n)$, $|t| = (1, nm)$ and $|x_i|$ are arbitrary lifts of a_i to A' for $i \in \{1, \dots, r-1\}$. Then the graded ring homomorphism $(\mathcal{R}[t, t^{-1}], A') \rightarrow (\mathcal{R}, A)$ taking x_i to x_i for $i = 1, \dots, r$ and t to 1 induces an isomorphism of stacks, since the order of $|t|$ is infinite. But the graded homomorphism $(\mathcal{R}[t, t^{-1}], A') \rightarrow (\mathcal{R}', A/K)$ taking x_i to x_i for $i = 1, \dots, r-1$, x_r to 1 and t to x_r also induces an isomorphism, since also $|x_r|$ has infinite order in $(\mathcal{R}[t, t^{-1}], A')$. This concludes the proof. \square

Remark 3.3. Let X be an algebraic stack associated to a homogeneous coordinate ring (\mathcal{R}, A) satisfying (a) and (b) from Definition 3.1, but not necessarily (c). Then X has finite stabilisers if and only if also (c) is satisfied. This follows easily from Proposition 3.2, which holds also if condition (c) is omitted.

Remark 3.4. Another consequence of Proposition 3.2 is that we always may choose homogeneous coordinates for our basic toric stacks such that the weights of the invertible coordinate functions are zero. This shows that a basic toric stack always is a product of a basic toric stack without torus factors and a torus.

Remark 3.5. From Proposition 3.2, it is also easy to determine when a basic toric stack is an orbifold. Let $(\mathcal{R}, A, \mathbf{a} = (a_1, \dots, a_r))$ be homogeneous coordinates for a basic toric stack X and let E be the support of the toric divisors in X . Let A_{rig} be the subgroup of A generated by the weights a_1, \dots, a_r . Then $X \setminus E$ is a product of a torus and the classifying stack $B(A/A_{\text{rig}})^\vee$. In particular, the basic toric stack X is an orbifold if and only if the weights a_1, \dots, a_r generate A .

Note that there is a canonical graded homomorphism $(\mathcal{R}, A_{\text{rig}}, \mathbf{a}) \rightarrow (\mathcal{R}, A, \mathbf{a})$ induced by the inclusion $A_{\text{rig}} \hookrightarrow A$. This corresponds to the *rigidification* $\rho: X \rightarrow X_{\text{rig}}$ of X (cf. [AOV08, Appendix A]). Here ρ is a gerbe banded by $(A/A_{\text{rig}})^\vee$ and X_{rig} is a basic toric orbifold.

Remark 3.6. Let X be a basic toric stack with homogeneous coordinates $(\mathcal{R}, A, \mathbf{a})$, and let $(\mathcal{R}_0, 0, \mathbf{0})$ denote the weight-zero part of \mathcal{R} together with the trivial grading by the trivial group. We have an obvious graded homomorphism $(\mathcal{R}_0, 0, \mathbf{0}) \rightarrow (\mathcal{R}, A, \mathbf{a})$ induced by inclusion. This corresponds to the coarse space $\pi: X \rightarrow X_{\text{cs}}$ of X . Note that according to our convention that toric stacks be smooth, the space X_{cs} is not a toric stack in general. It is, however, an affine toric variety (over S).

Remark 3.7. If we order the coordinate functions, then the set \mathbf{E} of toric divisors on a basic toric stack X inherits an ordering, and we get a standard pair (X, \mathbf{E}) . Indeed, this kind of standard pair is prototypical and, in §5, we will see that any standard pair with diagonalisable stabilisers is locally a basic toric stack.

3.2 Toric orbifolds

As with toric varieties, the gluing together of basic toric stacks can be described combinatorially. We review the parts of the theory we need in this article, restricting the discussion to toric orbifolds with no torus factors.

Let N be a lattice of rank n , and consider it as a subset of the vector space $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. By a cone σ in $N_{\mathbb{R}}$, we will always mean a *polyhedral, rational and strictly convex* cone. We write $\tau \preceq \sigma$ if τ is a face of σ . By $\sigma(1)$, we mean the set of one-dimensional faces, also called the *extremal rays*, of σ . Recall that σ is called *simplicial* if the cardinality of $\sigma(1)$ equals the dimension of the subspace of $N_{\mathbb{R}}$ spanned by $\sigma(1)$.

Given a fan Σ in $N_{\mathbb{R}}$, we denote the set of *rays*, that is, the set of one-dimensional cones, in Σ by $\Sigma(1)$. A fan is *simplicial* if all its cones are. We will frequently consider the free abelian group $\mathbb{Z}^{\Sigma(1)}$ on the set of rays in a fan Σ . An element $c_1\rho_1 + \dots + c_r\rho_r \in \mathbb{Z}^{\Sigma(1)}$, with $c_i \in \mathbb{Z}$ and $\rho_i \in \Sigma(1)$, is called *effective* if all coefficients c_i are greater than or equal to zero.

DEFINITION 3.8. An *orbifold fan* is a triple $\Sigma = (N, \Sigma, \beta)$, where N is a finitely generated free abelian group, Σ is a *simplicial* fan in $N_{\mathbb{R}}$ such that $|\Sigma|$ spans $N_{\mathbb{R}}$ and $\beta: \mathbb{Z}^{\Sigma(1)} \rightarrow N$ is a group homomorphism taking each generator $\rho \in \Sigma(1)$ to a non-zero lattice point on the ray ρ .

Given an orbifold fan $\Sigma = (N, \Sigma, \beta)$, we construct a toric orbifold via the *Cox construction*. Denote the dual $\text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ by M . Then the Cartier dual of M over S is an n -dimensional torus, which we denote by T_N . Its cocharacter and character groups may be canonically identified with N and M , respectively. The morphism β induces a homomorphism of algebraic groups $T_{\mathbb{Z}^{\Sigma(1)}} \rightarrow T_N$, which fits into an exact sequence

$$1 \rightarrow \Delta(\Sigma) \rightarrow T_{\mathbb{Z}^{\Sigma(1)}} \rightarrow T_N \rightarrow 1,$$

where the exactness at the term T_N is ensured by the fact that $|\Sigma|$ spans $N_{\mathbb{R}}$. Now consider the lattice $\mathbb{Z}^{\Sigma(1)}$ and the corresponding space $\mathbb{R}^{\Sigma(1)}$. Given a cone $\sigma \in \Sigma$, we have a corresponding cone $\tilde{\sigma}$ in $\mathbb{R}^{\Sigma(1)}$ spanned by the rays $\rho \in \sigma(1)$ viewed as generators in $\mathbb{R}^{\Sigma(1)}$. Collectively, the cones $\tilde{\sigma}$, for $\sigma \in \Sigma$, form a fan $\tilde{\Sigma}$ in $\mathbb{R}^{\Sigma(1)}$. Denote the corresponding toric variety, or rather family of toric varieties over S , by $X_{\tilde{\Sigma}}$.

We give an explicit description of the family $X_{\tilde{\Sigma}}$ of varieties. The *total coordinate ring* associated to Σ is the sheaf of rings $\mathcal{R} = \mathcal{O}_S[x_{\rho} \mid \rho \in \Sigma(1)]$. The *irrelevant ideal* is the ideal sheaf

$$B(\Sigma) = \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma \rangle,$$

where $x^{\hat{\sigma}}$ denotes the product of all elements x_{ρ} with $\rho \notin \sigma(1)$. Let $\mathbb{A}_S^{\Sigma(1)} = \text{Spec}_{\mathcal{O}_S} \mathcal{R}$ be the relative spectrum and $Z(\Sigma)$ be the closed subscheme associated to the irrelevant ideal $B(\Sigma)$. The scheme $X_{\tilde{\Sigma}}$ is simply $\mathbb{A}_S^{\Sigma(1)} \setminus Z(\Sigma)$. Note that the torus $T_{\mathbb{Z}^{\Sigma(1)}}$ is embedded in $X_{\tilde{\Sigma}}$ in a natural way, and the action of $\Delta(\Sigma)$ on $T_{\mathbb{Z}^{\Sigma(1)}}$ extends to $X_{\tilde{\Sigma}}$.

DEFINITION 3.9 (The Cox construction). Let $\Sigma = (N, \Sigma, \beta)$ be an orbifold fan, and consider the group $\Delta(\Sigma)$ acting on the scheme $X_{\tilde{\Sigma}}$ over the base scheme S as defined above. The *toric orbifold* X_{Σ} associated to Σ is defined as the stack quotient $[X_{\tilde{\Sigma}}/\Delta(\Sigma)]$.

Just like in the case with usual toric varieties, there is an order-reversing correspondence between cones in $\Sigma = (N, \Sigma, \beta)$ and orbit closures in X_{Σ} . Given a cone $\sigma \in \Sigma$, we have a closed variety $V(\langle x_{\rho}, \rho \in \sigma(1) \rangle)$ in $X_{\tilde{\Sigma}}$. Since this closed variety is $\Delta(\Sigma)$ -invariant, it descends to a

closed substack $V(\sigma)$ of X_{Σ} . In the particular case when we have a ray $\rho \in \Sigma(1)$, the substack $V(\rho) \subset X_{\Sigma}$ is a prime divisor, and we denote it by D_{ρ} . The divisor D_{ρ} is a smooth Cartier divisor. More generally, if $\psi = c_1\rho_1 + \dots + c_r\rho_r$ is an element of $\mathbb{Z}^{\Sigma(1)}$, we let D_{ψ} denote the divisor $c_1D_{\rho_1} + \dots + c_rD_{\rho_r}$. Such a divisor is called a *toric divisor*, and is a simple normal crossings divisor.

3.3 Morphisms of toric orbifolds

Next we describe morphisms of orbifold fans and toric orbifolds. Our definition is different from, but equivalent to, the one given by Iwanari in [Iwa09a].

Recall that a morphism of fans $f: (N, \Sigma) \rightarrow (N', \Sigma')$ is a group homomorphism $f: N \rightarrow N'$ such that the induced map $f_{\mathbb{R}} = f \otimes_{\mathbb{Z}} \mathbb{R}$ maps each cone $\sigma \in \Sigma$ into a cone $\sigma' \in \Sigma'$. This extends to orbifold fans as follows.

DEFINITION 3.10. Consider the orbifold fans $\Sigma = (N, \Sigma, \beta)$ and $\Sigma' = (N', \Sigma', \beta')$. A *morphism* $\Sigma \rightarrow \Sigma'$ of orbifold fans is a pair (f, \hat{f}) of group homomorphisms fitting into a commutative square

$$\begin{array}{ccc} \mathbb{Z}^{\Sigma(1)} & \xrightarrow{\hat{f}} & \mathbb{Z}^{\Sigma'(1)} \\ \beta \downarrow & & \downarrow \beta' \\ N & \xrightarrow{f} & N' \end{array}$$

such that both $f: (N, \Sigma) \rightarrow (N', \Sigma')$ and $\hat{f}: (\mathbb{Z}^{\Sigma(1)}, \tilde{\Sigma}) \rightarrow (\mathbb{Z}^{\Sigma'(1)}, \tilde{\Sigma}')$ are morphisms of fans. Since \hat{f} is uniquely determined by f , we often omit \hat{f} from the notation, and simply say that $f: \Sigma \rightarrow \Sigma'$ is a morphism of orbifold fans.

It is easy to see that a morphism $f: \Sigma \rightarrow \Sigma'$ of orbifold fans induces a corresponding equivariant morphism of pairs $(X_{\tilde{\Sigma}}, \Delta(\Sigma)) \rightarrow (X_{\tilde{\Sigma}'}, \Delta(\Sigma'))$, which, in turn, induces a 1-morphism $X_{\Sigma} \rightarrow X_{\Sigma'}$ of toric orbifolds. This gives a functor from the category of orbifold fans to the category of orbifolds over a base scheme S , and we call its essential image the *category of toric orbifolds* (without torus factors).

The simplest example of a toric morphism is that of toric open immersions, which correspond to subfans of stacky fans. Let $\Sigma = (N, \Sigma, \beta)$ be an orbifold fan. A *subfan* $\Sigma' \subset \Sigma$ is a triple $\Sigma' = (N, \Sigma', \beta')$, where $\Sigma' \subset \Sigma$ is a subset, which is a fan in its own right, and β' is the restriction of β to $\mathbb{Z}^{\Sigma'(1)}$. The canonical map $\Sigma' \rightarrow \Sigma$, which is the identity on N , corresponds to an open immersion $X_{\Sigma'} \rightarrow X_{\Sigma}$. We say that $X_{\Sigma'}$ is a *toric open substack* of X_{Σ} . The toric substacks corresponding to orbifold fans with a single maximal cone $\sigma \in \Sigma$ are of particular importance. We denote the corresponding substack, which is a *basic* toric orbifold, by U_{σ} .

Remark 3.11. Let $\Sigma = (N, \Sigma, \beta)$ be an orbifold fan. Then the coarse space of the toric orbifold X_{Σ} coincides with the toric variety X_{Σ} associated to the fan Σ . The association $(N, \Sigma, \beta) \mapsto (N, \Sigma)$ extends to a functor from the category of orbifold fans to the category of usual fans, and this functor corresponds to the functor taking a toric orbifold to its coarse space.

Remark 3.12. Let Σ be a simplicial fan on a lattice N . Then there is a canonical function $\beta_{\text{can}}: \mathbb{Z}^{\Sigma(1)} \rightarrow N$ taking $\rho \in \Sigma(1)$ to the non-zero lattice point u_{ρ} on ρ which is closest to the origin. The orbifold X_{can} associated to the orbifold fan $(N, \Sigma, \beta_{\text{can}})$ is the *canonical stack* of the toric variety $X = X_{\Sigma}$. Similarly as in the previous remark, the association $(N, \Sigma) \mapsto (N, \Sigma, \beta_{\text{can}})$ extends to a functor.

3.4 Toric stacky blow-ups

For smooth toric varieties, blow-ups at orbit closures correspond to star subdivisions. This generalises to toric orbifolds (see [EM12]). We recall the definition here.

DEFINITION 3.13. Let $\Sigma = (N, \Sigma, \beta)$ be an orbifold fan. Let σ be a cone in Σ and let $v = \sum_{\rho \in \sigma(1)} \beta(\rho)$. Denote the ray generated by v by ρ_0 . We define the *star subdivision* of the orbifold fan Σ along σ as $\Sigma^*(\sigma) = (N, \Sigma^*(v), \beta')$. Here $\Sigma^*(v)$ denotes the subdivision of the fan Σ obtained by adding the ray ρ_0 and subdividing each cone containing it, as described in [CLS11, §11.1]. The function β' is the extension of β to $\mathbb{Z}^{\Sigma^*(v)}$ taking the ray ρ_0 to v . There is a canonical map $\Sigma^*(\sigma) \rightarrow \Sigma$, which is the identity on N . The ray ρ_0 is called the *exceptional ray* of the star subdivision.

If X_Σ is the toric orbifold corresponding to the orbifold fan Σ , and σ is a cone in Σ , then the map $X_{\Sigma^*(\sigma)} \rightarrow X_\Sigma$ corresponding to the star subdivision is the blow-up of X_Σ with centre $V(\sigma)$. The divisor D_{ρ_0} on $X_{\Sigma^*(\sigma)}$ corresponding to the exceptional ray ρ_0 is the exceptional divisor of the blow-up.

DEFINITION 3.14. Let $\Sigma = (N, \Sigma, \beta)$ be an orbifold fan and $\rho = \{\rho_1, \dots, \rho_r\} \subset \Sigma(1)$ a set of rays. For each $\rho_i \in \rho$, we assign a weight d_i , which is a positive integer. Denote the function taking each ray to its weight by \mathbf{d} . Consider the group homomorphism $\beta': \mathbb{Z}^{\Sigma(1)} \rightarrow N$ defined by

$$\beta'(\rho) = \begin{cases} \mathbf{d}(\rho)\beta(\rho) & \text{if } \rho \in \rho, \\ \beta(\rho) & \text{otherwise.} \end{cases}$$

We denote the orbifold fan given by the triple (N, Σ, β') by $\Sigma_{\mathbf{d}^{-1}\rho}$. The natural morphism $\Sigma_{\mathbf{d}^{-1}\rho} \rightarrow \Sigma$ of orbifold fans, which is the identity map on the underlying group N , is called the *root construction* of Σ with respect to the rays in ρ with weights \mathbf{d} .

The terminology in the definition above is, of course, motivated by its relation to the root stack of the corresponding toric stacks. Using the same notation as in the definition above, we let $\pi: X' \rightarrow X$ be the morphism of toric orbifolds associated to the root fan $\Sigma_{\mathbf{d}^{-1}\rho} \rightarrow \Sigma$. On both X and X' we have toric divisors corresponding to the rays ρ_1, \dots, ρ_r . Denote the sets of such divisors by $\mathbf{D} = \{D_1, \dots, D_r\}$ and $\mathbf{D}' = \{D'_1, \dots, D'_r\}$, respectively. Then each divisor D'_i is a d_i th root of π^*D_i , and this structure identifies $X' \rightarrow X$ with the root stack $X_{\mathbf{d}^{-1}\mathbf{D}} \rightarrow X$, where we consider \mathbf{d} as a function on \mathbf{D} in the obvious way.

In terms of homogeneous coordinates, the root stack of a basic toric stack has the following description. Let X be a basic toric stack with homogeneous coordinates $(\mathcal{O}_S[x_1, \dots, x_r], A, \mathbf{a})$. Assume that \mathbf{D} is a set of toric divisors corresponding to the coordinates x_1, \dots, x_s for some $s \leq r$. Denote the generators of the group \mathbb{Z}^s by e_1, \dots, e_s and define the group

$$A_{\mathbf{d}^{-1}\mathbf{a}} = A \oplus \mathbb{Z}^s / \langle d_1 e_1 - a_1, \dots, d_s e_s - a_s \rangle,$$

which we think of as the group obtained from A by formally adjoining the roots $e_i = a_i/d_i$. Also, let $\mathbf{a}' = (e_1, \dots, e_s, a_{s+1}, \dots, a_r)$. Then the homogeneous coordinates of $X_{\mathbf{d}^{-1}\mathbf{D}}$ are given by

$$(\mathcal{O}_S[x_1^{1/d_1}, \dots, x_s^{1/d_s}, x_{s+1}, \dots, x_r], A_{\mathbf{d}^{-1}\mathbf{a}}, \mathbf{a}')$$

and the map $X_{\mathbf{d}^{-1}\mathbf{D}} \rightarrow X$ corresponds to the map of graded rings taking x_i to $x_i = (x_i^{1/d_i})^{d_i}$.

Remark 3.15. Any toric stack $X = X_\Sigma$ can be built up from the corresponding toric variety $X_{cs} = X_\Sigma$ as follows. There are canonical morphisms

$$X \rightarrow X_{\text{rig}} \rightarrow X_{\text{can}} \rightarrow X_{\text{cs}},$$

where the first is a gerbe (the rigidification of X), the second is a root stack and the third is the canonical stack associated to X_Σ . We describe this in terms of basic toric stacks. Let X be a basic toric stack with homogeneous coordinates $(\mathcal{O}_S[x_1, \dots, x_r], A, (a_1, \dots, a_r))$. Then its rigidification X_{rig} is a basic toric orbifold with homogeneous coordinates $(\mathcal{O}_S[x_1, \dots, x_r], A_{\text{rig}}, (a_1, \dots, a_r))$, with $A_{\text{rig}} = \langle a_1, \dots, a_r \rangle$ (see Remark 3.5).

For each $i = 1, \dots, r$, we define $A_i = A_{\text{rig}} / \langle a_1, \dots, \widehat{a}_i, \dots, a_r \rangle$, where \widehat{a}_i indicates that the element a_i is omitted. Then we have an exact sequence

$$0 \rightarrow A_{\text{can}} \rightarrow A_{\text{rig}} \xrightarrow{\varphi} A_1 \times \dots \times A_r \rightarrow 0,$$

where φ is the morphism induced by the canonical projections $A \rightarrow A_i$. Denote the order of A_i by d_i , and note that $a_i^{d_i} \in A_{\text{can}}$ for each $i = 1, \dots, r$. Furthermore, we have a graded homomorphism

$$(\mathcal{O}_S[x_1^{d_1}, \dots, x_r^{d_r}], A_{\text{can}}, (a_1^{d_1}, \dots, a_r^{d_r})) \rightarrow (\mathcal{O}_S[x_1, \dots, x_r], A_{\text{rig}}, (a_1, \dots, a_r))$$

given by obvious inclusion of sheaves of \mathcal{O}_S -algebras. It is straightforward to verify that this corresponds to a \mathbf{d} th root stack in the toric divisors, where $\mathbf{d} = (d_1, \dots, d_r)$, and that the corresponding morphism $X_{\text{rig}} \rightarrow X_{\text{can}}$ of basic toric orbifolds is the canonical morphism to the canonical stack.

3.5 Multiplicity, independency and smoothness

The toric destackification algorithm, which is described in the next section, is based on the well-known toric desingularisation algorithm described in for instance [CLS11, § 11]. In particular, the *multiplicity* of a cone plays an important role. Here we will briefly recall the main properties of multiplicities. We will also introduce the related concept of *independency* of toric divisors.

As usual, we let $\Sigma = (N, \Sigma, \beta)$ be an orbifold fan and $\sigma \in \Sigma$ a cone. Let ρ_1, \dots, ρ_r be the rays in $\sigma(1)$, and let u_i be the non-zero lattice point on the ray ρ_i which is closest to the origin. We associate the parallelotope

$$P_\sigma = \left\{ \sum_{i=1}^r \lambda_i u_i \mid 0 \leq \lambda_i < 1 \right\}$$

to the cone σ . Then the number of lattice points in P_σ is called the *multiplicity* of σ and is denoted by $\text{mult}(\sigma)$. The multiplicity satisfies the basic property $\text{mult}(\tau) \mid \text{mult}(\sigma)$ if $\tau \preceq \sigma$. It should be noted that the stacky structure β plays no part in the definition of multiplicity. In particular, the multiplicity of a cone is preserved by the root construction. A cone σ is called *smooth* provided that the multiplicity $\text{mult}(\sigma)$ equals one. An orbifold fan is *smooth* provided that all its cones are smooth.

The *multiplicity* $\text{mult}(\xi)$ at a point $\xi \in X_\Sigma$ in the toric orbifold is defined as the multiplicity of the cone spanned by the rays corresponding to the toric divisors passing through ξ . The multiplicity is one at ξ if and only if the corresponding point in the coarse space X_Σ is smooth over the base. In particular, the coarse space is smooth if and only if the orbifold fan is smooth.

The following proposition describes how to read off the multiplicity from the homogeneous coordinates of a basic toric orbifold.

PROPOSITION 3.16. *Let X be a basic toric orbifold with homogeneous coordinates*

$$(\mathcal{O}_S[x_1, \dots, x_r], A, \mathbf{a})$$

and let (N, Σ, β) be an orbifold fan defining X . Let σ be the (unique) maximal cone in Σ . Then $\text{mult}(\sigma) = |A_{\text{can}}|$, where A_{can} is defined as in Remark 3.15.

Proof. Let u_1, \dots, u_r be the non-zero lattice points closest to the origin on the rays $\rho_1, \dots, \rho_r \in \Sigma(1)$. Let $(N, \Sigma, \beta_{\text{can}})$ be the orbifold fan describing the canonical stack. Then $\beta_{\text{can}}(\rho_i) = u_i$. By [CLS11, (11.1.5)], the multiplicity of σ is given by the index $[N : \mathbb{Z}u_1 + \dots + \mathbb{Z}u_r]$. Hence, the result follows from the exact sequence

$$0 \rightarrow M \xrightarrow{\beta'_{\text{can}}} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\Sigma(1)}, \mathbb{Z}) \rightarrow A_{\text{can}} \rightarrow 0, \quad \beta'_{\text{can}} = \text{Hom}_{\mathbb{Z}}(\beta_{\text{can}}, \mathbb{Z})$$

from the Cox construction for the canonical stack X_{can} . □

Singularities on toric varieties only occur at the intersections of the toric divisors. Generally singularities get worse at points where more divisors intersect in the sense that the multiplicity grows. We introduce the concept of independency to capture the idea of nice divisors which do not make singularities worse.

DEFINITION 3.17. Let $\Sigma = (N, \Sigma, \beta)$ be an orbifold fan, $\sigma \in \Sigma$ a cone and $\rho \in \sigma(1)$ a ray. We say that ρ is *independent* at σ if $\text{mult} \tau = \text{mult} \sigma$, where τ is the face of σ spanned by the rays $\sigma(1) \setminus \rho$. A ray ρ is *independent* if it is independent at all cones σ containing ρ .

PROPOSITION 3.18. *Let $\Sigma = (N, \Sigma, \beta)$ be an orbifold fan and let σ be a cone in Σ . Then σ is smooth if and only if all rays in $\sigma(1)$ are independent at σ . In particular, the fan Σ is smooth if and only if all rays in $\Sigma(1)$ are independent.*

Proof. Let $\sigma \in \Sigma$. Assume that σ is smooth. Then $\text{mult} \sigma' = \text{mult} \sigma = 1$ for any facet $\sigma' \prec \sigma$. In particular, any ray in $\sigma(1)$ is independent at σ .

Conversely, assume that all rays in $\sigma(1)$ are independent at σ . Denote the rays in $\sigma(1)$ by ρ_1, \dots, ρ_r and let u_i denote the non-zero lattice points closest to the origin at ρ_i for each $i = 1, \dots, r$. Assume that $P = \lambda_1 u_1 + \dots + \lambda_r u_r$ is a lattice point in the parallelotope P_σ . By the assumption that all rays are independent, we have $\text{mult} \tau = \text{mult} \sigma$ for each facet τ of σ . In particular, the point P is contained in P_τ for each τ . Since the lattice points u_1, \dots, u_r are linearly independent in $N_{\mathbb{R}}$, this forces each $\lambda_i = 0$. Hence, the only lattice point in P_σ is the origin, that is, $\text{mult} \sigma = 1$. □

PROPOSITION 3.19. *Let X be a basic toric orbifold with homogeneous coordinates*

$$(\mathcal{O}_S[x_1, \dots, x_r], A, (a_1, \dots, a_r))$$

and let (N, Σ, β) be an orbifold fan defining X . Denote the rays of Σ by ρ_1, \dots, ρ_r and let σ denote the (unique) maximal cone of Σ . Then ρ_i is independent at σ if and only if the intersection $\langle a_1, \dots, \hat{a}_i, \dots, a_r \rangle \cap \langle a_i \rangle$ is trivial.

Proof. We assume, without loss of generality, that $i = r$. Let τ be the cone spanned by the rays $\rho_1, \dots, \rho_{r-1}$. By Proposition 3.16, the open substack $U_\tau \subset U_\sigma = X$ has homogeneous coordinates

$$(\mathcal{O}_S[x_1, \dots, x_r][x_r^{-1}], A', (\bar{a}_1, \dots, \bar{a}_r)),$$

where $A' = A/\langle a_r \rangle$. Define A_{can} and A_i as in Remark 3.15, and similarly for A'_{can} and A'_i . Since the morphism $\langle a_r \rangle \rightarrow A_r$ is surjective, the snake lemma gives us the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K & \longrightarrow & \langle a_r \rangle & \longrightarrow & A_r \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A_{\text{can}} & \longrightarrow & A & \longrightarrow & A_1 \times \cdots \times A_r \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A'_{\text{can}} & \longrightarrow & A' & \longrightarrow & A'_1 \times \cdots \times A'_{r-1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

with exact rows and columns. By Proposition 3.16, we have $\text{mult}(\sigma) = |A_{\text{can}}|$ and $\text{mult}(\tau) = |A'_{\text{can}}|$. Thus, it follows from Definition 3.17 and the diagram above that the ray ρ_r is independent at σ if and only if K vanishes. This is obviously equivalent to the condition stated in the proposition. \square

4. Toric destackification

In this section, we give an algorithm for functorial *destackification* (Definition 2.3) of toric orbifolds. Recall that for an orbifold, a destackification is roughly given by a sequence of *smooth stacky blow-ups* (see § 2.2) such that the modified orbifold becomes a root stack over its coarse space. For a toric orbifold this translates to the combinatorial problem of subdividing an *orbifold fan* (Definition 3.8) using *star subdivisions at cones* (Definition 3.13) and *roots of rays* (Definition 3.14), such that the resulting fan becomes *smooth* (see § 3.5). If we do not care about functoriality, this can be achieved by using an algorithm which is very similar to the classical algorithm for resolving singularities of a simplicial toric variety (see [CLS11, § 11]). We illustrate the procedure by giving an example.

Example 4.1. Consider the orbifold fan given in Figure 4.1(a). The maximal cone $\sigma = \text{Cone}(\rho_1, \rho_2)$ has multiplicity 5. To obtain a subdivision of the fan into cones with lower multiplicity, we subdivide σ at a ray passing through a non-zero lattice point in the parallelotope P_σ . This can be achieved by first taking the third root at the ray ρ_1 and then taking the star subdivision at σ , which yields the fan in Figure 4.1(b). Note that the exceptional ray ρ_3 passes through the lattice point $(1, 1)$, which indeed lies in P_σ . The situation has improved, since the new maximal cones have multiplicities 1 and 3, respectively. To complete the destackification, we take the second root of ρ_2 followed by a star subdivision at $\text{Cone}(\rho_2, \rho_3)$, which yields a smooth orbifold fan. The last steps are not shown in the figure.

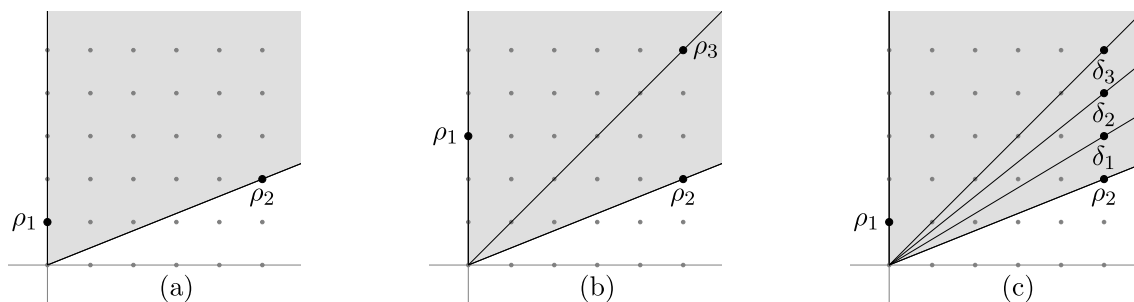


FIGURE 4.1. A stacky fan corresponding to a basic toric orbifold (a) together with the partial destackifications (b) and (c) described in Examples 4.1 and 4.4, respectively.

The problem with the naive algorithm outlined in Example 4.1 is that it is not *functorial* (Definition 2.6) with respect to toric open immersions, as illustrated in the next example. In particular, this means that it will be hard to use it as a building block for a global destackification algorithm which works for stacks which are only locally toric.

Example 4.2. The toric suborbifolds U_{ρ_1} and U_{ρ_2} of the toric orbifold associated to the orbifold fan given in Figure 4.1(a) are isomorphic. Yet, corresponding suborbifolds in Figure 4.1(b) are non-isomorphic since we have stackiness along the divisor corresponding to ρ_1 , but not along the divisor corresponding to ρ_2 . This shows that destackification described in Example 4.1 cannot be obtained from running a destackification algorithm which is functorial with respect to toric open immersions. In fact, both the orbifolds U_{ρ_1} and U_{ρ_2} are smooth toric varieties and any sensible destackification algorithm should leave such varieties unmodified.

The problem illustrated in Example 4.2 is really intrinsic to the root construction; taking a root modifies the stack along a whole divisor, even at points lying over smooth points of the coarse space. On the other hand, it is in general not possible to destackify by just using star subdivisions at cones, as shown by the following example (cf. [Kol07, Claim 2.29.2]).

Example 4.3. Let us, for simplicity, assume that we are working over a field k . Then the basic toric orbifold associated to the orbifold fan in Figure 4.1 has homogeneous coordinate ring $(k[x_1, x_2], C_5, (1, 3))$. Blowing up the origin yields two charts with homogeneous coordinate rings $(k[x_1, x_2/x_1], C_5, (1, 2))$ and $(k[x_1/x_2, x_2], C_5, (3, 3))$, respectively. But the first chart is isomorphic to the toric orbifold we started with since the weight vector $(1, 3)$ can be obtained from $(1, 2)$ by multiplying with the unit 3 and swapping the coordinates. Thus, no improvement towards destackification has been achieved.

Our solution to the problem is to admit only stacky blow-ups in centres lying over the original problematic locus. In particular, this means that the first stacky blow-up must be an ordinary blow-up.

Example 4.4. We describe how the toric stack given by the orbifold fan in Figure 4.1(a) can be destackified using only stacky blow-ups with loci lying over the original problematic locus. Instead of taking the third root of ρ_1 and subdividing as in Figure 4.1(a), we take three subsequent subdivisions at the cones $\text{Cone}(\rho_1, \rho_2)$, $\text{Cone}(\rho_1, \delta_1)$ and $\text{Cone}(\rho_1, \delta_2)$ with δ_1, δ_2 and δ_3 denoting the exceptional rays of each subdivision. The resulting fan is pictured in Figure 4.1(c). Note that

Cone (ρ_1, δ_3) is identical to Cone (ρ_1, ρ_3) from Figure 4.1(b). In particular, it has multiplicity 1. The other cones of Figure 4.1(c) have multiplicities 1, 5 and 5 respectively when enumerated from left to right. The cone Cone (δ_1, ρ_2) can be destackified by a single star subdivision (this is not shown in the figure). But, as was shown in Example 4.3, the cone Cone (δ_2, δ_1) is isomorphic to Cone (ρ_1, ρ_2) in Figure 4.1(a) which we started with. Hence, it may seem that no progress has been made. However, since both the rays δ_1 and δ_2 correspond to divisors lying over the original problematic locus, we may now root them. This allows us to complete the destackification by proceeding as in Example 4.1 with the rays δ_2 and δ_1 taking the roles of ρ_1 and ρ_2 , respectively (this is not shown in the figure).

Note that the philosophy behind this solution is similar to the one behind classical strong desingularisation algorithms. We relax the functoriality requirement and do not demand the process to be functorial with respect to open immersions after each stacky blow-up. This requires us to somehow keep track of the history of the destackification process. We do this by adding additional structure to our toric orbifolds.

Firstly, we will assume that the rays of the orbifold fan are ordered. Note that the ordering of the rays also induces a total ordering on the set of cones. This is induced by the lexicographic ordering of the power set of the set of rays. The ordering of the rays ensures that the pair (X, \mathbf{E}) , where X is the toric stack and \mathbf{E} is the set of toric divisors, is a *standard pair* in the sense of Definition 2.1. When taking roots and star subdivisions of cones, we use similar conventions regarding the ordering of the rays of the new stacky fans as was described in § 2.2.

Secondly, we make use of a *distinguished structure* (Definition 2.7) on our toric orbifold. Combinatorially, the distinguished structure corresponds to a subset of the set of rays in the orbifold fan. As we will see, these rays represent divisors which lie over the original problematic locus.

Roughly, Algorithm B, which is the main toric destackification algorithm, works as follows. At each iteration of the main loop one problematic cone is identified. This cone is subdivided with a single star subdivision and the exceptional ray is marked as distinguished. The resulting cones are then improved by invoking Algorithm A, which performs a sequence of *admissible* stacky blow-ups (Definition 2.7). Combinatorially, admissible stacky blow-ups correspond to roots of distinguished rays and star subdivisions of cones containing at least one distinguished ray. Note that the admissible stacky blow-ups precisely correspond to stacky blow-ups in centres lying over the problematic cone identified in the first step of the iteration.

Example 4.5. In principle, the destackification described in Example 4.4 corresponds to a single iteration of Algorithm B. The first blow-up creates the exceptional ray δ_1 , which is marked as distinguished. Since the exceptional ray of each admissible star subdivision is distinguished, all subsequent star subdivisions and roots described in the example are distinguished.

Note that the actual output of Algorithm B will be slightly different (and considerably harder to draw) than that described in Example 4.4. The output depends on the somewhat arbitrary choice of ordering made in Step A2.

In Example 4.4, we could make all cones smooth via admissible stacky blow-ups after a single initial blow-up creating a distinguished exceptional ray. This is always possible for two-dimensional cones, but not for cones of higher dimension. Instead we focus on making all distinguished rays *independent* (Definition 3.17). Since independent rays do not contribute to the multiplicity of the cone, this effectively reduces the problem of destackifying the cone to the problem of destackifying a cone of lower dimension.

ALGORITHM A (Partial toric destackification). The input of the algorithm is an orbifold fan Σ_0 with an ordered set of rays having a distinguished structure. The output is a sequence

$$\Sigma_n \rightarrow \cdots \rightarrow \Sigma_0$$

of admissible stacky blow-ups, with the property that all distinguished rays of Σ_n are independent. The construction is functorial with respect to isomorphisms $\Sigma'_0 \rightarrow \Sigma_0$ of stacky fans preserving the ordering and the distinguished structure. We use the notation $\Sigma_i = (N, \Sigma_i, \beta_i)$ in the description of the algorithm.

- A0.** [Initialise.] Set $i = 0$.
- A1.** [Choose a cone.] Let \mathcal{S} be the set of cones $\sigma \in \Sigma_i$ such that $\sigma(1)$ contains a distinguished divisor and such that the relative interior of the parallelotope P_σ contains a lattice point. Order the cones $\sigma \in \mathcal{S}$ first by number of non-distinguished rays in $\sigma(1)$, then by multiplicity and finally by the ordering induced by the ordering on $\Sigma_i(1)$. If \mathcal{S} is empty, the algorithm terminates. Otherwise let σ_i be the largest element of \mathcal{S} with respect to the ordering described above.
- A2.** [Choose a formal sum of rays.] Let \mathcal{P} be the set of formal sums $\psi \in \mathbb{Z}^{\sigma_i(1)}$ such that the ray $\mathbb{R}_{\geq 0}\beta_i(\psi)$ passes through a lattice point in the relative interior of the parallelotope P_{σ_i} . This set is non-empty by our choice of σ_i in the previous step. Let ψ_i be the smallest element of \mathcal{P} with respect to the lexicographic ordering of $\mathbb{Z}^{\sigma_i(1)}$ induced by the ordering of $\Sigma_i(1)$.
- A3.** [Root distinguished rays.] Assume that $\psi_i = d_1\rho_1 + \cdots + d_s\rho_s + c_1\delta_1 + \cdots + c_r\delta_r$, with ρ_j and δ_j being distinct non-distinguished and distinguished rays, respectively. Let $\Sigma_{i+1} \rightarrow \Sigma_i$ be the root construction $(\Sigma_i)_{c_1^{-1}\delta_1, \dots, c_r^{-1}\delta_r} \rightarrow \Sigma_i$ and $\psi_{i+1} = d_1\rho_1 + \cdots + d_s\rho_s + \delta_1 + \cdots + \delta_r$. Increment i by one. Note that after this step all distinguished rays in the support of ψ_i have coefficient one. Also, the transformation rule asserts that $\beta_i(\psi_i) = \beta_{i-1}(\psi_{i-1})$.
- A4.** [Perform a stacky star subdivision.] Let σ_i be the cone generated by the support of ψ_i . Let $\Sigma_{i+1} \rightarrow \Sigma_i$ be the stacky star subdivision $\Sigma_i(\sigma_i) \rightarrow \Sigma_i$ and denote the exceptional ray by ε_{i+1} . Furthermore, let $\psi_{i+1} = \psi_i - \sum_{\rho \in \sigma_i(1)} \rho + \varepsilon_{i+1}$ and then increment i by 1. Note that after this step the support of ψ_i contains just one distinguished ray ε_i , which occurs with coefficient one. Also, the transformation rule asserts that $\beta_i(\psi_i) = \beta_{i-1}(\psi_{i-1})$.
- A5.** [Iterate inner loop.] While the support of ψ_i contains more than one ray, repeat from Step A4.
- A6.** [Iterate main loop.] Repeat from Step A1.

Proof of correctness of Algorithm A. Functoriality is clear, since all choices in the algorithm depend only on properties preserved by isomorphisms.

If σ is a cone containing a distinguished, non-independent ray δ , then there is a face σ' of σ containing δ with $P_{\sigma'}$ containing a lattice point in its relative interior. Hence, the algorithm does not halt prematurely.

It remains to prove that the algorithm halts. For notational convenience, we assume, without loss of generality, that $i = 0$ at the beginning of an iteration of the main loop and $i = n$ when the iteration ends.

Denote the cone generated by the support of ψ_0 by σ_0 , and let τ_0 be any cone in Σ_0 of maximal dimension containing σ_0 . Using the notation in Step A3, we have

$$\tau_0 = \text{Cone}(\rho_1, \dots, \rho_s, \delta_1, \dots, \delta_r, \nu_1, \dots, \nu_t)$$

for some rays ν_1, \dots, ν_t . By maximality of σ_0 with respect to the ordering defined in Step A1, we have $\text{mult}(\tau_0) = \text{mult}(\sigma_0)$. Define τ_{i+1} recursively as any choice of cone of maximal dimension

in the subdivision of τ_i such that τ_{i+1} has the same number of non-distinguished rays as τ_i . For $i \geq 2$, we have

$$\tau_i = \text{Cone}(\rho_1, \dots, \rho_s, \delta_1, \dots, \widehat{\delta}_k, \dots, \delta_r, \nu_1, \dots, \nu_t, \varepsilon_i),$$

where $\widehat{\delta}_k$ indicates that the ray δ_k should be omitted from the list for some k with $1 \leq k \leq r$.

The transformation rule for the elements ψ_i asserts that $\beta_{i+1}(\psi_{i+1}) = \beta_i(\psi_i)$ throughout a whole iteration of the main loop. In particular, we have $\beta_n(\psi_n) = \beta_n(\varepsilon_n) = \beta_0(\psi_0)$. But the ray through $\beta_0(\psi_0)$ passes through a lattice point in $P_{\sigma_0} \subset P_{\tau_0}$ by choice of ψ_0 . For the same reason as for the traditional algorithm for toric varieties, it follows that the multiplicity of τ_n is strictly smaller than $\text{mult } \tau_0$. Since any cone produced in the iteration of the main loop is either a face of τ_n for some choice of sequence τ_0, \dots, τ_n or has fewer non-distinguished rays, it follows that all new cones are smaller than σ_0 with respect to the ordering defined in Step A2. Since σ_0 has been removed, this process cannot continue indefinitely, and the algorithm eventually stops. \square

The inner workings of Algorithm A are best understood by considering some examples.

Example 4.6. We give an explicit example of the formal sum ψ_i associated to a cone σ_i as described in A2. For notational convenience, we assume that $i = 0$. Assume that σ_0 is a cone spanned by the three rays ρ_1, ρ_2, ρ_3 and that the marked lattice points on the rays are given by

$$\beta_0(\rho_1) = (6, 3, 3), \quad \beta_0(\rho_2) = (0, 2, 1), \quad \beta_0(\rho_3) = (0, 0, 1).$$

Then the non-zero lattice points on the rays in $\sigma_0(1)$ closest to the origin are given by

$$u_1 = (2, 1, 1), \quad u_2 = (0, 2, 1), \quad u_3 = (0, 0, 1)$$

and we see that the multiplicity of σ_0 equals $|\det(u_1, u_2, u_3)| = 4$. Hence, the parallelotope P_{σ_0} contains three non-trivial lattice points. It is easily verified that these are given by

$$v_1 = (1, 2, 2), \quad v_2 = (0, 1, 1), \quad v_3 = (1, 1, 1).$$

Note that only v_1 and v_3 lie in the relative interior of P_{σ_0} . These points can also be expressed as

$$v_1 = \frac{1}{12}\beta_0(2\rho_1 + 9\rho_2 + 9\rho_3), \quad v_3 = \frac{1}{12}\beta_0(2\rho_1 + 3\rho_2 + 3\rho_3).$$

Hence, $\psi_0 = 2\rho_1 + 3\rho_2 + 3\rho_3$ since this expression is smaller than $2\rho_1 + 9\rho_2 + 9\rho_3$ lexicographically.

Example 4.7. We continue the previous example by tracing what happens when executing the Steps A3 through A5 of Algorithm A. The results depend on which rays in σ_0 are distinguished, and we consider three different cases. In the first case all rays are distinguished and we let $\delta_1 = \rho_1, \delta_2 = \rho_2, \delta_3 = \rho_3$. In the second case the rays $\delta_1 = \rho_2, \delta_2 = \rho_3$ are distinguished. In the third case only the ray $\delta_1 = \rho_3$ is distinguished. Table 4.1 shows how the formal sum ψ_i changes with i .

The resulting fans are illustrated in Figure 4.2. The key things to note about these fans are the following.

(i) Since the lattice point $\beta_i(\psi_i)$ is independent of i during the execution of the Steps A3 through A5, the ray $\varepsilon := \varepsilon_i$ is the same in all three cases when the inner loop terminates. Note also that ε passes through v_3 , which lies in the parallelotope P_{σ_0} as explained in Example 4.6. Hence, any cone containing ε and two of the rays in $\sigma_0(1)$ must have strictly lower multiplicity than σ_0 . These cones are shaded in Figure 4.2.

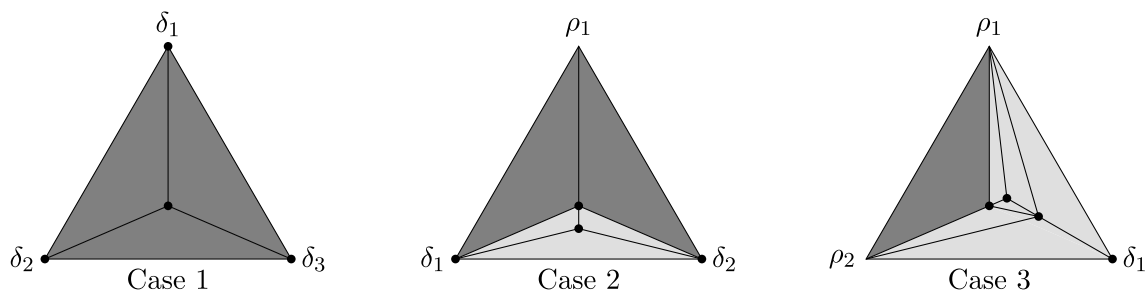


FIGURE 4.2. Examples of the subdivision procedure from Algorithm A in the three different cases described in Example 4.7. In each case only the intersection of the respective fan with the plane through the points $\beta_0(\sigma_0(1))$ is shown, and the resulting intersection is affinely transformed such that the cone σ_0 is pictured as an equilateral triangle. The maximal cones which are guaranteed to have lower multiplicity than σ_0 are shaded and the distinguished rays are marked by black dots.

TABLE 4.1. The formal sum ψ_i for the three examples described in Example 4.7.

	Case 1	Case 2	Case 3
ψ_0	$2\delta_1 + 3\delta_2 + 3\delta_3$	$2\rho_1 + 3\delta_1 + 3\delta_2$	$2\rho_1 + 3\rho_2 + 3\delta_1$
ψ_1	$\delta_1 + \delta_2 + \delta_3$	$2\rho_1 + \delta_1 + \delta_2$	$2\rho_1 + 3\rho_2 + \delta_1$
ψ_2	$\varepsilon = \varepsilon_2$	$\rho_1 + \varepsilon_2$	$\rho_1 + 2\rho_2 + \varepsilon_2$
ψ_3	—	$\varepsilon = \varepsilon_3$	$\rho_2 + \varepsilon_3$
ψ_4	—	—	$\varepsilon = \varepsilon_4$

(ii) For the other maximal cones in the subdivision we have no control over the multiplicity, which might well be higher than what we started with. However, each of these cones has fewer non-distinguished rays. The distinguished rays are marked by black dots in Figure 4.2.

This shows that all maximal cones in the subdivision are smaller than the cone σ_0 with respect to the ordering introduced in Step A1. Hence, we do get an improvement towards destackification.

We conclude the section by describing the functorial toric destackification algorithm. We leave the (easy) proof of correctness to the reader, since the algorithm is a special case of the more general Algorithm E.

ALGORITHM B (Functorial toric destackification). The input of the algorithm is an orbifold fan Σ_0 together with a total ordering on the set $\Sigma_0(1)$ of rays. The output is a sequence

$$\Sigma_n \rightarrow \cdots \rightarrow \Sigma_0$$

of stacky modifications such that all rays in $\Sigma_n(1)$ are independent. In particular, all cones in Σ_n are smooth. The construction is functorial in the obvious sense with respect to isomorphisms $\Sigma'_0 \rightarrow \Sigma_0$ preserving the ordering of the rays and subfans $\Sigma''_0 \subseteq \Sigma_0$.

B0. [Initialise.] Set $i = 0$.

B1. [Choose a cone.] Consider the set \mathcal{S} of cones σ in Σ_i with the property that none of the rays in $\sigma(1)$ are independent at σ . If this set is empty, then all rays in Σ_i are independent and the algorithm terminates. Choose a cone $\sigma \in \mathcal{S}$ of maximal dimension. If several such cones exist, choose the largest one with respect to the natural ordering on the cones in Σ_i .

- B2.** [Create distinguished ray.] Let $\Sigma_{i+1} \rightarrow \Sigma_i$ be the star subdivision of Σ_i in σ . Give Σ_{i+1} a distinguished structure by letting the exceptional ray from the subdivision be the only distinguished ray. Increment i by one.
- B3.** [Resolve the cone.] Invoke Algorithm A and append the output to the sequence. Increment i by the length of this output.
- B4.** [Iterate.] Forget the distinguished structure, and iterate from Step B1.

5. Local homogeneous coordinates

In §3, we introduced *basic toric stacks*. Here we will show that every smooth tame stack with diagonalisable stabilisers is locally of this form in a certain sense. This will allow us to use *local homogeneous coordinates* even for non-toric stacks, which in turn will allow us to generalise the toric destackification algorithm. We start by making a precise definition of what we mean by a stack being locally toric.

DEFINITION 5.1. Let X be an algebraic stack which is smooth over a scheme S , and let $\xi \in X$ be a point. By a *toric chart* of X over S at ξ , we mean a diagram

$$\begin{array}{ccc} & X' & \\ f \swarrow & & \searrow g \\ X & & X'' \end{array}$$

of algebraic stacks over S , together with a point $\xi' \in X'$. The data are required to satisfy the following properties.

- (i) The stack X'' is a basic toric stack over S .
- (ii) The point ξ' maps to ξ in X and to a point $\xi'' \in X''$ lying in the intersection of the prime toric divisors of X'' .
- (iii) The maps f and g are étale and stabiliser-preserving.

A homogeneous coordinate ring of X'' is called a *local homogeneous coordinate ring* at ξ . Assume that E is a simple normal crossings divisor on X and that Z is a closed substack of X having simple normal crossings with E . Then we say that E and Z are *compatible* with the toric chart if the pull-back of E to X' coincides with the pull-back of a toric divisor on X'' , and the pull-back of Z to X' coincides with the pull-back of an intersection of prime toric divisors on X'' .

To prove that a smooth tame stack with diagonalisable stabilisers has a toric chart at every point, we need a version of the structure theorem for tame algebraic stacks which takes smoothness into account. We give such a theorem in Appendix A. We will also need the following lemma.

LEMMA 5.2. *Let $X = \text{Spec } A$ be an affine scheme over an affine base scheme S , and let G be a finite, linearly reductive, locally free group scheme over S acting on X . Let $\xi \in X$ be a point, and let $D(f)$ be a distinguished open subscheme of X containing the orbit of ξ . Then there is a refinement $\xi \in D(g) \subset D(f)$ such that g is an invariant section which is a multiple of f .*

Proof. Let $\pi: X \rightarrow X/G = \text{Spec } A_0$ be the coarse quotient, where $A_0 \subset A$ is the ring of invariant sections. The map π is integral and therefore closed. The set $\pi(V(f))$ does not contain $\pi(\xi)$, by the assumption that the orbit of ξ is contained in $D(f)$. Let $D(h)$, with $h \in A_0$, be a distinguished open neighbourhood of $\pi(\xi)$ in the complement of $\pi(V(f))$ in X/G . This pulls back to an open subset, also denoted by $D(h)$, satisfying $\xi \in D(h) \subset D(f)$. The condition $D(h) \subset D(f)$ implies that $\text{rad}(h) \subset \text{rad}(f)$. Hence, there is a power $g = h^n$ of h which is a multiple of f . \square

Now we are ready for the main result of this section.

PROPOSITION 5.3. *Let X be an algebraic stack with finite inertia and diagonalisable geometric stabilisers. Assume that X is smooth and quasi-separated over a scheme S . Then X admits toric charts over S at each of its points. Furthermore, if E is a simple normal crossings divisor on X , and Z is a closed substack of X having simple normal crossings with E , then the toric charts may be chosen such that they are compatible with E and Z .*

Proof. The question is local on X with respect to stabiliser-preserving étale morphisms. Thus, by Propositions A.8 and A.9, we may assume that X is of the form $[U/\Delta]$, where U is an affine scheme which is smooth over S and Δ is a diagonalisable group acting on U . Furthermore, we may assume that ξ lifts to a point $\xi' \in U$ which is fixed under the Δ -action.

The Δ -action corresponds to a grading on \mathcal{O}_U by the character group Δ^\vee , which is a finite abelian group. Choose homogeneous global sections f_1, \dots, f_n of \mathcal{O}_U such that the differentials df_1, \dots, df_n form a basis of $\Omega_{U/S} \otimes_{\mathcal{O}_U} \kappa(\xi')$. Consider the map $\mathcal{O}_S[x_1, \dots, x_n] \rightarrow \mathcal{O}_U$ taking x_i to f_i . We give the polynomial ring a Δ^\vee -graded structure, by letting x_i have the same degree as f_i . This gives an equivariant map $\widehat{g}: U \rightarrow \mathbb{A}_S^n$ over S . By construction, the canonical map

$$\Omega_{\mathbb{A}_S^n/S} \otimes_{\mathcal{O}_S[x_1, \dots, x_n]} \kappa(\xi') \rightarrow \Omega_{U/S} \otimes_{\mathcal{O}_U} \kappa(\xi')$$

is an isomorphism. Since U is smooth over S , it follows that \widehat{g} is étale at ξ' by [EGAIV, Corollary 17.11.2]. Denote the corresponding map $[U/\Delta] \rightarrow [\mathbb{A}_S^n/\Delta]$ of stacks by g . The map g is representable, so the stabiliser of ξ injects into the stabiliser of $g(\xi)$. Since ξ' is fixed by the action of Δ , the stabiliser at ξ is Δ , so the map of stabilisers must be an isomorphism. Since the locus where f is étale and stabiliser-preserving is open [Ryd13, Proposition 6.5], we may as well assume that $[U/\Delta] \rightarrow [\mathbb{A}_S^n/\Delta]$ is étale and stabiliser-preserving, after shrinking U invariantly by using Lemma 5.2 if necessary. Finally, we simply remove the prime toric divisors from $[\mathbb{A}_S^n/\Delta]$ which do not contain $g(\xi)$.

Now we turn to the statement about the simple normal crossings divisors. Let E_1, \dots, E_r be the components of E passing through ξ . They correspond to locally principal homogeneous ideals I_i in \mathcal{O}_U . Also, denote the homogeneous ideal corresponding to Z by I . Next we choose our sections f_1, \dots, f_n one by one in a way such that the differentials df_i remain linearly independent in $\Omega_{U/S} \otimes_{\mathcal{O}_U} \kappa(\xi')$. First we pick homogeneous f_i from I_i for $1 \leq i \leq r$. Then we pick homogeneous f_{r+1}, \dots, f_s from I with s as large as possible. Finally, we pick the remaining homogeneous sections from \mathcal{O}_U . By the normal crossings assumption, we get compatibility in a neighbourhood of ξ' , which we may assume is Δ -invariant by Lemma 5.2. \square

6. Conormal invariants

In the last section, we established that any standard pair (X, \mathbf{E}) with diagonalisable stabilisers locally looks like a basic toric stack. This basic toric stack is described by a set of combinatorial

parameters, which we will call the *toric type*. In principle, the toric type captures the information of the stacky fan describing the basic toric stack together with how the rays in this stacky fan correspond to the components of \mathbf{E} . As we will see, the toric type at a given point can be recovered in a coordinate-free manner.

In the destackification algorithms, several different invariants are used in order to determine appropriate loci to blow up. In this section, we develop a common framework for studying a class of invariants, which we will call *conormal invariants*. By definition, the value of a conormal invariant at a given point depends only on the toric type. Given a standard pair (X, \mathbf{E}) , a conormal invariant induces an upper semi-continuous function on the underlying topological space $|X|$ taking values in some ordered set. Since we usually want to blow up the locus where the function obtains its maximum, we need to assert that this locus is smooth. We also need to assert invariance under base change and compositions with smooth, stabiliser-preserving morphisms and rigidifications in order to get the right functorial properties. We will establish simple combinatorial criteria for conormal invariants to satisfy these properties.

Let (X, \mathbf{E}) be a standard pair over a scheme S , and assume that X has diagonalisable stabilisers. The stabiliser at a geometric point $\xi: \text{Spec } \bar{k} \rightarrow X$ will be denoted Δ_ξ . Denote the pull-back of X along the composition $\text{Spec } \bar{k} \rightarrow X \rightarrow S$ by $X_{\bar{k}}$. Then the morphism ξ factors as

$$\text{Spec } \bar{k} \rightarrow B\Delta_\xi \hookrightarrow X_{\bar{k}} \rightarrow X.$$

The map $\text{Spec } \bar{k} \rightarrow X_{\bar{k}}$ is a section of the natural projection. By Lemma A.10, this implies that the canonical monomorphism $B\Delta_\xi \hookrightarrow X_{\bar{k}}$ is a closed immersion. The invariants which we are interested in will be derived from the conormal bundle $\mathcal{N}_{B\Delta_\xi/X_{\bar{k}}}$.

Let $A(\xi)$ denote the character group of Δ_ξ . Since Δ_ξ is assumed to be diagonalisable, the category of coherent \mathcal{O}_{Δ_ξ} -modules is simply equivalent to the category of $A(\xi)$ -graded \bar{k} -vector spaces. In particular, the sheaf $\mathcal{N}_{B\Delta_\xi/X_{\bar{k}}}$ splits into rank-1 locally free subsheaves. In general, this splitting is non-canonical, but each divisor in the set \mathbf{E} passing through the point ξ canonically determines a rank-1 subbundle of $\mathcal{N}_{B\Delta_\xi/X_{\bar{k}}}$.

PROPOSITION 6.1. *Let (X, \mathbf{E}) be a standard pair over a scheme S and assume that X has diagonalisable stabilisers. Given a geometric point $\xi: \text{Spec } \bar{k} \rightarrow X$, we let E^1, \dots, E^s be the components of \mathbf{E} passing through ξ . Let $g_i: B\Delta_\xi \hookrightarrow E_{\bar{k}}^i$ denote the canonical morphism to the fibre of the component E^i . Then the conormal bundle $\mathcal{N}_{B\Delta_\xi/X_{\bar{k}}}$ splits as a direct sum*

$$\mathcal{N}_{B\Delta_\xi/X_{\bar{k}}} = g_1^* \mathcal{N}_{E_{\bar{k}}^1/X_{\bar{k}}} \oplus \dots \oplus g_s^* \mathcal{N}_{E_{\bar{k}}^s/X_{\bar{k}}} \oplus \mathcal{N}_{\text{res}}, \tag{6.1}$$

where each summand $g_i^* \mathcal{N}_{E_{\bar{k}}^i/X_{\bar{k}}} V_i$ has rank 1.

Proof. By passing to the fibre, we may, without loss of generality, assume that $S = \text{Spec } \bar{k}$. Let $Z_0 = X$ and define Z_i recursively by means of the cartesian diagrams

$$\begin{array}{ccc} Z_i & \longrightarrow & Z_{i-1} \\ h_i \downarrow & & \downarrow \\ E^i & \longrightarrow & X \end{array}$$

Since we assume that the divisors E^i intersect transversally, each Z_i is smooth and we have canonical isomorphisms $\mathcal{N}_{Z_i/Z_{i-1}} \simeq h_i^* \mathcal{N}_{E^i/X}$ by [EGAIV, Proposition 17.13.2]. Now consider the increasing filtration

$$B\Delta_\xi \hookrightarrow Z_s \hookrightarrow \dots \hookrightarrow Z_0 = X$$

of closed immersions between stacks which are smooth over S . We denote the various compositions by $f_i: B\Delta_\xi \rightarrow Z_i$. By [EGAIV, Proposition 16.9.13], we have short exact sequences

$$0 \rightarrow f_i^* \mathcal{N}_{Z_i/Z_{i-1}} \rightarrow \mathcal{N}_{B\Delta_\xi/Z_{i-1}} \rightarrow \mathcal{N}_{B\Delta_\xi/Z_i} \rightarrow 0$$

for each $i \in \{1, \dots, s\}$. Since the group Δ_ξ is linearly reductive, these sequences split, and we get a decomposition

$$\mathcal{N}_{B\Delta_\xi/X} = f_1^* \mathcal{N}_{Z_1/Z_0} \oplus \dots \oplus f_r^* \mathcal{N}_{Z_s/Z_{s-1}} \oplus \mathcal{N}_{B\Delta_\xi/Z_s}.$$

But the maps g_i factors through h_i , which implies that we get canonical isomorphisms $g_i^* \mathcal{N}_{E^i/X} \simeq f_i^* \mathcal{N}_{Z_i/Z_{i-1}}$. We therefore get the desired decomposition by letting $\mathcal{N}_{\text{res}} = \mathcal{N}_{B\Delta_\xi/Z_s}$. Since the substacks E^i are effective Cartier divisors, the bundles $\mathcal{N}_{E^i/X}$ are locally free of rank 1. □

For the purpose of constructing the invariants needed in the destackification algorithm, we only need some combinatorial information extracted from the conormal representation. The next step is to give a precise description of this information.

Given a finite set \mathbf{E} , we denote its associated pointed set by \mathbf{E}_+ . That is, the pointed set \mathbf{E}_+ is the disjoint union of \mathbf{E} with the singleton set $\{*\}$, with $*$ regarded as the distinguished point in \mathbf{E}_+ .

Consider the class of pairs (A, \mathbf{v}) , where A is a finite abelian group and \mathbf{v} is an *unordered* sequence of pairs $v_i = (a_i, e_i) \in A \times \mathbf{E}_+$ such that each element of \mathbf{E} occurs at most once in the sequence e_1, \dots, e_r . Such a pair (A, \mathbf{v}) is considered equivalent to another pair (A', \mathbf{v}') provided that there exists a group isomorphism $\varphi: A \rightarrow A'$ such that $\mathbf{v}' = ((\varphi(a_1), e_1), \dots, (\varphi(a_r), e_r))$ for some suitable ordering of the elements in the sequence \mathbf{v}' .

DEFINITION 6.2. The set $U(\mathbf{E})$ of *toric types* parametrised by a finite set \mathbf{E} is defined as the set of equivalence classes of pairs (A, \mathbf{v}) with A a finite abelian group and \mathbf{v} an unordered sequence of elements $v_i = (a_i, e_i) \in A \times \mathbf{E}_+$ as described above. The element a_i is called the *weight* of v_i . If $e_i \in \mathbf{E}$, we say that v_i is *marked* by e_i . Otherwise, we say that v_i is *unmarked*.

The conormal representation lets us associate a toric type to each point of a standard pair (X, \mathbf{E}) .

DEFINITION 6.3. Let (X, \mathbf{E}) be a standard pair and let $\xi \in X$ be a point in X . Let $\mathcal{N}_1 \oplus \dots \oplus \mathcal{N}_r$ be a splitting of the conormal bundle $\mathcal{N}_{B\Delta_\xi/X_{\bar{k}}}$ into locally free sheaves of rank 1. Denote the character group of the stabiliser at ξ by A and let a_i be the character corresponding to \mathcal{N}_i .

We assume that the splitting is compatible with the splitting given in (6.1) in the sense that $\mathcal{N}_i = g_i^* \mathcal{N}_{E^i/X_{\bar{k}}}$ for $1 \leq i \leq s$ using the notation and the indexing from Proposition 6.1. Let

$$v_i = \begin{cases} (a_i, E^i) & \text{if } 1 \leq i \leq s, \\ (a_i, *) & \text{otherwise.} \end{cases}$$

We define the *toric type* of (X, \mathbf{E}) at ξ as the element in $U(\mathbf{E})$ represented by $(A, (v_1, \dots, v_r))$ in $U(\mathbf{E})$.

Remark 6.4. Note that the unordered sequence a_1, \dots, a_r referred to in Definition 6.3 depends only on the class $[\mathcal{N}_{B\Delta_\xi/X_{\bar{k}}}]$ of the conormal bundle in the Grothendieck group $K_0(\text{Coh}(B\Delta_\xi))$. Alternatively, one could instead consider the class of the derived pull-back of the cotangent

complex to the Grothendieck group of the triangulated category of perfect complexes on $B\Delta_\xi$. This is proved in Appendix C.

Next we investigate how the toric type varies over a basic toric stack.

PROPOSITION 6.5. *Let S be a scheme and let X be a basic toric stack with homogeneous coordinate ring $(\mathcal{O}_S[x_1, \dots, x_r], A, \mathbf{a})$. Assume that $1 \leq s \leq r$, and let $\mathbf{E} = (E^1, \dots, E^s)$ with $E^i = V(x_i)$. We consider the standard pair $(X, \mathbf{E})/S$.*

Let ξ be an arbitrary point in X . Then we have a surjection $\varphi: A \rightarrow A(\xi)$ to the character group of the stabiliser at a geometric point representing ξ . The kernel of φ is generated by the elements a_i such that $\xi \notin V(x_i)$. The toric type at ξ is given by

$$(A(\xi), (v_1, \dots, v_r)), \quad v_i = (\varphi(a_i), e_i),$$

where $e_i = E^i$ if $i \leq s$ and the divisor E^i passes through ξ and $*$ otherwise.

Proof. Since the field \bar{k} is algebraically closed, the map ξ factors through $\text{Spec}_{\mathcal{O}_S} \mathcal{O}_S[x_1, \dots, x_r]$. Let α_i be the image of x_i through the corresponding map

$$\Gamma(\mathcal{O}_S[x_1, \dots, x_r]) \rightarrow \bar{k}.$$

Then we have $\alpha_i = 0$ precisely when ξ passes through $V(x_i)$.

Consider the atlas $\tilde{X}_{\bar{k}} = \text{Spec } \bar{k}[x_1, \dots, x_r]$ of $X_{\bar{k}}$. The closed immersion $B\Delta_\xi \hookrightarrow X_{\bar{k}}$ corresponds to the slice of the action groupoid at the closed subscheme $V(I) \subset \tilde{X}_{\bar{k}}$ defined by the ideal $I = (x_1 - \alpha_1, \dots, x_r - \alpha_r)$. In other words, we have the cartesian diagram

$$\begin{array}{ccc} \Delta_\xi & \xrightarrow{g'} & \tilde{X}_{\bar{k}} \times_{\bar{k}} \Delta \\ \downarrow & & \downarrow f \\ \text{Spec } \bar{k} & \xrightarrow{g} & \tilde{X}_{\bar{k}} \times_{\bar{k}} \tilde{X}_{\bar{k}} \end{array}$$

The map g corresponds to the \bar{k} -algebra map $\bar{k}[x_1, \dots, x_r, y_1, \dots, y_r] \rightarrow \bar{k}$ taking x_i and y_i to α_i , and the map f corresponds to the \bar{k} -algebra map

$$\bar{k}[x_1, \dots, x_r, y_1, \dots, y_r] \rightarrow \bar{k}[x_1, \dots, x_r][A]$$

taking x_i to x_i and y_i to $a_i x_i$. It follows that the map g' corresponds to

$$\bar{k}[x_1, \dots, x_r][A] \rightarrow \bar{k}[A]/(\alpha_i - a_i \alpha_i)$$

taking x_i to α_i , where i ranges from 1 to r . Since the relation $\alpha_i = a_i \alpha_i$ is trivial if $\alpha_i = 0$ and equivalent to $a_i = 1$ otherwise, the right-hand side is the group algebra $\bar{k}[A(\xi)]$ in the statement of the proposition. The conormal representation is the \bar{k} -vector space I/I^2 , which has the elements $u_i = (x_i - \alpha_i) + I^2$ for $i \in \{1, \dots, r\}$ as basis. Since u_i has weight $\varphi(a_i)$ and corresponds to the divisor $V(x_i)$ precisely when $\alpha_i = 0$, the result follows. \square

Example 6.6. Let k be a field and (X, \mathbf{E}) the standard pair with X being the basic toric stack with homogeneous coordinates $(k[x_1, x_2, x_3], C_2 \times C_2, ((1, 0), (0, 1), (1, 1)))$ and \mathbf{E} being the sequence (E^1, E^2) with $E^i = V(x_i)$ for $i = 1, 2$. Table 6.1 shows how the toric type varies over the strata defined by the toric divisors. Note that the toric types in the rows 1 through 7 are distinct whereas the toric type in row 8 is identical to the toric type in row 7.

TABLE 6.1. The table shows how the conormal representation varies over the strata defined by the toric divisors of the basic toric stack in Example 6.6. The stratum $\bigcap_{s_i=1} V(x_i) \setminus \bigcup_{s_i=0} V(x_i)$ corresponds to the row labelled by the binary word $s_1s_2s_3$.

	$s_1s_2s_3$	A	a_1	a_2	a_3	e_1	e_2	e_3
1.	111	$C_2 \times C_2$	(1, 0)	(0, 1)	(1, 1)	E^1	E^2	*
2.	110	C_2	1	1	0	E^1	E^2	*
3.	101	C_2	1	0	1	E^1	*	*
4.	100	0	0	0	0	E^1	*	*
5.	011	C_2	0	1	1	*	E^2	*
6.	010	0	0	0	0	*	E^2	*
7.	001	0	0	0	0	*	*	*
8.	000	0	0	0	0	*	*	*

Motivated by Proposition 6.5, we introduce a partial ordering on the set $U(\mathbf{E})$ for each finite set \mathbf{E} . Let X be a basic toric stack and let \mathbf{E} be a sequence of toric divisors on X . By design, the ordering on $U(\mathbf{E})$ makes the function taking a point in X to the toric type at that point upper semi-continuous.

DEFINITION 6.7. Let $\alpha \in U(\mathbf{E})$ be a toric type represented by the triple (A, \mathbf{v}) with $\mathbf{v} = (v_1, \dots, v_r)$ and $v_i = (a_i, e_i)$. Given $\alpha' \in U(\mathbf{E})$, we write $\alpha \geq \alpha'$ if there exists a subset $J \subset \{1, \dots, r\}$ such that α' is represented by $(A', (v'_1, \dots, v'_r))$ with $A' = A/\langle a_i \rangle_{i \in J}$ and

$$v'_i = \begin{cases} (\bar{a}_i, e_i) & \text{if } i \notin J, \\ (0, *) & \text{if } i \in J, \end{cases}$$

where \bar{a}_i denotes the image of a_i through the canonical morphism $A \rightarrow A'$.

We are now ready define what we mean by a *conormal invariant*.

DEFINITION 6.8. A *conormal invariant* ι is a rule which to each finite totally ordered set \mathbf{E} associates an order-preserving function $\iota_{\mathbf{E}}: U(\mathbf{E}) \rightarrow W(\mathbf{E})$ to a partially ordered set $W(\mathbf{E})$ and, to each injective order-preserving function $f: \mathbf{E} \rightarrow \mathbf{E}'$ associates an order-preserving function $W(f): W(\mathbf{E}) \rightarrow W(\mathbf{E}')$ such that the diagram

$$\begin{array}{ccc} U(\mathbf{E}) & \xrightarrow{\iota_{\mathbf{E}}} & W(\mathbf{E}) \\ U(f) \downarrow & & \downarrow W(f) \\ U(\mathbf{E}') & \xrightarrow{\iota_{\mathbf{E}'}} & W(\mathbf{E}') \end{array}$$

commutes.

For each standard pair (X, \mathbf{E}) , we get a function $\iota_{(X, \mathbf{E})}: |X| \rightarrow W(\mathbf{E})$ taking a point $\xi \in X$ to $\iota_{\mathbf{E}}(\alpha)$, where α is the toric type at ξ . We call the function $\iota_{\mathbf{E}}(\alpha)$ the *realisation* of the conormal invariant for the standard pair (X, \mathbf{E}) .

Remark 6.9. Note that a conormal invariant is simply a natural transformation $U \rightarrow W$ between functors from the category of finite totally ordered sets and injective order-preserving functions to the category of partially ordered sets.

For most of the conormal invariants actually used by the destackification algorithm, the partially ordered set $W(\mathbf{E})$ will simply be the set \mathbb{N} of natural numbers with its standard ordering.

Example 6.10. Let $(X, \mathbf{E})/S$ be a standard pair and let $\xi \in X$ be a point with toric type (A, \mathbf{v}) . The following example illustrates that several well-known invariants can be thought of as realisations of conormal invariants $U \rightarrow \mathbb{N}$, where we regard \mathbb{N} as the constant functor.

- (i) The *relative dimension* of X over S at ξ is the length of the sequence \mathbf{v} .
- (ii) The *order* of the stabiliser at ξ is the order of A .
- (iii) The *multiplicity* at ξ is the order of A_{can} , with A_{can} defined by the exact sequence in Remark 3.15.

Example 6.11. For the conormal invariants given in Example 6.10, the ordered set \mathbf{E} played no part. Here we give some examples where \mathbf{E} does play a part.

- (i) The identity natural transformation $U \rightarrow U$ can be regarded as the *universal conormal invariant*. Its realisation for the standard pair (X, \mathbf{E}) evaluated at a point ξ is simply the toric type at ξ .
- (ii) Let $2^{\mathbf{E}}$ denote the power set of \mathbf{E} endowed with the lexicographical ordering induced by the ordering on \mathbf{E} . Consider the conormal representation given by the natural transformation $U(\mathbf{E}) \rightarrow 2^{\mathbf{E}}$ which takes (A, \mathbf{v}) to the set of elements in \mathbf{E} marking some element in \mathbf{v} . Its realisation for the standard pair (X, \mathbf{E}) evaluated at a point ξ is the set of components of \mathbf{E} passing through ξ .

Example 6.12. Let ι_1, \dots, ι_n be conormal invariants taking values in the totally ordered sets W_1, \dots, W_n . Then we can form its *lexicographical composition* $\iota_1 \times \dots \times \iota_n$ taking values in the product set $W_1 \times \dots \times W_n$ endowed with the lexicographical ordering.

Next we investigate the functorial properties of conormal invariants. Let $(X, \mathbf{E})/S$ be a standard pair and $\iota: U \rightarrow W$ a conormal invariant. As described in §2.5, we get a new standard pair $(X_{S'}, \mathbf{E}_{S'})/S'$ given any base change $S' \rightarrow S$. Note that we have a naturally defined one-to-one correspondence between the ordered sets \mathbf{E} and the base change $\mathbf{E}_{S'}$ induced by pull-back. Hence, we get a diagram

$$\begin{array}{ccc}
 |X_{S'}| & \xrightarrow{\iota_{(X_{S'}, \mathbf{E}_{S'})}} & W(\mathbf{E}_{S'}) \\
 \downarrow & & \downarrow \\
 |X| & \xrightarrow{\iota_{(X, \mathbf{E})}} & W(\mathbf{E})
 \end{array} \tag{6.2}$$

Similarly, given a *smooth* morphism $f: X' \rightarrow X$, we get a standard pair $(X', \mathbf{E}')/S$ and a naturally defined order-preserving function $\mathbf{E}' \rightarrow \mathbf{E}$. This function is injective, but not necessarily surjective since we allow ourselves to omit empty divisors from \mathbf{E}' . This gives us a diagram

$$\begin{array}{ccc}
 |X'| & \xrightarrow{\iota_{(X', \mathbf{E}')}} & W(\mathbf{E}') \\
 \downarrow & & \downarrow \\
 |X| & \xrightarrow{\iota_{(X, \mathbf{E})}} & W(\mathbf{E})
 \end{array} \tag{6.3}$$

We want to establish conditions when the two diagrams above commute.

DEFINITION 6.13. We define the following two properties which a conormal invariant $\iota: U \rightarrow W$ may have.

(P1) For any finite ordered set \mathbf{E} and any $(A, \mathbf{v}) \in U(\mathbf{E})$, we have $\iota(A, \mathbf{v}) = \iota(A, \mathbf{v}')$, where \mathbf{v}' is the subsequence of \mathbf{v} consisting of the elements distinct from the trivial element $(0, *) \in A \times \mathbf{E}_+$.

(P2) For any finite ordered set \mathbf{E} and any $(A, \mathbf{v}) \in U(\mathbf{E})$, we have $\iota(A, \mathbf{v}) = \iota(A_{\text{rig}}, \mathbf{v})$, where A_{rig} is the subgroup of A generated by the weights of \mathbf{v} . That is, the subgroup $A_{\text{rig}} = \langle a_1, \dots, a_r \rangle$ if $\mathbf{v} = ((a_1, e_1), \dots, (a_r, e_r))$.

PROPOSITION 6.14. Let $\iota: U \rightarrow W$ be a conormal invariant and let $(X, \mathbf{E})/S$ be a standard pair with diagonalisable stabilisers. Consider the 2-commutative diagrams

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ & \searrow & \downarrow \\ & & S \end{array} \quad \begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

where f is smooth and the square is 2-cartesian. Let \mathbf{F} be the pull-back of \mathbf{E} along f and let \mathbf{E}' be the pull-back of \mathbf{E} along g . Then $\iota_{(X', \mathbf{E}')/S'} = \iota_{(X, \mathbf{E})/S} \circ |g|$. Furthermore, we have the equality $\iota_{(Y, \mathbf{F})/S} = \iota_{(X, \mathbf{E})/S} \circ |f|$ under any of the following circumstances.

- (i) The morphism f is étale and stabiliser-preserving.
- (ii) The morphism f is smooth and stabiliser-preserving and ι satisfies property (P1).
- (iii) The morphism f is a gerbe and ι satisfies property (P2).

Proof. Let $\xi': \text{Spec } \bar{k} \rightarrow X'$ be a geometric point. Since we have a canonical isomorphism between $X' \times_S \text{Spec } \bar{k}$ and $X \times_S \text{Spec } \bar{k}$, functoriality with respect to base change follows immediately.

We explore the other functoriality properties by examining the first diagram. Let $\xi: \text{Spec } \bar{k} \rightarrow Y$ be a geometric point. By the previous paragraph, we may, without loss of generality, assume that $S = \text{Spec } \bar{k}$. Denote the toric type at ξ , as defined in Definition 6.3, by

$$(A, (v_1, \dots, v_r)), \quad v_i = (a_i, e_i).$$

We may assume that v_1, \dots, v_s are marked by the components E^1, \dots, E^s of \mathbf{E} and that v_i is unmarked for $i > s$. Similarly, we denote the toric type at $f \circ \xi$ by

$$(A', (v'_1, \dots, v'_{r'})), \quad v'_i = (a'_i, e'_i)$$

and assume that v'_1, \dots, v'_s are marked by the components F^1, \dots, F^s , where F^i denotes the pull-back of E^i . The elements v'_i for $i > s$ are unmarked. We have 2-commutative diagrams

$$\begin{array}{ccc} B\Delta_\xi & \xrightarrow{b} & Y \\ a \downarrow & & \downarrow f \\ B\Delta_{f \circ \xi} & \longrightarrow & X \end{array} \quad \begin{array}{ccc} B\Delta_\xi & \xrightarrow{g_i} & F^i \longrightarrow Y \\ a \downarrow & & \downarrow f_i \quad \downarrow f \\ B\Delta_{f \circ \xi} & \xrightarrow{h_i} & E^i \longrightarrow X \end{array} \tag{6.4}$$

where the rightmost square is 2-cartesian and i ranges from 1 to s .

First assume that the morphism f is stabiliser-preserving. Then we can identify Δ_ξ with $\Delta_{f \circ \xi}$ and assume that a is the identity map. In particular, the character groups A and A' are equal. Since conormal bundles commute with flat base change, we get

$$h_i^* \mathcal{N}_{E^i/X} \cong g_i^* f_i^* \mathcal{N}_{E^i/X} \cong g_i^* \mathcal{N}_{F^i/Y}$$

from the right-hand diagram. It follows that $a_i = a'_i$ for $1 \leq i \leq s$.

We introduce the shorthand notation \tilde{Y} for the fibre product $B\Delta_\xi \times_X Y$ and denote the canonical projections to $B\Delta_\xi$ and Y by p and q , respectively. Note that p is smooth and q is a regular immersion. Consider the sequence $B\Delta_\xi \xrightarrow{\sigma} \tilde{Y} \xrightarrow{q} Y$ of morphisms, where σ is the canonical morphism induced by b . Since σ is a regular immersion, being a section of p , we get an exact sequence

$$0 \rightarrow \sigma^* \mathcal{N}_{\tilde{Y}/Y} \rightarrow \mathcal{N}_{B\Delta_\xi/Y} \rightarrow \mathcal{N}_{B\Delta_\xi/\tilde{Y}} \rightarrow 0 \tag{6.5}$$

by [EGAIV, Proposition 16.9.3]. The bundle $\mathcal{N}_{\tilde{Y}/Y}$ is the pull-back of $\mathcal{N}_{B\Delta_\xi/X}$ along the projection p . Since σ is a section of p , this allows us identify the first term in (6.5) with $\mathcal{N}_{B\Delta_\xi/X}$. Consider the 2-commutative square

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{p} & B\Delta_\xi \\ \downarrow & & \downarrow \pi \\ \tilde{Y}_{cs} & \longrightarrow & S \end{array}$$

induced by passing to coarse spaces. Since p is stabiliser-preserving and π is a gerbe, this square is 2-cartesian. It follows that the conormal bundle $\mathcal{N}_{B\Delta_\xi/\tilde{Y}}$ associated to the section σ is isomorphic to the pull-back $\pi^* \mathcal{N}_{S/\tilde{Y}_{cs}}$ of the conormal bundle $\mathcal{N}_{S/\tilde{Y}_{cs}}$ associated to σ_{cs} . Hence, the sequence (6.5) can be rewritten as

$$0 \rightarrow \mathcal{N}_{B\Delta_\xi/X} \rightarrow \mathcal{N}_{B\Delta_\xi/Y} \rightarrow \pi^* \mathcal{N}_{S/\tilde{Y}_{cs}} \rightarrow 0.$$

Since the bundle $\pi^* \mathcal{N}_{S/\tilde{Y}_{cs}}$ is defined over the base field S , it must correspond to the trivial character. It follows that we can choose the indexing of the toric types such that $a_i = a'_i$ whenever $1 \leq i \leq r$ and $a'_i = 0$ whenever $i > r$. In particular, this implies (ii). Furthermore, the difference $r' - r$ is easily seen to be equal to the relative dimension of f , which implies (i).

Next we instead assume that f is a gerbe. In this case we have $r' = r$ since the relative dimensions of X and Y over S are equal. Furthermore, all squares in the diagram (6.4) are 2-cartesian. We have a surjective group homomorphism $\Delta_\xi \rightarrow \Delta_{f \circ \xi}$, which allows us to identify A with a subgroup of A' . Since conormal bundles commute with flat base change, we get

$$\mathcal{N}_{B\Delta_\xi/Y} = a^* \mathcal{N}_{B\Delta_{f \circ \xi}/X}.$$

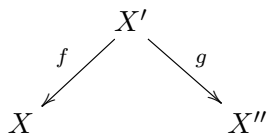
The pull-back functor $a^*: \text{Coh}(B\Delta_{f \circ \xi}) \rightarrow \text{Coh}(B\Delta_\xi)$ corresponds to the obvious functor from the category of A -graded vector spaces to the category of A' -graded vector spaces, which preserves the grading. This implies that $a'_i = a_i$ for $1 \leq i \leq r$ for a suitable indexing of the toric types. A similar argument applied to the right-hand diagram in (6.4) shows that we also have $e'_i = e_i$, which proves (iii). □

As a direct consequence of Proposition 6.14, it follows that realisations of conormal invariants are upper semi-continuous.

PROPOSITION 6.15. *Let $\iota: U \rightarrow W$ be a conormal invariant and let (X, \mathbf{E}) be a standard pair with diagonalisable stabilisers. Then the realisation $\iota_{(X, \mathbf{E})/S}$ is an upper semi-continuous function. In particular, the locus where $\iota_{(X, \mathbf{E})/S}$ obtains a maximum is a closed subset of $|X|$.*

Proof. Since the property of being semi-continuous is preserved under post composition by order-preserving functions, it is enough to verify semi-continuity for the toric type. Furthermore, it

is enough to verify that the locus $Z \subset X$ where the toric type is maximal is closed. If X is a basic toric stack and \mathbf{E} a subset of the toric divisors, this is clear from the computation of toric types for basic toric stacks given in Proposition 6.5 and the construction of the order relation in Definition 6.7. In general, we may choose a toric chart



as in Definition 5.1, at any point $\xi \in X$ by Proposition 5.3. Since the property of Z being closed can be verified locally on X , we may assume that f is surjective. Let Z' and Z'' denote the loci where the toric type is maximal in X' and X'' , respectively. Since both f and g are étale and stabiliser-preserving, we have $Z' = f^{-1}(Z) = g^{-1}(Z'')$ by Proposition 6.14(i). Since X'' is a basic toric stack, the locus Z'' is closed. Hence, Z' is closed by continuity of g . Since f is submersive, it follows that also Z is closed, which concludes the proof. \square

In the destackification algorithm we need to blow up X in a locus which is maximal with respect to some conormal invariant. Since we only want blow-ups with smooth centres, we need a criterion to ensure that the maximal locus has a structure of a smooth substack. If the base S is reduced, it is obvious that there can be at most one such structure, but in the general case this is not so clear. Fortunately, there exists a simple combinatorial condition, which is easy to verify in practice, which ensures both of these properties.

DEFINITION 6.16. Let $\iota: U \rightarrow W$ be a conormal invariant. We say that ι is *smooth* provided that for each finite ordered set \mathbf{E} and each triple $\alpha, \alpha', \alpha'' \in U(\mathbf{E})$ such that $\alpha \geq \alpha'$ and $\alpha \geq \alpha''$, the condition $\iota(\alpha) = \iota(\alpha') = \iota(\alpha'')$ implies that $\iota(\alpha) = \iota(\alpha' \wedge \alpha'')$.

This definition has a rather combinatorial flavour. To get a feeling for what it means geometrically, it is useful to note that the order relation \geq on the set $U(\mathbf{E})$ corresponds to *generalisation* in the following sense: if (X, \mathbf{E}) is a standard pair and $Z_\alpha \subseteq X$ denotes the locus with toric type $\alpha \in U(\mathbf{E})$, then $\alpha' \leq \alpha$ implies that Z_α lies in the closure of $Z'_{\alpha'}$. Thus, Definition 6.16 has the following interpretation: let (X, \mathbf{E}) be a standard pair and consider a smooth conormal invariant ι . Assume that $\xi \in X$ is a point where (the realisation of) ι assumes its maximum. If the invariant ι remains constant when we generalise in two different directions separately, then it must remain constant if we generalise in both these directions at the same time.

Example 6.17. All the conormal invariants given in Examples 6.10 and 6.11 are smooth, but since we are not going to use this fact, its proof is left to the reader.

Example 6.18. To understand the condition in Definition 6.16, it is useful to study a situation where it is not satisfied. Let τ be the conormal invariant which takes a toric type represented by (A, \mathbf{v}) to the number of elements $v_i = (a_i, e_i)$ in the sequence \mathbf{v} satisfying $a_i \neq 0$ and $e_i = *$. Consider the case where $\mathbf{E} = (E^1, E^2)$ and the toric type α is represented by (A, \mathbf{v}) , where $A = C_2 \times C_2$ and \mathbf{v} has three elements $v_i = (a_i, e_i)$ with

$$a_1 = (1, 0), \quad a_2 = (0, 1), \quad a_3 = (1, 1), \quad e_1 = E^1, \quad e_2 = E^2, \quad e_3 = *.$$

Then $\tau(\alpha) = 1$. The basic toric stack over the field k associated to this toric type has homogeneous coordinate ring $(k[x_1, x_2, x_3], A, (a_1, a_2, a_3))$ with $E^1 = V(x_1)$ and $E^2 = V(x_2)$. One verifies that the invariant τ assumes the value 1 at the locus $(V(x_1) \cup V(x_2)) \cap V(x_3)$ and 0 outside this locus. In other words, if we start in the origin, the invariant τ remains constant if we generalise in the x_1 or the x_2 directions separately, but not if we generalise in both these directions at the same time. In particular, the locus where τ assumes its maximum is not smooth.

As a side remark, the conormal invariant τ actually plays a role in the destackification algorithm and will be introduced as the *toroidal index* in Definition 7.4 in the next section.

PROPOSITION 6.19. *Let $(X, \mathbf{E})/S$ be a standard pair with diagonalisable stabilisers and let $\iota: U \rightarrow W$ be a smooth conormal invariant. Let m be a maximal value for $\iota_{(X, \mathbf{E})/S}$. Then the locus where $\iota_{(X, \mathbf{E})/S}$ obtains m has a unique structure of a smooth substack of X having normal crossings with \mathbf{E} .*

Proof. The question is local on the base, so we may assume that $S = \text{Spec } R$ is affine. By a standard limit argument, we may also assume that R is noetherian.

We start by investigating the situation locally. Let X be a basic toric stack with homogeneous coordinates

$$(R[x_1, \dots, x_r], A, (a_1, \dots, a_r))$$

and let $\mathbf{E} = (E^1, \dots, E^s)$, with $E^i = V(x_i)$ for some $s \leq r$.

Now let $\alpha = (A, \mathbf{v})$ be the toric type at the origin, and let $\beta = (A', \mathbf{v}')$ be the greatest lower bound of the set $\{\alpha' \mid \alpha \geq \alpha', \iota(\alpha) = \iota(\alpha')\}$. Such an element exists since the set is finite, and it is contained in the set by the smoothness hypothesis for ι . Let $J \subseteq \{1, \dots, r\}$ be the maximal subset defining the relation $\beta \leq \alpha$ as in Definition 6.7, and let K be the kernel of the canonical group homomorphism $A \rightarrow A'$. Furthermore, define $O = \{1, \dots, s\} \setminus J$ and $P = \{s+1, \dots, r\} \setminus J$. From the explicit description of the toric type for a basic toric stack given in Proposition 6.5, it is easy to see that the maximal locus for $\iota_{(X, \mathbf{E})/S}$ is $Z = V(x_i \mid i \in O \cup P)$. In particular, the substack Z is smooth. The locus $F = V(x_i \mid i \in O)$ is the intersection of the divisors E^i containing Z .

If S is reduced, the substack $Z \subset X$ is clearly the only substack structure on the underlying space $|Z|$ of the required form. If S is non-reduced, we can, by the noetherian hypothesis, factor the map $S_{\text{red}} \hookrightarrow S$ into a finite sequence of nilpotent thickenings defined by square zero ideals. It is enough to show that the substack structure of Z_{red} lifts uniquely at each step. This reduces the situation to the following deformation problem:

$$\begin{array}{ccccccc}
 Z & \cdots & \longrightarrow & F^{\subset} & \longrightarrow & X & \longrightarrow & S \\
 \uparrow & & & \uparrow & & \uparrow & & \uparrow \\
 Z_0 & \subset & \longrightarrow & F_0 & \subset & \longrightarrow & X_0 & \longrightarrow & S_0
 \end{array}$$

where the map $S_0 \rightarrow S$ is a nilpotent thickening defined by a square zero ideal N . We want to show that the stack Z , together with the dashed arrows, is essentially the only stack fitting into the diagram, in a way such that the leftmost square becomes cartesian and the stack becomes smooth over S . Note that, since we require Z to have normal crossings with \mathbf{E} , we deform Z inside F and not inside X . Let I be the ideal $\langle x_i \mid i \in P \rangle$ in the homogeneous coordinate ring of F_0 and let \mathcal{I} be the corresponding ideal in \mathcal{O}_{F_0} . Let \mathcal{M} be the sheaf of \mathcal{O}_{Z_0} -modules

$\text{Hom}_{\mathcal{O}_{Z_0}}(\mathcal{I}/\mathcal{I}^2, N \otimes \mathcal{O}_{Z_0})$. Then the set of objects completing the diagram is a torsor under the group $H^0(Z_0, \mathcal{M})$. The sheaf \mathcal{M} corresponds to the graded R' -module

$$M^\bullet = \text{Hom}_{R'}^\bullet(I/I^2, N \otimes R'),$$

where R' is the A -graded ring $R[x_1, \dots, x_s]/\langle x_i \mid i \in O \cup P \rangle$. The global sections functor factors through the pushforward functor π_* , where $\pi: Z_0 \rightarrow (Z_0)_{\text{cs}}$ is the map to the coarse space, and $\pi_*\mathcal{M}$ is simply the degree-zero part of M^\bullet , viewed as an $(R')^0$ -module. But the homogeneous elements of R' have degrees in K , whereas I is generated by homogeneous elements with degrees not in K . It follows that the degree-zero part of M^\bullet is the zero module, which shows that the lift of Z_0 is unique.

Now let $X' \rightarrow X$ be an étale stabiliser-preserving map. Denote the pull-backs of Z_0 and Z by Z'_0 and Z' , respectively. The natural map $Z'_0 \rightarrow Z_0$ is also étale and stabiliser-preserving. By flatness, the sheaf \mathcal{M} pulls back to the sheaf $\mathcal{M}' = \text{Hom}_{\mathcal{O}_{Z'_0}}(\mathcal{I}'/(\mathcal{I}')^2, N \otimes \mathcal{O}_{Z'_0})$, where \mathcal{I}' is the ideal sheaf defining Z'_0 in X_0 . The square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ (X')_{\text{cs}} & \longrightarrow & X_{\text{cs}} \end{array}$$

formed by the maps to the coarse spaces is cartesian and the horizontal maps are étale. Hence, also $H^0(Z'_0, \mathcal{M}') = 0$, and we get unicity for the closed substack $Z' \subset X'$. A general stack (X'', E'') satisfying the standard hypothesis can be covered by stacks as X' above. The unicity of $Z' \subset X'$ asserts that the stack structure descends to a closed substack Z'' of X'' , as desired. \square

7. Outline of the algorithms

In this section, we outline the destackification algorithms. We also introduce the various conormal invariants used by the algorithms and describe how they are used. All invariants we define, except the *divisorial type*, take values among the natural numbers.

DEFINITION 7.1 (Independency index). Let $\alpha = (A, \mathbf{v})$ be a toric type with $\mathbf{v} = (v_1, \dots, v_r)$ and $v_j = (a_j, e_j)$. An element v_j of the sequence \mathbf{v} is called *independent* if the intersection

$$\langle a_j \rangle \cap \langle a_1, \dots, \widehat{a}_j, \dots, a_r \rangle \tag{7.1}$$

is the trivial subgroup. The *independency index* of α is the number of elements in the sequence \mathbf{v} which are *not* independent.

The following definition is a direct generalisation of the concept of independency for toric divisors on toric stacks (see Definition 3.17).

DEFINITION 7.2. Let $(X, \mathbf{E})/S$ be a standard pair with diagonalisable stabilisers. Let ξ be a point in X and E a component of \mathbf{E} passing through ξ . Assume that the toric type at ξ is (A, \mathbf{v}) . Then E is said to be *independent at ξ* provided that the element in \mathbf{v} marked by E is independent in the sense of Definition 7.1. The component E is said to be *independent* if it is independent at each of its points.

TABLE 7.1. The table shows some conormal invariants computed for various toric types. The columns marked by ι , τ and δ contain the *independency index*, the *toroidal index* and the *divisorial index*, respectively. The column marked by δ_i , for $i \in \{1, 2, 3, 4\}$, contains the E^i -*divisorial index*.

	A	a_1	a_2	a_3	a_4	e_1	e_2	e_3	e_4	ι	τ	δ	δ_1	δ_2	δ_3	δ_4
1.	$C_3 \times C_3$	(1, 0)	(0, 1)	(0, 0)	(0, 0)	E^1	E^2	*	*	0	0	0	0	0	-	-
2.	$C_3 \times C_3$	(1, 0)	(0, 1)	(0, 0)	(0, 0)	*	*	*	*	0	2	2	-	-	-	-
3.	$C_3 \times C_3$	(1, 0)	(0, 1)	(0, 2)	(0, 0)	E^1	E^2	*	*	2	1	0	0	2	-	-
4.	$C_6 \times C_2$	(1, 0)	(5, 1)	(3, 1)		E^1	E^2	*		3	1	0	2	1	-	-
5.	$C_6 \times C_6$	(1, 0)	(5, 3)	(0, 2)		E^1	E^2	E^3		2	0	0	0	0	0	-
6.	$C_6 \times C_2$	(1, 0)	(1, 1)	(0, 0)		E^1	E^2	*		2	0	0	0	0	0	-
7.	$C_3 \times C_3$	(1, 0)	(0, 1)	(1, 1)	(2, 0)	E^1	E^2	*	*	4	2	0	3	1	-	-

The independency index measures how far the coarse space X_{cs} is from being smooth. In particular, the invariant vanishes at a point $\xi \in |X|$ precisely when X_{cs} is smooth at the corresponding point. This easily follows from Propositions 3.18 and 3.19 by using local homogeneous coordinates as described in §5. Thus, one of the main objectives of the destackification process is to make this invariant vanish.

Example 7.3. Table 7.1 gives several examples where the various conormal invariants introduced in this section are computed. An example where the independency index does not vanish is given in row 3. Here v_1 and v_4 are independent whereas v_2 and v_3 are not. Hence, the independency index is 2.

Although the independency index is a smooth conormal invariant, in the sense of Definition 6.16, it is not fruitful to just repeatedly blow up the locus where the invariant assumes its maximum, as was demonstrated in Example 4.3. Instead we would like to use the combinatorial approach described in Algorithm A. This requires that we have enough globally defined divisors to work with. We introduce a conormal invariant that quantifies this.

DEFINITION 7.4 (Toroidal index). Let $\alpha = (A, \mathbf{v})$ be a toric type with $\mathbf{v} = (v_1, \dots, v_r)$ and $v_i = (a_i, e_i)$. The *toroidal index* of α is the number of elements in the sequence \mathbf{v} which are unmarked and have non-trivial weight. That is, the number of elements v_i of the form $v_i = (a_i, *)$ with $a_i \neq 0$. If the toroidal index is zero at a point $\xi \in X$, we say that $(X, \mathbf{E})/S$ is *toroidal at ξ* . The standard pair $(X, \mathbf{E})/S$ is *toroidal* if it is toroidal at each of its points.

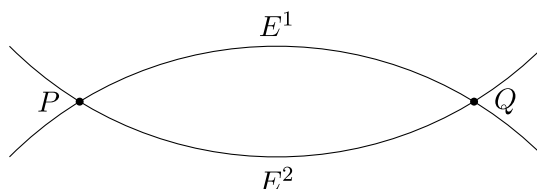
Remark 7.5. This definition of *toroidal stack* is closely related to the classical definition of toroidal variety given in [KKMS73]. By using local homogeneous coordinates, it is easy to see that a stack is toroidal at a point ξ precisely when it has a toric chart at ξ , which is compatible with \mathbf{E} in the sense of Definition 5.1, such that $\mathbf{E}(\xi)$ is in one-to-one correspondence with the toric divisors of the chart. If S is the spectrum of a field, this implies that the pair (X_{cs}, U_{cs}) is a toroidal variety in the sense of [KKMS73], where U_{cs} is the coarse space of the complement of the support of \mathbf{E} . More generally, if S is a scheme, then $(X_{cs}, U_{cs})/S$ is a flat family of toroidal varieties. It should be noted that since we assume that our toric stacks are simplicial, not every toroidal variety can be constructed in this way.

Let (X, \mathbf{E}) be a standard pair. The following example illustrates that in order to obtain a destackification in the sense of Definition 2.3, it is necessary to bring both the independency index and the toroidal index to zero. That this is also sufficient, at least if the components of \mathbf{E} are connected, will be proved in Proposition 8.7.

Example 7.6. Let C be a nodal cubic in the projective plane \mathbb{P}_k^2 , where k is a field, and denote the node by $P \in \mathbb{P}_k^2$. By using the construction from [MO05, Theorem 4.1], which is a generalisation of the root construction, it is possible to construct a stack X with coarse space \mathbb{P}^2 such that the stabiliser over the smooth locus of C is μ_3 . Although the coarse space of X is smooth, the standard pair (X, \emptyset) is not toroidal. The toric type at the point lying over the node of C will be as in row 2 of Table 7.1. Note that although X is a root stack over its coarse space over an étale neighbourhood of P , it is not possible to describe X as a root stack of X_{cs} in any simple normal crossings divisor.

Since the toric destackification process is essentially a global approach, some care must be taken when destackifying stacks which are not toric, but only toroidal. This is illustrated by the following example.

Example 7.7. Consider a two-dimensional toroidal stack (X, \mathbf{E}) , where \mathbf{E} has two components E^1 and E^2 that intersect at two points P and Q .



Assume that the independency index is 2 at P and Q . Clearly, we must blow up both P and Q during the destackification process, but not necessarily at the same time. Locally, at each of the points P and Q , the stack X is isomorphic to toric stacks, but these stacks need not be isomorphic to each other. Thus, it might be necessary to apply different combinatorial recipes to destackify the points. Even if they are isomorphic, the components E^1 and E^2 may play different roles, so the order of the components are important.

The example shows that we need an invariant which captures the combinatorial recipe for destackification. In principle, we use the stacky cone describing the toric stack to which X is locally isomorphic at the point in question. We shall, however, use a more algebraic description.

DEFINITION 7.8 (Divisorial type). Let \mathbf{E} be a finite set. The set $T(\mathbf{E})$ of *divisorial types* parametrised by \mathbf{E} is the subset of the set $U(\mathbf{E})$ of toric types consisting of elements represented by pairs (A, \mathbf{v}) such that:

- (i) all elements in \mathbf{v} are marked by some divisor in \mathbf{E} ;
- (ii) no element in \mathbf{v} is independent in the sense of Definition 7.1;
- (iii) the group A is generated by the weights of the elements of \mathbf{v} .

Given a toric type (A, \mathbf{v}) , its associated *divisorial type* is the pair (A', \mathbf{v}') , where \mathbf{v}' is the subsequence of \mathbf{v} of elements which are marked and not independent and A' is the subgroup generated by the weights of \mathbf{v}' .

Example 7.9. The divisorial type associated to the toric type in row 4 in Table 7.1 is $(C_6 \times C_2, (a_1, E_1), (a_2, E_2))$, with $a_1 = (1, 0)$ and $a_2 = (5, 1)$. Note that this is the same as the divisorial type for the toric type in row 5. Indeed, since v_3 is independent, it is omitted from the divisorial type and the subgroup of $C_6 \times C_6$ generated by $(1, 0)$ and $(5, 3)$ is isomorphic to $C_6 \times C_2$ with $(1, 0)$ and $(5, 3)$ in $C_6 \times C_6$ corresponding to $(1, 0)$ and $(5, 1)$ in $C_6 \times C_2$ under this isomorphism. The divisorial type for row 6 is given by $(C_6 \times C_2, (a_1, E_1), (a_2, E_2))$, with $a_1 = (1, 0)$ and $a_2 = (1, 1)$. Although the set of divisors and the isomorphism class of the group are the same as for row 4, this divisorial type is different since there is no automorphism of $C_6 \times C_2$ taking the pair $(1, 0), (5, 1)$ to the pair $(1, 0), (1, 1)$.

The set $T(\mathbf{E})$ of divisorial types parametrised by \mathbf{E} inherits a partial ordering from the partial ordering on $U(\mathbf{E})$. We refine this partial ordering to a total ordering. First we give an alternative description of the set $T(\mathbf{E})$. Let (A, \mathbf{v}) be a divisorial type with $\mathbf{v} = v_1, \dots, v_r$ and $v_i = (a_i, e_i)$. By Definition 7.8(i), each element e_i is an element E^i of \mathbf{E} and we may assume that indexing is chosen such that e_1, \dots, e_r is decreasing. The sequence a_1, \dots, a_r of weights defines a group homomorphism $\varphi: \mathbb{Z}^r \rightarrow A$. By Definition 7.8(ii), this is surjective and therefore fits into an exact sequence

$$0 \rightarrow K \xrightarrow{f} \mathbb{Z}^r \xrightarrow{\varphi} A \rightarrow 0 \tag{7.2}$$

with K being free of rank r since A is finite. Note that the divisorial type (A, \mathbf{v}) is uniquely determined by the subsequence (E^1, \dots, E^s) of \mathbf{E} together with f up to composition by an automorphism of K . By choosing a basis of K , we may view f as a non-singular r -by- r matrix with integer entries and the automorphisms of K as elements in the unimodular group $GL_r(\mathbb{Z})$. A coset in $Mat_{r \times r}(\mathbb{Z})/GL_r(\mathbb{Z})$ has a unique representative on *Hermite normal form* (see [Coh93, §2.4.2]). Recall that a non-singular matrix $H = (a_{ij})$ is on Hermite normal form if it is upper triangular, with $a_{ii} > 0$ for all i , and $a_{ii} > a_{ij} \geq 0$ for $j > i$. This allows us to uniquely associate a pair (\mathbf{E}', H) to each element of $T(\mathbf{E})$, where \mathbf{E}' is a subsequence of \mathbf{E} of some length r and H is a non-singular r -by- r matrix with integer entries on Hermite normal form. We order such pairs first lexicographically on \mathbf{E}' and then lexicographically on the entries of H . Here the entries are ordered first by rows, with high row numbers being more significant, and then by columns, with low column numbers being more significant. We illustrate the ordering by an example.

Example 7.10. The Hermite normal forms of the matrices, as discussed above, associated to the examples in the rows 4, 5 and 6 in Table 7.1 are computed to

$$\begin{pmatrix} 6 & 2 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 6 & 2 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 6 & 4 \\ 0 & 2 \end{pmatrix},$$

respectively. In particular, the divisorial type of the example in row 6 is larger than the divisorial type for the others.

It is easy to see that this ordering is a refinement of the partial ordering inherited by $U(\mathbf{E})$ and that taking the associated divisorial type of an arbitrary toric type gives a conormal invariant $U \rightarrow E$.

Remark 7.11. To any divisorial type (A, \mathbf{v}) , we get an associated orbifold fan $\Sigma = (N, \Sigma, \beta)$ with a unique maximal cone such that the rays in $\Sigma(1)$ are labelled by the markings on the elements of \mathbf{v} . Indeed, consider the exact sequence 7.2. Define N as the dual $\text{Hom}_{\mathbb{Z}}(K, \mathbb{Z})$ of K and β as the dual $\text{Hom}_{\mathbb{Z}}(f, \mathbb{Z})$ of f . The cones of Σ are the cones spanned by all subsets of $\{\beta(u_1), \dots, \beta(u_s)\}$, where u_i denotes the dual of the standard basis of \mathbb{Z}^s .

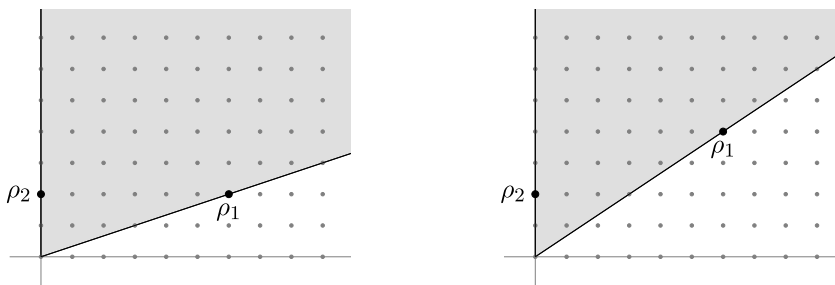


FIGURE 7.1. The stacky fans associated to the divisorial types associated to the toric types in rows 4 and 5 (left-hand picture) and row 6 (right-hand picture) of Table 7.1.

Example 7.12. The stacky fans associated to the divisorial types described in Example 7.10 are pictured in Figure 7.1. Running Algorithm B on the orbifold fan in the left-hand picture yields a destackification in three steps (this is illustrated in Figure 8.3). The corresponding destackification for the orbifold fan in the right-hand picture terminates after a single blow-up. This illustrates that different combinatorial recipes may be needed for different divisorial types.

The techniques described so far are enough to solve the destackification problem in the toroidal case. The procedure is described by Algorithm E in the next section if we omit Step E5. If we do not have a toroidal structure when we start, we need to create one. One problem is that the toroidal index is not a smooth conormal invariant, as was demonstrated in Example 6.18. Instead of using the toroidal index directly, we introduce a coarser invariant, the *divisorial index*, which may be thought of as a smoothed version of the toroidal index.

DEFINITION 7.13 (Divisorial index). Let $\alpha = (A, \mathbf{v})$ be a toric type and let $A_{\text{div}} \subseteq A$ be the subgroup generated by the weights of the elements of \mathbf{v} which are marked by some divisor. The *divisorial index* of α is the number of elements v_i in the sequence \mathbf{v} with weights a_i satisfying $a_i \notin A_{\text{div}}$. If the divisorial index is zero at ξ , we say that the standard pair $(X, \mathbf{E})/S$ is *divisorial at ξ* . Furthermore, we say that the pair $(X, \mathbf{E})/S$ is *divisorial* if it is divisorial at each of its points.

Remark 7.14. Geometrically, the property for a stack of being divisorial can be understood as follows: each of the components E^i of \mathbf{E} gives rise to a \mathbb{G}_m -torsor F^i , and the fibre product $F = F^1 \times_X \cdots \times_X F^m$ is a \mathbb{G}_m^r -torsor. If X is an orbifold, the pair (X, \mathbf{E}) is divisorial precisely when F is an algebraic space. This is equivalent to the stacky locus being contained in the divisor \mathbf{E} . In general, the pair (X, \mathbf{E}) is divisorial precisely when F is a gerbe.

Classically, a scheme is called *divisorial* [SGA6, Definition 2.2.5] if it has an ample family of line bundles. This is equivalent to the scheme having a \mathbb{G}_m^n -torsor, for some n , whose total space is quasi-affine (see [Hau02, Theorem 1] for varieties and [Gro13, Corollary 5.5] for the generalisation to stacks). Hence, our notion of divisorial stack is related, but not equivalent, to the classical definition.

The process of modifying X such that it becomes divisorial is straightforward, and described in Algorithm C. But to modify a divisorial stack such that it becomes toroidal is trickier. It turns out that, in general, this is not possible by just using ordinary blow-ups; root stacks are needed. It seems that the easiest approach is to interleave the process of reducing the toroidal index with

the process of reducing the independency index. Simply put, we just ignore the fact that (X, \mathbf{E}) is not toroidal, and use exactly the same algorithm as in the toroidal case. The distinguished divisors we create will, in general, not be independent in this case, but they will have the weaker property of being *divisorially independent*.

DEFINITION 7.15. We assume that $(X, \mathbf{E})/S$ is divisorial. Let $\xi \in X$ be a point and let E be a component of \mathbf{E} passing through ξ . Assume that the toric type at ξ is (A, \mathbf{v}) with $\mathbf{v} = (v_1, \dots, v_r)$ and $v_i = (a_i, e_i)$. Choose the indexing such that v_1 is marked by E and v_2, \dots, v_s are the other marked elements of \mathbf{v} . We say that E is *divisorially independent* at ξ provided that the intersection $\langle a_1 \rangle \cap \langle a_2, \dots, a_s \rangle$ is the trivial subgroup. A component of \mathbf{E} not passing through ξ is considered divisorially independent at ξ by default.

Note that the property for a component of \mathbf{E} of being independent at a point $\xi \in X$ does not depend on the other components of \mathbf{E} , whereas the property of being divisorially independent does.

This reduces the problem to modifying (X, \mathbf{E}) such that the divisorially independent divisors become independent. This is achieved by Algorithm D. Or, rather, the algorithm ensures that either the divisor becomes independent or the toroidal index drops. In either case we get an improvement, which allows us to solve the problem by repeating the procedure.

The main invariant used by Algorithm D is slightly more subtle than the others.

DEFINITION 7.16 (*E*-divisorial index). Let $\alpha = (A, \mathbf{v}) \in U(\mathbf{E})$ be a toric type with divisorial index zero, with $\mathbf{v} = (v_1, \dots, v_r)$ and $v_i = (a_i, e_i)$. Fix a component E of \mathbf{E} . If none of the elements of \mathbf{v} are marked by E , then the *E*-divisorial index of α is defined to be zero. Otherwise, we choose the indexing such that $e_1 = E$ and consider the groups

$$A_{\text{div}} = \langle a_i \mid e_i \neq * \rangle, \quad A_{\text{div}}^1 = \langle a_i \mid e_i \neq * \text{ and } i \neq 1 \rangle.$$

Then the subquotient $A^E = A_{\text{div}}/A_{\text{div}}^1$ of A is cyclic and generated by the canonical image \bar{a}_1 of a_1 in A^E . Since the divisorial index is assumed to be zero, each weight a_i for any element has a canonical image \bar{a}_i in A^E . Let c_i be the smallest non-negative number such that $\bar{a}_i = c_i \bar{a}_1$. The *E*-divisorial index is defined as the sum

$$c = \sum_{e_j=*} c_j.$$

Example 7.17. Consider the toric type in row 7 in Table 7.1. Using the notation from Definition 7.16, we have $A^{E^1} \cong C_3$. Choose the isomorphism such that the image of a_1 is 1 in C_3 . Then the images of a_3 and a_4 are 1 and 2, respectively. Hence, the E^1 -divisorial index is 3.

The *E*-divisorial index measures how far a divisorially independent divisor E is from being actually independent. More precisely, the following is true: let (X, \mathbf{E}) be a standard pair and let E^i be a component of \mathbf{E} . Choose a point $\xi \in E^i$, where (X, \mathbf{E}) is divisorial and E^i divisorially independent. Then E^i is independent at ξ if and only if the E^i -divisorial index vanishes at ξ .

We conclude the section by summarising the properties of the conormal invariants introduced here.

PROPOSITION 7.18. All of the following conormal invariants satisfy the properties (P1) and (P2) defined in § 6:

- (a) *independency index*;
- (b) *toroidal index*;
- (c) *divisorial type*;
- (d) *divisorial index*;
- (e) *E-divisorial index for any component $E \in \mathbf{E}$* .

Of these, the invariants (a), (d) and (e) are smooth. The invariants (b) and (c) are not smooth, but they are locally constant on the locus where the independency index obtains its maximum.

Proof. Let $(A, \mathbf{v}) \in U(\mathbf{E})$ be a toric type with $\mathbf{v} = (v_1, \dots, v_r)$ and $v_j = (a_j, e_j)$. We choose the indexing and integers s, t such that $e_j = *$ precisely when $j > s$ and $v_j = (0, *)$ precisely when $j > t$. We denote the subgroup generated by a_1, \dots, a_r by A_{rig} .

Let ι be one of the conormal invariants in (a)–(e). To see that the conditions (P1) and (P2) are satisfied for ι , we need to verify that $\iota(A, \mathbf{v}) = \iota(A_{\text{rig}}, (v_1, \dots, v_s))$. For the independency index this is clear since the definition only involves A_{rig} and the sequence a_1, \dots, a_r and since the condition (7.1) is trivially verified whenever $a_j = 0$. The conditions are equally easy to verify for the other invariants and we leave the details to the reader.

Now consider a toric type (A', \mathbf{v}') which is dominated by (A, \mathbf{v}) and use the notation $\mathbf{v}' = (v'_1, \dots, v'_r)$ and $v'_j = (a'_j, e'_j)$. By definition, we obtain (A', \mathbf{v}') by choosing a subset $J \subset \{1, \dots, r\}$ and letting

$$A' = A / \langle a_j \mid j \in J \rangle, \quad a'_j = \bar{a}_j, \quad e'_j = \begin{cases} * & \text{if } j \in J, \\ e_j & \text{otherwise,} \end{cases}$$

where \bar{a}_j denotes the canonical image of a_j in A' . Let ι denote one of the conormal invariants in (a), (d) or (e). In each case we state the precise condition on J under which we have $\iota(A, \mathbf{v}) = \iota(A', \mathbf{v}')$:

- (a) $\langle a_j \rangle \cap \langle a_1, \dots, \hat{a}_j, \dots, a_r \rangle = 0$ for each $j \in J$;
- (d) $a_j \in A_{\text{div}} = \langle a_1, \dots, a_s \rangle$ for each $j \in J$;
- (e) $a_j \in \langle a_1, \dots, \hat{a}_k, \dots, a_s \rangle$ for each $j \in J$, where k corresponds to the given divisor.

From these characterisations, it is clear that the smoothness condition of Definition 6.16 is satisfied for all three invariants.

The last statement follows from the fact that the independency index must drop if either the toroidal index or the divisorial type drops. □

8. The destackification algorithm

In this section, we give the full details of the destackification algorithms outlined in the previous section. We fix some notation which will be used throughout the section.

Let $(X_0, \mathbf{E}_0)/S$ be a standard pair with diagonalisable stabilisers over a quasi-compact scheme S . We wish to construct a smooth, stacky blow-up sequence

$$\Pi: (X_n, \mathbf{E}_n) \rightarrow \dots \rightarrow (X_0, \mathbf{E}_0).$$

Each stacky blow-up $\pi_i: (X_{i+1}, \mathbf{E}_{i+1}) \rightarrow (X_i, \mathbf{E}_i)$ in the sequence will have a centre Z_i which is determined by some smooth conormal invariant ι as described in Proposition 6.19. Different invariants ι will be used at different stages of the algorithm.

Let $\xi_i \in X_i$ be a point, which we assume lies in Z_i . Typically, we wish to investigate how the value of some conormal invariant ι under consideration at a lift ξ_{i+1} of ξ_i to X_{i+1} relates to the value of ι at ξ_i . Assume that the toric type of X_i at ξ_i is (A, \mathbf{v}) , where $\mathbf{v} = (v_1, \dots, v_r)$ and $v_j = (a_j, e_j)$. It is convenient to work with local homogeneous coordinates, which we may do in light of the locally toric description of standard pairs given in Proposition 5.3 together with the functoriality properties of conormal invariants described in Proposition 6.14. Hence, we may just as well assume that X_i is a basic toric stack with homogeneous coordinates

$$(\mathcal{O}_S[x_1, \dots, x_r], A, (a_1, \dots, a_r)) \tag{8.1}$$

and that ξ_i lies in the locus where the toric type is maximal, which is $V(x_j \mid a_j \neq 0 \text{ or } e_i \neq *)$. The substack Z_i is of the form

$$Z_i = V(x_j \mid j \in J), \quad J \subseteq \{1, \dots, r\}, \tag{8.2}$$

where the subset J depends on the toric type (A, \mathbf{v}) as well as the conormal invariant ι under consideration. The blow-up X_{i+1} is covered by $|J|$ coordinate patches indexed by $p \in J$. The coordinates for these patches are

$$(\mathcal{O}_S[\tilde{x}_1, \dots, \tilde{x}_r], \tilde{A}, (\tilde{a}_1, \dots, \tilde{a}_r)) \tag{8.3}$$

with

$$\tilde{A} = A, \quad \tilde{x}_j = \begin{cases} x_j/x_p & \text{if } j \in J \setminus \{p\}, \\ x_j & \text{otherwise,} \end{cases} \quad \tilde{a}_j = \begin{cases} a_j - a_p & \text{if } j \in J \setminus \{p\}, \\ a_j & \text{otherwise.} \end{cases} \tag{8.4}$$

In these coordinates $V(\tilde{x}_j)$ is the strict transform of $V(x_j)$ if $j \neq p$ and $V(\tilde{x}_p)$ is the exceptional divisor, which we denote by F . We see that the toric type at the patch is bounded above by $(\tilde{A}, \tilde{\mathbf{v}})$, with $\tilde{\mathbf{v}} = (\tilde{v}_j)$ and $\tilde{v}_j = (\tilde{a}_j, \tilde{e}_j)$, where \tilde{e}_j is given by

$$\tilde{e}_j = \begin{cases} F & \text{if } j = p, \\ \pi_i^{-1}(e_j) & \text{if } e_j \in \mathbf{E}_i \text{ and } j \neq p, \\ * & \text{otherwise.} \end{cases} \tag{8.5}$$

ALGORITHM C (Divisorialification). The input of the algorithm is a standard pair (X, \mathbf{E}) over a quasi-compact scheme S , with X having diagonalisable stabilisers. The output of the algorithm is a smooth, ordinary blow-up sequence

$$\Pi: (X_n, \mathbf{E}_n) \rightarrow \dots \rightarrow (X_0, \mathbf{E}_0) = (X, \mathbf{E})$$

such that $(X_n, \mathbf{E}_n)/S$ is divisorial. The construction is functorial (see Definition 2.6) with respect to arbitrary base change $S' \rightarrow S$ and with respect to morphisms $X' \rightarrow X$ which are either gerbes or smooth and stabiliser-preserving.

- C0.** [Initialise.] Set $i = 0$.
- C1.** [Finished?] Let Z_i be the locus in X_i where the divisorial index of the standard pair (X_i, \mathbf{E}_i) is maximal. If $Z_i = X_i$, then the algorithm terminates.
- C2.** [Blow up.] Let $\pi_i: (X_{i+1}, \mathbf{E}_{i+1}) \rightarrow (X_i, \mathbf{E}_i)$ be the blow-up of X_i in Z_i .
- C3.** [Iterate.] Increment i by 1 and iterate from Step C1.

Proof of correctness of Algorithm C. We will show that the maximum of the divisorial index decreases strictly after each iteration of the algorithm. This cannot continue forever, so the algorithm eventually halts. Since the divisorial index vanishes generically, it must vanish

FUNCTORIAL DESTACKIFICATION

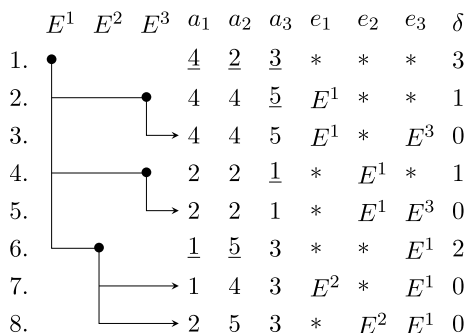


FIGURE 8.1. The blow-up tree produced by Algorithm C for the input described in Example 8.1. The first three columns correspond to the three blow-ups performed by the algorithm and, at each blow-up, the coordinate patches marked by • are replaced by one or more patches. These columns are labelled by the exceptional divisor of the corresponding blow-up. The indices j in the set J , as defined in (8.2), are indicated by underlining the corresponding weight a_j . The column marked by δ contains the divisorial index.

identically when the algorithm halts, which proves that (X_n, \mathbf{E}_n) is indeed divisorial. The functoriality properties follow directly from the corresponding functoriality properties of conormal invariants described in Proposition 6.14.

Assume that we are in Step C2 and fix a point $\xi_i \in Z_i$ with toric type (A, \mathbf{v}) . We use the notation from the introduction of the section. To verify that the divisorial index decreases, we may work locally and without loss of generality assume that X_i is a basic toric stack with homogeneous coordinates as in (8.1). Let $A_{\text{div}} \subset A$ be the subgroup of A generated by the weights a_j such that $e_j \neq *$. Recall from Definition 7.13 that the divisorial index is the number of indices $j \in \{1, \dots, r\}$ such that $a_j \notin A_{\text{div}}$. It follows that the set J , as defined in (8.2), corresponding to the divisorial index is simply $J = \{j \mid a_j \notin A_{\text{div}}\}$.

Now consider the coordinate patch of X_{i+1} corresponding to $p \in J$, with coordinates as in (8.3). From (8.4) and (8.5), we see that the subgroup $\tilde{A}_{\text{div}} \subseteq A$ generated by the weights \tilde{a}_j for which $\tilde{e}_j \neq *$ is generated by A_{div} and a_p . It follows that $\tilde{a}_j \notin \tilde{A}_{\text{div}}$ implies that $a_j \notin A_{\text{div}}$. Moreover, we have $\tilde{a}_p \in \tilde{A}_{\text{div}}$ whereas $a_p \notin A_{\text{div}}$. Hence, the divisorial index at each point in $(X_{i+1}, \mathbf{E}_{i+1})$ is strictly smaller than the divisorial index at ξ_i , which concludes the proof. \square

Example 8.1. Let k be a field and consider the basic toric stack (X, \mathbf{E}) with homogeneous coordinates $(k[x_1, x_2, x_3], C_6, (4, 2, 3))$ and $\mathbf{E} = \emptyset$. The blow-up tree generated by Algorithm C with this input is described in Figure 8.1. The algorithm terminates after three iterations. Note that the third blow-up does not modify the underlying stack. However, it is non-trivial in the sense that it modifies the ordered set of divisors.

The following example illustrates that divisorialification can be non-trivial even if the stack we start with has a smooth coarse space.

Example 8.2. Consider the standard pair (X, \emptyset) described in 7.6. The divisorialification algorithm will first blow up the point lying over the node of C , which has divisorial index 2. Note that the strict transform \tilde{C} of the divisor C is smooth. The divisorial index vanishes away from \tilde{C} and is one at each point of \tilde{C} . Hence, the algorithm terminates after blowing up \tilde{C} in the second iteration. In this particular case, we actually even get a destackification of (X, \emptyset) .

Recall the definition of a distinguished structure on a standard pair from Definition 2.7.

ALGORITHM D (Divisorialification along distinguished divisors). The input of the algorithm is a divisorial stack with distinguished structure $(X, \mathbf{E}, \mathbf{D})$ over a quasi-compact base scheme S . The output of the algorithm is a smooth, ordinary blow-up sequence

$$\Pi: (X_n, \mathbf{E}_n, \mathbf{D}_n) \rightarrow \cdots \rightarrow (X_0, \mathbf{E}_0, \mathbf{D}_0) = (X, \mathbf{E}, \mathbf{D}),$$

with each blow-up being admissible (see Definition 2.7). The resulting triple $(X_n, \mathbf{E}_n, \mathbf{D}_n)$ has the property that for each component D of \mathbf{D}_n , the D -divisorial index, in the sense of Definition 7.16, vanishes. Furthermore, each of the centres Z_i in the blow-up sequence is contained in exactly one of the components of \mathbf{D}_i and transversal to all other components of \mathbf{E}_i . The construction is functorial (see Definition 2.6) with respect to arbitrary base change $S' \rightarrow S$ and with respect to morphisms $X' \rightarrow X$ which are either gerbes or smooth and stabiliser-preserving.

D0. [Initialise.] Let $i = 0$.

D1. [Finished?] Let D'_i denote the oldest component of \mathbf{D}_i for which the D'_i -divisorial index does not vanish identically. If no such component exists, the algorithm terminates. Otherwise, we let Z_i be the smooth substack of X_i where the D'_i -divisorial index assumes its maximal value.

D2. [Blow up maximal locus.] Let $\pi_i: (X_{i+1}, \mathbf{E}_{i+1}, \mathbf{D}_{i+1}) \rightarrow (X_i, \mathbf{E}_i, \mathbf{D}_i)$ be the blow-up of X_i in Z_i .

D3. [Iterate.] Increment i by one and iterate from Step D1.

Proof of correctness of Algorithm D. Assume that we are in iteration i . Let $\xi_i \in Z_i$ be a point in the centre of the blow-up, and let ξ_{i+1} be an arbitrary lifting of ξ_i to the exceptional locus. We will prove the following three statements.

(i) The \tilde{D} -divisorial index at ξ_{i+1} for the strict transform \tilde{D} of any component D of \mathbf{D}_i is not larger than the D -divisorial index at ξ_i .

(ii) In the case where the component in the previous statement is D'_i , the index is strictly smaller.

(iii) The F -divisorial index at ξ_{i+1} , where F is the exceptional divisor of the blow-up, is strictly smaller than the D'_i -divisorial index at ξ_i .

Together these statements prove that the algorithm terminates with the right exit condition. Indeed, let N be the maximum of the D -divisorial index where D ranges through the components of \mathbf{D}_0 . At each iteration i , let w_j be the number of components D of \mathbf{D}_i such that the maximum of the D -divisorial index is j . Then the N -tuple $(w_N, w_{N-1}, \dots, w_1)$ decreases strictly in lexicographical ordering with each iteration of the algorithm.

We use the notation from the introduction of the section. To verify the statements above, we may work locally and assume that X_i is a basic toric stack with homogeneous coordinates as in (8.1). We choose the indexing such that the divisor D'_i as defined in Step D1 is $V(x_1)$. For each $k \in \{1, \dots, r\}$, we define the subgroup

$$A_{\text{div}}^k = \langle a_j \mid j \neq k, e_j \neq * \rangle$$

and, for each $j \in \{1, \dots, r\}$, we define c_j^k as the smallest natural numbers such that $a_j \equiv c_j^k a_k \pmod{A_{\text{div}}^k}$. Recall from Definition 7.16 that if $V(x_k)$ corresponds to a component E of \mathbf{E} , the E -divisorial index is given by $\sum c_j^k$, where the sum is taken over the indices j such that $e_j = *$.

	D^2	D^3	a_1	a_2	a_3	a_4	e_1	e_2	e_3	e_4	ι	τ	δ_1	δ_2	δ_3	
1.	•		(0, 1)	<u>(1, 0)</u>	<u>(1, 1)</u>	<u>(2, 1)</u>	E^1	D^1	*	*	4	2	3	-	-	
2.	—	•	(0, 1)	<u>(1, 0)</u>	(0, 1)	<u>(1, 1)</u>	E^1	D^2	*	*	4	2	0	1	-	
3.		—	→	(0, 1)	(1, 0)	(0, 1)	(0, 1)	E^1	D^3	*	*	3	2	0	0	0
4.		—	→	(0, 1)	(0, 2)	(0, 1)	(1, 1)	E^1	D^2	*	D^3	4	1	0	0	0
5.	—	•	(0, 1)	(0, 2)	<u>(1, 1)</u>	<u>(1, 0)</u>	E^1	D^1	D^2	*	4	1	0	1	-	
6.		—	→	(0, 1)	(0, 2)	(1, 1)	(0, 2)	E^1	D^1	D^3	*	4	1	0	0	0
7.		—	→	(0, 1)	(0, 2)	(0, 1)	(1, 0)	E^1	D^1	D^2	D^3	4	0	0	0	0
8.	—	→	(0, 1)	(2, 2)	(2, 0)	(2, 1)	E^1	D^1	*	D^2	4	1	0	0	0	

FIGURE 8.2. Blow-up tree produced by Algorithm D as described in Example 8.3. The last five columns contain the independency index, the toroidal index and the D_j -divisorial indices with j equal to 1, 2 or 3, respectively.

The set J , defined as in (8.2), with respect to the D' -divisorial index is the set of indices j such that $c_j^1 \neq 0$. Fix $p \in J$ and consider the corresponding patch of X_{i+1} with homogeneous coordinates as in (8.3). We define \tilde{A}_{div}^k and \tilde{c}_j^k similarly as above.

Now assume that $V(x_k)$ corresponds to a component D of the distinguished divisor D_i . When computing the $\pi_i^{-1}(D)$ -divisorial index, we may assume that $k \neq p$. Then $\tilde{A}_{\text{div}}^k = A_{\text{div}}^k + \langle a_p \rangle$. It follows that for any j such that $1 \leq j \leq r$, we have

$$\tilde{a}_j \equiv c_j^k a_k \pmod{\tilde{A}_{\text{div}}^k},$$

so $\tilde{c}_j^k \leq c_j^k$. By taking the sum over all j such that $\tilde{e}_j = *$, we see that the $\pi_i^{-1}(D)$ -divisorial index is bounded above by the D -divisorial index, which proves the first statement. In the particular case when $k = 1$, one verifies that $\tilde{c}_p^1 = 0$, whereas $c_p^1 \neq 0$, which proves the second statement.

Finally, we investigate the F -divisorial index where F is the exceptional divisor, which is given by $V(y_p)$. We have $\tilde{A}_{\text{div}}^p = A_{\text{div}}^p + \langle a_1 - a_p \rangle$, so $\tilde{a}_p \equiv a_p \equiv a_1 \pmod{\tilde{A}_{\text{div}}^p}$. For $j \in J$, we get

$$\tilde{a}_j \equiv a_j - a_p \equiv (c_j^1 - 1)\tilde{a}_p \pmod{\tilde{A}_{\text{div}}^p},$$

which proves that $\tilde{c}_j^p < c_j^1$, and the third statement follows. □

Example 8.3. Let k be a field. Figure 8.2 illustrates the blow-up tree produced by Algorithm D when applied to the standard pair $(X, \mathbf{E}, \mathbf{D})$ with distinguished structure \mathbf{D} defined as follows: the stack X is the basic toric stack with homogeneous coordinates $(k[x_1, \dots, x_4], (C_3)^2, \mathbf{a})$, where $\mathbf{a} = ((0, 1), (1, 0), (1, 1), (2, 1))$. We let $E^1 = V(x_1)$ and $D^1 = V(x_2)$. The ordered sets \mathbf{E} and \mathbf{D} are defined as (E^1, D^1) and (D^1) , respectively.

LEMMA 8.4. *Let $(X, \mathbf{E}, \mathbf{D})$ be a divisorial stack with distinguished structure over a quasi-compact scheme S , and let*

$$\Pi: (X_n, \mathbf{E}_n, \mathbf{D}_n) \rightarrow \dots \rightarrow (X_0, \mathbf{E}_0, \mathbf{D}_0) = (X, \mathbf{E}, \mathbf{D})$$

be the output of Algorithm D applied to $(X, \mathbf{E}, \mathbf{D})$. Let $\xi_0 \in X_0$ be a point at which all distinguished divisors are divisorially independent in the sense of Definition 7.15. For each i

such that $0 < i \leq n$, let $\xi_i \in X_i$ be a lift of ξ_{i-1} such that the toroidal index (see Definition 7.4) at ξ_i is the same as the toroidal index at ξ_{i-1} . Then all components of \mathbf{D}_i are divisorially independent at ξ_i . In particular, all components of \mathbf{D}_n are independent at ξ_n in the sense of Definition 7.2.

Proof. Note that a divisor E is independent at a point if and only if it is divisorially independent and the E -divisorial index is zero. This is an easy consequence of the definitions. Therefore, the last statement of the lemma follows from the second last statement.

We fix i and assume that all components of \mathbf{D}_i are divisorially independent at ξ_i . We want to prove that all components of \mathbf{D}_{i+1} are divisorially independent at ξ_{i+1} .

As usual, we may work locally and assume that X_i is a basic toric stack. We use the same notation as in the proof of correctness of Algorithm D. We need only consider the patch corresponding to $p = 1$ of the blow-up, since the toroidal index is lower at all other patches. But, at this patch, we have $\tilde{e}_j \in \mathbf{E}_{i+1}$ (or $\tilde{e}_j \in \mathbf{D}_{i+1}$) precisely when $e_j \in \mathbf{E}_i$ (or $e_j \in \mathbf{D}_i$). Furthermore, we have $\tilde{a}_j = a_j$ whenever $e_j \neq *$. From this it follows that the distinguished divisors are divisorially independent at ξ_{i+1} provided that the distinguished divisors are divisorially independent at ξ_i , as desired. \square

Example 8.5. Consider the standard pair $(X, \mathbf{E}, \mathbf{D})$ from Example 8.3. Note that the divisors in \mathbf{E} are *divisorially independent* (see Definition 7.15) since the group $(C_3)^2$ is a direct sum of the cyclic groups generated by the weights a_1 and a_2 of the coordinate functions x_1 and x_2 , respectively. Hence, Lemma 8.4 applies. From Figure 8.2, we see that the a_1 coefficient has been reduced to zero in the weights of the third and fourth coordinate functions in chart 3. Hence, the divisor D_3 is *independent* at this chart (see Definition 7.1). At all other terminal charts, i.e., the charts 4, 6, 7 and 8, the property of divisorial independence is destroyed. Note, however, that at these charts the toroidal index has dropped.

Let (X, \mathbf{E}) be a standard pair. Recall that we do not require the components of the sequence \mathbf{E} to be connected (see Definition 2.1). This poses a small technical problem at the final stage of the destackification process. To deal with this, we introduce the concept of a *refinement* of the sequence \mathbf{E} .

DEFINITION 8.6. Let (X, \mathbf{E}) and (X, \mathbf{E}') be standard pairs. Assume that the sequence \mathbf{E} is indexed by the set $[s] = \{1, \dots, s\}$ and \mathbf{E}' by the set $[s'] = \{1, \dots, s'\}$. We say that \mathbf{E}' is a *refinement* of \mathbf{E} if there exists a surjective function $f: [s'] \rightarrow [s]$ such that for each $i \in [s]$, the component E^i in \mathbf{E} is the disjoint union $\bigcup_j (E')^j$, with $(E')^j \in \mathbf{E}'$, where the union is taken over the set $f^{-1}(\{i\})$.

The next proposition shows that, up to a refinement of the divisor \mathbf{E} , a divisorial standard pair (X, \mathbf{E}) is destackified (cf. Definition 2.3) provided that the independency index vanishes everywhere.

PROPOSITION 8.7. Let $(X, \mathbf{E})/S$ be a divisorial stack over a quasi-compact scheme S , and assume that the independency index is everywhere zero at X . Let $\pi: X \rightarrow X_{\text{cs}}$ be the coarse space. Then $(X_{\text{cs}}, \mathbf{E}_{\text{cs}})/S$ is a standard pair. In particular, the stack X_{cs} is smooth and \mathbf{E}_{cs} has simple normal crossings.

Moreover, there exists a refinement \mathbf{E}' of \mathbf{E} , in the sense of Definition 8.6, such that the following holds.

- (i) The divisor \mathbf{E}' is the \mathbf{d} th root of $\pi^*\mathbf{E}'_{\text{cs}}$ for some sequence \mathbf{d} of positive integers indexed by the components of \mathbf{E}' .
- (ii) The canonical factorisation $X \rightarrow (X_{\text{cs}})_{\mathbf{d}^{-1}\mathbf{E}'_{\text{cs}}} \rightarrow E_{\text{cs}}$ makes X a gerbe over the stack $(X_{\text{cs}})_{\mathbf{d}^{-1}\mathbf{E}'_{\text{cs}}}$.

Proof. That X_{cs} is smooth and that \mathbf{E}_{cs} has normal crossings can be verified étale locally on X_{cs} . Hence, we may assume that X is a basic toric stack with homogeneous coordinates as in (8.1). We use the notation

$$\mathcal{R} := \mathcal{O}_S[x_1, \dots, x_r].$$

We choose the coordinates such that the components of \mathbf{E} are $V(x_j)$ for $j \in \{1, \dots, s\}$. Let $A_{\text{rig}} = \langle a_1, \dots, a_r \rangle$. By the assumption that the independency index is everywhere zero, we have $A_{\text{rig}} = \bigoplus_{j=1}^r A_j$ with each group A_j finite and cyclic. By the assumption that (X, \mathbf{E}) is divisorial, the group A_j vanishes for each $j > s$.

Denote the order of A_j by d_j . Then the coarse space is the relative spectrum of the invariant ring

$$\mathcal{R}' := \mathcal{O}_S[x_1^{d_1}, \dots, x_s^{d_s}, x_{s+1}, \dots, x_r].$$

In particular, the coarse space is smooth since this is a polynomial ring. The coarse space E_{cs}^j of a component E^j of \mathbf{E} corresponding to the j th coordinate is $V(x_j^{d_j})$. Hence also E_{cs}^j is smooth, and the divisors $E_{\text{cs}}^1, \dots, E_{\text{cs}}^j$ have simple normal crossings. This proves that $(X_{\text{cs}}, \mathbf{E}_{\text{cs}})$ is a standard pair.

From the local coordinates, we also see that E^j is a d_j th root of $\pi^{-1}(E_{\text{cs}}^j)$. Globally, the coefficient d_j is locally constant on E^j . By refining \mathbf{E} such that d_j becomes constant on each component, we may assume that E^j is the d_j th root of $\pi^{-1}(E_{\text{cs}}^j)$ globally, which proves (i). This gives the factorisation in (ii) by the universal property of root stacks. The fact that X is a gerbe over $(X_{\text{cs}})_{\mathbf{d}^{-1}\mathbf{E}_{\text{cs}}}$ can again be verified locally. Here it follows from the sequence of homomorphisms between the local homogeneous coordinate rings of graded \mathcal{O}_S -algebras

$$(R', 0) \rightarrow (R, A_{\text{rig}}) \rightarrow (R, A);$$

here the first map corresponds to the root stack and the second map corresponds to the gerbe (cf. Remark 3.5). □

Remark 8.8. Let (X, \mathbf{E}) be a standard pair. It is easy to see that any refinement \mathbf{E}' of \mathbf{E} can be obtained via a sequence of blow-ups, at least if we do not care about the ordering of \mathbf{E}' . Indeed, we only have to blow up a suitable subset of the components of \mathbf{E}' one after another. Notice that each such blow-up is trivial in the sense that it does not modify X since the centres of the blow-ups are effective Cartier divisors.

Remark 8.9. A naive choice of refinement of \mathbf{E} in Proposition 8.7 would be to take \mathbf{E}' such that each component of \mathbf{E}' is connected. The problem with this choice is that it in general cannot be made functorial, even with respect to isomorphisms. To see this, let X be a two-dimensional orbifold admitting a non-trivial involution σ . Assume that P and Q are distinct points on X which are exchanged by σ and that the corresponding points on X_{cs} are isolated singularities. Then it is easy to see that any functorial destackification algorithm must blow up the points P and Q simultaneously, which results in a disconnected divisor.

Remark 8.10. Let (X, \mathbf{E}) be a divisorial standard pair and assume that the independency index is everywhere vanishing. By Remark 8.8, there is a refinement \mathbf{E}' of \mathbf{E} as in Proposition 8.7,

but, by Remark 8.9, some care must be taken when constructing a refinement algorithm in order to get the right functorial properties. A simple algorithm which works is to take each of the components E^i of \mathbf{E} and repeatedly blow up the locus where the order of the corresponding weight a_i in the toric type is maximal until the order becomes zero. This gives a refinement \mathbf{E}' such that the order of a_i is constant for each component $(E')^i$ of \mathbf{E}' . We leave to the reader to verify that this gives an algorithm with the right functorial properties and that \mathbf{E}' is a suitable choice of refinement for the purpose of Proposition 8.7.

ALGORITHM E (Destackification). The input of the algorithm is a divisorial stack (X, \mathbf{E}) over a quasi-compact base scheme S . The output of the algorithm is a destackification (see Definition 2.3)

$$\Pi: (X_n, \mathbf{E}_n) \rightarrow \cdots \rightarrow (X_0, \mathbf{E}_0) = (X, \mathbf{E}).$$

The construction is functorial (see Definition 2.6) with respect to arbitrary base change $S' \rightarrow S$ and with respect to morphisms $X' \rightarrow X$ which are either gerbes or smooth and stabiliser-preserving.

- E0.** [Initialise.] Let $i = 0$.
- E1.** [Find the worst locus.] The main conormal invariant of the algorithm is the invariant $U(\mathbf{E}_i) \rightarrow \mathbb{N} \times \mathbb{N} \times T(\mathbf{E}_i)$ composed lexicographically, in the sense of Example 6.12, by the independency index (see Definition 7.1), the toroidal index (see Definition 7.4) and the divisorial type (see Definition 7.8). Let Z_i be the locus in X_i where this invariant obtains its maximum. If $Z_i = X_i$, we perform the algorithm indicated in Remark 8.10 and then terminate.
- E2.** [Blow up Z_i .] By construction of the main invariant in Step E1, the divisorial type is constant at Z_i . Let Σ be the orbifold fan with ordered rays associated to this divisorial type in the sense of Remark 7.11. Let $(X_{i+1}, \mathbf{E}_{i+1}) \rightarrow (X_i, \mathbf{E}_i)$ be the blow-up of (X_i, \mathbf{E}_i) in the centre Z_i . Denote the exceptional divisor by \mathbf{D}_{i+1} and mark it as a distinguished divisor. Also, let Σ_{i+1} be the star subdivision $\Sigma^*(\sigma)$, where σ is the maximal cone in Σ . We label the exceptional ray δ by \mathbf{D}_{i+1} , and give Σ_{i+1} a distinguished structure by letting $\{\delta\}$ be the set of distinguished rays. Increment i by 1.
- E3.** [Perform toric destackification.] Perform Algorithm A on Σ_i , and denote the result by $\Sigma_{i+k} \rightarrow \cdots \rightarrow \Sigma_i$.
- E4.** [Perform corresponding stacky blow-ups.] Perform the corresponding stacky blow-ups on $(X_i, \mathbf{E}_i, \mathbf{D}_i)$ to form the sequence

$$(X_{i+k}, \mathbf{E}_{i+k}, \mathbf{D}_{i+k}) \rightarrow \cdots \rightarrow (X_i, \mathbf{E}_i, \mathbf{D}_i).$$

At each step, a star subdivision corresponds to a blow-up in the intersection of the divisors labelling the rays of the subdivided cone. Rooting a ray corresponds to the taking of a root stack of the same order in the corresponding distinguished divisor. Also, the ray–divisor correspondence is extended in each step such that the exceptional ray of each subdivision or root corresponds to the exceptional divisor of the corresponding stacky blow-up. Increment i by k .

- E5.** [Eliminate divisorial index along distinguished divisors.] Perform Algorithm D on the triple $(X_i, \mathbf{E}_i, \mathbf{D}_i)$, and append the output of the algorithm to the blow-up sequence. Increment i by the length of the output. After this step we forget the distinguished structure.
- E6.** [Iterate.] Iterate from Step E1.

Proof of correctness of Algorithm E. When the algorithm terminates, the independence index is constant (zero) at X_i , according to the termination criterion in Step E1. Hence, by Proposition 8.7 and Remark 8.10, we only need to prove that the algorithm terminates.

We will prove that the maximum of the conormal invariant composed by the independency index, the toroidal index and the divisorial type decreases strictly with each iteration of the main loop of the algorithm. Note that the independency index decreases weakly with each iteration since all blow-ups performed during the iteration have centres which intersect the transforms of the divisors which were independent at the start of the iteration transversally. It is easy to see that also the toroidal index decreases weakly at each blow-up.

We examine how the invariant described above is affected during a single iteration of the main loop. For notational convenience, we assume that $i = 0$ at the start of the iteration and that $i = n$ at the end of the iteration. Assume that we are in Step E1 of the algorithm, and let ξ be any point in Z_0 . Since all blow-ups during a single iteration have centres lying above Z_0 , it is enough to show that any point in X_n lying over ξ has either strictly lower independence index, or strictly lower toroidal index, than ξ at the end of the iteration.

Using the notation from the beginning of the section, we denote the toric type at ξ by (A, \mathbf{v}) , where $\mathbf{v} = (v_1, \dots, v_r)$ and $v_j = (a_j, e_j)$. Choose the indexing of \mathbf{v} such that, for some integers s, t satisfying $1 \leq s \leq t \leq r$, the following holds.

- (i) The component v_j is independent in the sense of Definition 7.1 if and only if $j > t$.
- (ii) For indices j such that $1 \leq j \leq s$, we have $e_j \in \mathbf{E}_0$.
- (iii) For indices j such that $s < j \leq t$, we have $e_j = *$.

In particular, the independency index at ξ is t . Note that since (X_0, \mathbf{E}_0) is assumed to be divisorial, we have $a_j = 0$ for each $j > t$ such that $e_j = *$ by the assumption that v_j is independent. This implies that the toroidal index at ξ is $t - s$. The set J , as defined in (8.2), is equal to $\{1, \dots, t\}$.

The toroidal index and the independency index can be computed using local homogeneous coordinates, so we may just as well assume that X_0 is a basic toric stack. Using the indexing convention described above, the local homogeneous coordinates are given by

$$\mathcal{R} = \mathcal{O}_S[x_1, \dots, x_s, x_{s+1}, \dots, x_t, x_{t+1}, \dots, x_r].$$

By our assumption that v_j is independent for $j > t$, the group $A_{\text{rig}} = \langle a_1, \dots, a_r \rangle \subset A$ splits as a direct sum $A_{\text{rig}} = A' \oplus A''$, where $A' = \langle a_1, \dots, a_t \rangle$ and $A'' = \langle a_{t+1}, \dots, a_r \rangle$.

After the blow-up in Step E2, the stack X_1 is covered by $|J| = t$ patches. Choose any $p \in J$ and let $U_1 \subset X_1$ be the corresponding coordinate patch. The coordinate ring of U_1 is given by

$$\tilde{\mathcal{R}} = \mathcal{O}_S[\tilde{x}_1, \dots, \tilde{x}_s, \tilde{x}_{s+1}, \dots, \tilde{x}_t, \tilde{x}_{t+1}, \dots, \tilde{x}_r]$$

using the notation from (8.3). We define $\tilde{A}_{\text{rig}} \subset \tilde{A}$ and the subgroups \tilde{A}' and \tilde{A}'' similarly as above and note that $\tilde{A}' = A'$ and $\tilde{A}'' = A''$.

If $p > s$, then $\tilde{e}_p \neq *$ whereas $e_p = *$, so the toroidal index at the patch is strictly lower than the toroidal index at ξ . Hence, we may assume that $p \leq s$.

Next we consider the toric stacks X'_i associated to the stacky fans Σ_i . Recall that X'_0 is the basic toric stack associated to the divisorial type of (A, \mathbf{v}) . Note that the group A' is generated by a_1, \dots, a_s by the assumption that (X_0, \mathbf{E}_0) is divisorial. It follows that the homogeneous coordinates for X'_0 are

$$(\mathcal{R}' = \mathcal{O}_S[x_1, \dots, x_s], A', (a_1, \dots, a_s)).$$

By the assumption that $p \leq s$, the blow-up X'_1 contains a basic toric substack $U'_1 \subset X'_1$ with homogeneous coordinates

$$(\tilde{\mathcal{R}}' = \mathcal{O}_S[\tilde{x}_1, \dots, \tilde{x}_s], \tilde{A}', (\tilde{a}_1, \dots, \tilde{a}_s)).$$

Now execute the partial toric resolution in Step E3, and assume that it finishes in k steps. Also, perform the corresponding stacky blow-ups in Step E4. For each stacky blow-up $X_{i+1} \rightarrow X_i$ with $1 < i \leq k$, we choose an arbitrary patch U_{i+1} lying over U_i , with U_{i+1} being a basic toric stack. We assume by induction that U_i has homogeneous coordinates

$$(\hat{\mathcal{R}} = \mathcal{O}_S[\hat{x}_1, \dots, \hat{x}_r], \hat{A}, (\hat{a}_1, \dots, \hat{a}_t, \hat{a}_{t+1}, \dots, \hat{a}_r))$$

satisfying the following conditions.

- (i) The orbifold fan Σ_i has a maximal cone whose corresponding basic toric stack U'_i has homogeneous coordinates

$$(\mathcal{O}_S[\hat{x}_1, \dots, \hat{x}_s], \langle \hat{a}_1, \dots, \hat{a}_s \rangle, (\hat{a}_1, \dots, \hat{a}_s)).$$

- (ii) The toric divisor on U' corresponding to \hat{x}_j corresponds to a component E of \mathbf{E}_i if and only if $V(\hat{x}_j) = E$ in U_i .
- (iii) For each $j > s$, the divisor $V(\hat{x}_j)$ on U_i is either unmarked or independent.

It is straightforward to verify that these properties are preserved under the stacky blow-ups described in Step E4. In particular, they hold for U_{k+1} . By the exit condition of Algorithm A, any distinguished divisor $V(\hat{x}_j)$ on U'_{k+1} is independent. It follows that the corresponding divisor $V(\hat{x}_j)$ on U_{k+1} is divisorially independent at the points where the toroidal index has not dropped.

Now execute the subalgorithm in Step E5. From Lemma 8.4, we see that after this step all distinguished divisors are independent at points where the toroidal index has not dropped. Due to the first blow-up, there is at least one distinguished divisor going through every point lying over ξ . Thus, at points where the toroidal index has not dropped, the independency index is at most $t - 1$, which concludes the proof. □

Example 8.11. In a final example, we examine the output of Algorithm E when applied to the basic toric stack (X, \mathbf{E}) with homogeneous coordinates $(k[x_1, x_2, x_3], A, \mathbf{a})$. Here we let $A = C_6 \times C_2$, $\mathbf{a} = ((1, 0), (5, 1), (3, 1))$ and $\mathbf{E} = (E^1, E^2)$ with $E^1 = V(x_1)$ and $E^2 = V(x_2)$.

In Step E1, the origin $Z_0 = V(x_1, x_2, x_3)$ is identified as the most problematic locus. The divisorial type is given by the triple $(A, (a_1, E^1), (a_2, E^2))$ with the weights corresponding to E_1 and E_2 , respectively. Figure 8.3 shows the destackification of the corresponding orbifold fan, which is obtained by invoking Algorithm A from Step E3. This gives a combinatorial recipe for destackifying (X, \mathbf{E}) ; after the initial blow-up of Z_0 , we blow up the intersections $D^1 \cap E^2$ and $E^1 \cap D^1$ in turn. The resulting blow-up tree is shown in Figure 8.4. The key things to note are as follows.

- (i) If we forget the weight in the third column, i.e., the weight not corresponding to any divisor in \mathbf{E} , in the charts 1–7 of Figure 8.4, we precisely get the blow-up tree described in Figure 8.3. Since all the distinguished divisors in the output of Algorithm A are guaranteed to be independent, the distinguished divisors in Figure 8.4 are guaranteed to be *divisorially independent* at the charts 3, 4, 6 and 7. Hence, these charts will improve after invoking Algorithm D from Step E5.

- (ii) In the remaining terminal charts the toroidal index has dropped, which gives the desired improvement. Note that the blow-ups of the loci in charts 8 and 9 do nothing to further improve the situation; they are merely side effects of improving charts 2 and 5.

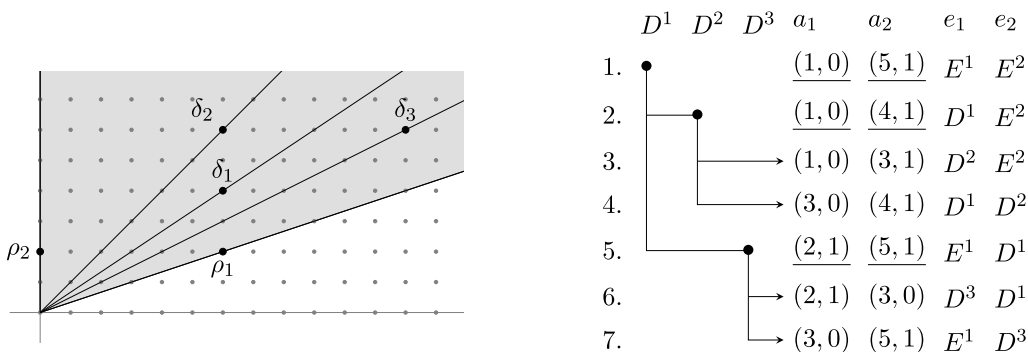


FIGURE 8.3. The left-hand picture shows the orbifold fan Σ_{i+k} directly after Step E3 in the first iteration of the main loop in Algorithm E for the input described in Example 8.11. The table to the right shows the corresponding blow-up tree. The weights are in the group $C_6 \times C_2$ and the divisors E^i and D^i correspond to the rays ρ_i and δ_i respectively from the fan in the left-hand picture.

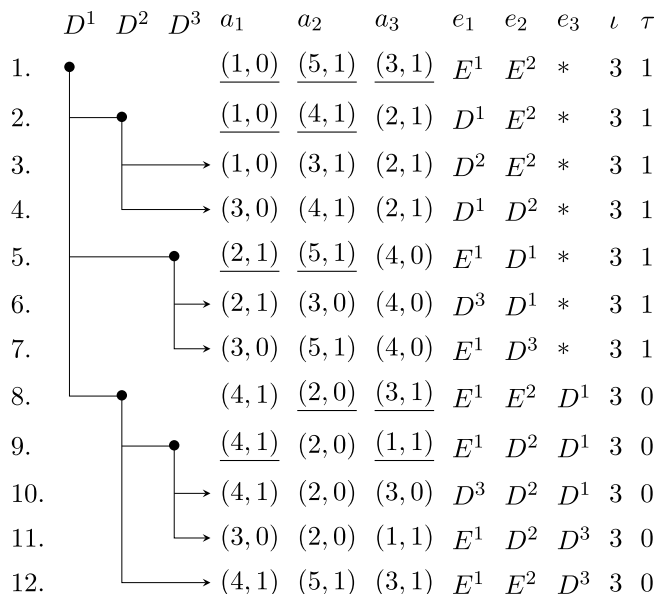


FIGURE 8.4. The blow-up tree produced by Steps E2–E4 of Algorithm E for the input described in Example 8.11. The weights are in the group $C_6 \times C_2$. The last two columns contain the independency index and the toroidal index, respectively. Note that the distinguished divisors D^1 , D^2 and D^3 are *divisorially independent* (Definition 7.15) at charts 3, 4, 6 and 7.

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Appendix A. Tame stacks

This appendix may be viewed as a supplement to §3 of the article [AOV08]. We start by recalling some of the main concepts. Let S be a scheme and let X be an algebraic stack which is quasi-

separated and locally of finite presentation over S . If X has finite inertia, there exists a coarse space $\pi: X \rightarrow X_{\text{cs}}$ with the map π being proper. Following [AOV08], we say that X is *tame* if the functor $\pi_* \text{QCoh } X \rightarrow \text{QCoh } X_{\text{cs}}$ is exact. We call a group scheme $G \rightarrow S$ *finite, linearly reductive* if G is finite, flat, locally of finite presentation and the fibres are linearly reductive. We say that an algebraic stack X has linearly reductive stabiliser at a point $\xi \in |X|$ if the stabiliser at one of, or equivalently any of, the k -points representing ξ is linearly reductive.

The following theorem is an extension of the main theorem of [AOV08].

THEOREM A.1. *Let S be a scheme, and let X be an algebraic stack which is quasi-separated and locally of finite presentation over S . Assume that X has finite inertia. Then the following conditions are equivalent.*

- (a) *The stack X is tame.*
- (b) *The stabilisers of X are linearly reductive.*
- (c) *There exist a covering $Y \rightarrow X_{\text{cs}}$ of the coarse space, which is faithfully flat and locally of finite presentation, a finite, linearly reductive group scheme $G \rightarrow Y$ and a G -space $U \rightarrow Y$, which is finite and finitely presented, together with an isomorphism*

$$[U/G] \simeq Y \times_{X_{\text{cs}}} X.$$

- (d) *The same as (c), but $Y \rightarrow X_{\text{cs}}$ can be assumed to be étale.*

If, in addition, the morphism $X \rightarrow S$ is assumed to be smooth, the above conditions are equivalent to the following condition.

- (e) *The same as (d), but U can be assumed to be smooth over S .*

The equivalence of the conditions (a)–(d) is [AOV08, Theorem 3.2]. Here we will prove that (e) is equivalent to the other conditions under the extra hypothesis, and give a simplification of the proof that (b) implies (d).

In [AOV08], it is proven that tame gerbes admit sections étale locally. The argument given is based on rigidification and the structure theory of linearly reductive groups. But, rather interestingly, the existence of an étale local section is a consequence of a much more elementary fact regarding gerbes in general.

PROPOSITION A.2. *Let S be a scheme and let X be an algebraic stack which is an fppf gerbe over S . Then the structure morphism $\pi: X \rightarrow S$ is smooth.*

Proof. The question is local on the base in the fppf topology, so we may assume that X is a classifying stack $B_S G$ for some group algebraic space G which is flat and locally of finite presentation over S . In particular, we have an atlas $S \rightarrow X$. Let $U \rightarrow X$ be a smooth atlas. Then the fibre product $U' = S \times_X U$ is an algebraic space which is smooth over S , and the projection $U' \rightarrow U$ is faithfully flat and locally of finite presentation. Hence, U is also smooth over S by [EGAIV, Proposition 17.7.7], and it follows that X is smooth over S . \square

In fact, from the proof we see that the structure morphism of a gerbe has all properties which are fppf local on the base and which *descend* fppf locally on the source. Note, however, that although for instance being étale is such a property when we restrict to morphisms of schemes, this is not the case when we consider morphisms of algebraic stacks. Indeed, the classifying stack $B\mu_p$ is not étale over the base if the base is a field of characteristic p .

One of the fundamental properties of finite linearly reductive groups acting on algebraic spaces is that taking quotients of invariant closed subspaces coincides with taking schematic images. We give a formulation of this property in terms of tame stacks.

PROPOSITION A.3. *Let $X \rightarrow S$ be a tame stack over a scheme S and let $\pi: X \rightarrow X_{\text{cs}}$ be its coarse space. Let $Z \subset X$ be a closed substack. Then the canonical map $Z \rightarrow \pi(Z)$ to the schematic image of Z under π is the coarse space of Z .*

Proof. The question may be verified after a faithfully flat base change of the coarse space. Thus, we may use Theorem A.1(c) to reduce to the case when $X_{\text{cs}} = \text{Spec } A$ for some ring A and $X = [U/G]$, where G is a linearly reductive group scheme over $\text{Spec } A$, and $U = \text{Spec } B$, where B is a finite A -algebra. Let $I \subset B$ be the G -invariant ideal defining Z . Then the coarse space of Z is $\text{Spec } (A/I)^G$ and the schematic image is given by $\text{Spec } A^G/I^G$. But the functor $-^G$ is exact since the group G is linearly reductive, so $(A/I)^G = A^G/I^G$, as desired. \square

Propositions A.2 and A.3 together imply the following corollary, which is a reformulation of [AOV08, Proposition 3.7]. This gives the simplification of the proof, that (b) implies (d) in Theorem A.1, that was promised earlier.

COROLLARY A.4. *Let $X \rightarrow S$ be a tame stack over a scheme S and let $\pi: X \rightarrow X_{\text{cs}}$ be the coarse space. Then the residual field at each point $\xi \in |X|$ coincides with the residual field of the point $\pi(\xi)$. In particular, every k -point in X_{cs} , with k a field, lifts to a K -point of X for some separable field extension K/k .*

Proof. Let $k = \kappa(\pi(\xi))$ be the residue field of $\pi(\xi)$. Let $X_k \rightarrow \text{Spec } k$ be the pull-back of π along $\text{Spec } k \hookrightarrow X_{\text{cs}}$. Then $X_k \rightarrow \text{Spec } k$ is a coarse space by [AOV08, Corollary 3.3(a)]. Furthermore, the induced map $X_k^{\text{red}} \rightarrow \text{Spec } k$ from the reduction of X_k is a coarse space by Proposition A.3, since the schematic image of X_k^{red} in $\text{Spec } k$ must be $\text{Spec } k$ itself. But the monomorphism $X_k^{\text{red}} \hookrightarrow X$, being a monomorphism from a reduced, locally noetherian singleton, is the residual gerbe at ξ . Hence, k is indeed the residue field at ξ . By Proposition A.2, the map $X_k^{\text{red}} \rightarrow \text{Spec } k$ is smooth, so it admits a section étale locally on $\text{Spec } k$. From this, the last statement of the proposition follows. \square

Before turning to condition (e) of Theorem A.1, we review what is meant by a *fixed point* for an action by an algebraic group G . Note that it is insufficient to just study the action of the topological group $|G|$, as is illustrated by the following basic example.

Example A.5. The group μ_2 has a natural action on $\text{Spec } \mathbb{C}$ over $\text{Spec } \mathbb{R}$. Topologically, the space $\text{Spec } \mathbb{C}$ has a single point, but it is not accurate to say that the point is fixed under the μ_2 -action. Rather, we wish to think of μ_2 as acting freely on $\text{Spec } \mathbb{C}$, making $\text{Spec } \mathbb{C}$ a μ_2 -torsor over $\text{Spec } \mathbb{R}$. In this case, we can think of $\text{Spec } \mathbb{C}$ having two different geometric points $\text{Spec } \mathbb{C} \rightarrow \text{Spec } \mathbb{C}$ over $\text{Spec } \mathbb{R}$, neither of which is fixed under the μ_2 -action.

If we are working with non-reduced group schemes, even this point of view does not work. This can be seen by instead considering the corresponding example over the function field $k = \mathbb{F}_p(X)$, and the group μ_p acting on $\text{Spec } k[Y]/(Y^p - X)$ over $\text{Spec } k$.

Instead we consider the correct, sheaf-theoretic definition based on the following proposition. We omit the proof, since it is an easy diagram chase.

PROPOSITION A.6. Let $R \rightrightarrows U$ be a groupoid of sheaves on a site and let $\xi: T \rightarrow U$ be a generalised point. Then the following statements are equivalent.

- (i) For any morphism $T' \rightarrow T$, the restriction of ξ to T' is the unique representative of its isomorphism class in the groupoid $R(T') \rightrightarrows U(T')$ viewed as a small category.
- (ii) The graph $\Gamma_\xi: T \hookrightarrow U \times T$ is invariant with respect to the groupoid $R \times T \rightrightarrows U \times T$.

If the groupoid $R \rightrightarrows U$ is an action groupoid for a group action $G \times U \rightarrow U$, then the above two statements are equivalent to the following statement.

- (iii) The canonical monomorphism $\text{Stab}(\xi) \rightarrow G \times T$ of groups over T is an isomorphism.

DEFINITION A.7. Let $R \rightrightarrows U$ and $\xi: T \rightarrow U$ be as in Proposition A.6. If the conditions given in the proposition are satisfied, we say that ξ is a *fixed point* for the groupoid $R \rightrightarrows U$. If the groupoid is algebraic and $\xi \in |U|$ is a point in U , we say that ξ is a *fixed point* if it may be represented by a morphism $\text{Spec } k \rightarrow U$ which is a fixed point in the above sense. It is easily verified that the choice of representative is irrelevant.

That condition (b) implies condition (d) of Theorem A.1 follows from the sharper [AOV08, Proposition 3.6]. In order to see that it also implies (e), we need to sharpen the formulation of the proposition somewhat more.

PROPOSITION A.8. Let S be a scheme and let X be an algebraic stack having finite inertia and being quasi-separated and locally of finite presentation over S . Denote the coarse space by $\pi: X \rightarrow X_{\text{cs}}$ and let $\xi \in |X|$ be a point. If the stabiliser at ξ is linearly reductive, then there exist an étale neighbourhood $Y \rightarrow X_{\text{cs}}$ of $\pi(\xi)$, a finite, linearly reductive group scheme $G \rightarrow Y$ acting on a finite scheme $U \rightarrow Y$ of finite presentation and an isomorphism $[U/G] \simeq Y \times_{X_{\text{cs}}} X$ of algebraic stacks. Furthermore, the point ξ lifts to a point $\xi' \in U$ which is fixed under the action of G .

All but the last sentence comes from the original statement, and although the last sentence is not explicitly stated, it follows from the proof. Indeed, the scheme $U \rightarrow Y$ is constructed in a way such that the diagram

$$\begin{array}{ccc} \text{Spec } k & \xrightarrow{\xi'} & U \\ \downarrow & & \downarrow \\ \text{B}_k G_\xi & \longrightarrow & Y \times_{X_{\text{cs}}} X \end{array}$$

becomes cartesian. Here $\xi: \text{Spec } k \rightarrow X$ is a morphism representing ξ , the vertical maps are G -torsors and G_ξ denotes the stabiliser at ξ . In particular, the point ξ' becomes the desired lifting according to the third condition of Proposition A.6 characterising fixed points. Finally, to see that this implies (e) in the case when X is smooth over the base, we apply the following proposition.

PROPOSITION A.9. Let U be an algebraic space which is flat, locally of finite presentation and quasi-separated over a scheme S . Assume that $R \rightrightarrows U$ is a groupoid which is flat and locally of finite presentation, and assume that the stack quotient $[U/R]$ is smooth over S . Then U is smooth over S at any point $\xi \in |U|$ which is a fixed point with respect to the groupoid $R \rightrightarrows U$.

Proof. Let $\xi: \text{Spec } k \rightarrow U$ be a geometric point representing ξ , and let $R_k \rightrightarrows U_k$ denote the pull-back of the groupoid along the morphism $\text{Spec } k \rightarrow U \rightarrow S$. Since ξ is a fixed point, the graph $\Gamma_\xi: \text{Spec } k \rightarrow U_k$ is invariant in the groupoid. Hence, the diagram

$$\begin{array}{ccc} \text{Spec } k & \xrightarrow{\Gamma_\xi} & U_k \\ \downarrow & & \downarrow \\ \text{B}_k\text{Stab}(\xi) & \xrightarrow{\iota} & [U_k/R_k] \end{array}$$

is 2-cartesian. The graph $\Gamma_\xi: \text{Spec } k \rightarrow U_k$ is a closed immersion since it is a rational point. Since the vertical maps in the diagram are faithfully flat and locally of finite presentation, it follows that also ι is a closed immersion, by descent. The stack $[U_k/R_k]$ is smooth over k since it is isomorphic to the pull-back $[U/R] \times_S \text{Spec } k$ and smoothness is stable under base change. The stack $\text{B}_k\text{Stab}(\xi)$ is smooth over k since it is a gerbe. It follows that ι is a regular immersion, so the same holds for the graph Γ_ξ , since the property of being a regular immersion is stable under flat base change. But then U_k must be regular at Γ_ξ . Since U is flat and of finite presentation over S , it follows that $U \rightarrow S$ is smooth at ξ . \square

We conclude the section with a technical lemma, which is not related to tame stacks, about closed points on stacks. In general, rational points on algebraic stacks need not be closed. For instance, the stack $[\mathbb{A}_k^1/\mathbb{G}_m]$ has an open rational point. But stacks with finite stabilisers are better behaved.

LEMMA A.10. *Let k be a field and X an algebraic stack which is locally of finite type and quasi-separated over $\text{Spec } k$. If X has point-wise finite stabilisers, then every point of finite type in X is closed. In particular, all rational points of X are closed.*

Proof. Let $\xi \in |X|$ be a point of finite type in X , and let $f: \mathcal{G}_\xi \hookrightarrow X$ be the inclusion of the residual gerbe at ξ . By the assumption that ξ is a point of finite type, the monomorphism f is locally of finite type. We want to show that f is a closed immersion.

We may assume that X is of finite type over k . Since X is quasi-separated and has finite stabilisers, we can choose a quasi-finite, flat covering $U \rightarrow X$ with U affine [Ryd11, Theorem 7.1]. Let $U_\xi = U \times_X \mathcal{G}_\xi$. Since U_ξ is quasi-finite over the field k , it must also be finite over k . Since we assume that U is affine, also the projection $U_\xi \rightarrow U$ must be finite. It follows that f is a finite monomorphism and therefore a closed immersion. \square

Appendix B. The cotangent complex of toric stacks

We wish to compute the cotangent complex for a basic toric stack X without torus factors over a scheme S . Assume that X has homogeneous coordinates $(\mathcal{O}_S[x_1, \dots, x_n], A, \mathbf{a})$, where $A = \mathbb{Z}/q_1\mathbb{Z} \times \dots \times \mathbb{Z}/q_s\mathbb{Z}$ and $a_i = (a_{1i}, \dots, a_{si})$ for $1 \leq i \leq n$.

The quasi-coherent \mathcal{O}_X -modules are in canonical one-to-one correspondence with the quasi-coherent A -graded $\mathcal{O}_S[x_1, \dots, x_n]$ -modules. Given $a \in A$, we denote by $\mathcal{O}_X(-a)$ the line bundle corresponding to the free $\mathcal{O}_S[x_1, \dots, x_n]$ -module of rank 1 generated in degree a .

PROPOSITION B.1. *Let X be a basic toric stack over a scheme S , and assume that X has homogeneous coordinates as described above. Then the cotangent complex $L_{X/S}$ is quasi-isomorphic to the perfect complex*

$$\mathcal{O}_X(-a_1) \oplus \dots \oplus \mathcal{O}_X(-a_n) \oplus \mathcal{O}_X^s \rightarrow \mathcal{O}_X^s$$

concentrated in cohomological degrees $[0, 1]$ and with differential given by the matrix

$$\begin{pmatrix} a_{11}x_1 & \cdots & a_{1n}x_n & q_1 & & 0 \\ \vdots & \ddots & \vdots & & \ddots & \\ a_{s1}x_1 & \cdots & a_{sn}x_n & 0 & & q_s \end{pmatrix}.$$

Proof. Due to the base-change properties of the cotangent complex, we may just as well assume that $S = \text{Spec } \mathbb{Z}$. Consider the \mathbb{Z}^s -graded coordinate ring $\mathcal{O}_{\mathbb{A}^n \times \mathbb{G}_m^s} = \mathbb{Z}[x_1, \dots, x_n, t_1^{\pm 1}, \dots, t_s^{\pm 1}]$ of the space $\mathbb{A}^n \times \mathbb{G}_m^s$, with x^i having degree a_i , viewed as a vector of integers, and t_j having degree q_j . The grading corresponds to an action of the torus \mathbb{G}_m^s . The stack quotient $[\mathbb{A}^n \times \mathbb{G}_m^s / \mathbb{G}_m^s]$ is equivalent to X , with the equivalence induced by slicing the action groupoid $\mathbb{A}^n \times \mathbb{G}_m^s \times \mathbb{G}_m^s \rightrightarrows \mathbb{A}^n \times \mathbb{G}_m^s$ at the closed subscheme $V(t_j = 1 \mid 1 \leq j \leq s)$ of $\mathbb{A}^n \times \mathbb{G}_m^s$. Indeed, then we get the Morita equivalent groupoid $\mathbb{A}^n \times \Delta \rightrightarrows \mathbb{A}^n$, with $\Delta = A^\vee$, which is the presentation of X corresponding to the original A -grading.

The atlas introduced above gives us a cartesian square

$$\begin{array}{ccc} \mathbb{A}^n \times \mathbb{G}_m^s \times \mathbb{G}_m^s & \xrightarrow{\alpha} & \mathbb{A}^n \times \mathbb{G}_m^s \\ \pi \downarrow & & q \downarrow \\ \mathbb{A}^n \times \mathbb{G}_m^s & \xrightarrow{q} & X \end{array}$$

of smooth morphisms. Here π denotes the projection on the first two factors and α denotes the action map. Choose coordinates $\mathcal{O}_{\mathbb{A}^n \times \mathbb{G}_m^s \times \mathbb{G}_m^s} = \mathbb{Z}[x_i, t_j^{\pm 1}, u_k^{\pm 1}]$. Then the action map corresponds to the ring homomorphism φ taking x_i to $x_i u_1^{a_{1i}} \cdots u_s^{a_{si}}$ and t_i to $t_i u_i^{q_i}$. We get an induced map of differentials

$$d\varphi: \alpha^* \Omega_{\mathbb{A}^n \times \mathbb{G}_m^s} \rightarrow \Omega_{\mathbb{A}^n \times \mathbb{G}_m^s \times \mathbb{G}_m^s / \mathbb{A}^n \times \mathbb{G}_m^s}$$

given by

$$dx_i \mapsto a_{1i} \varphi(x_i) u_1^{-1} du_1 + \cdots + a_{si} \varphi(x_i) u_s^{-1} du_s, \quad dt_j \mapsto q_j \varphi(t_j) u_j^{-1} du_j.$$

Since the above square is cartesian and α is flat, this descends to a map $\Omega_{\mathbb{A}^n \times \mathbb{G}_m^s} \rightarrow \Omega_{\mathbb{A}^n \times \mathbb{G}_m^s / X}$. We choose (dx_i, dt_j) as a basis for the left-hand side. An easy calculation gives that the elements $u_i^{-1} du$ descend to elements of $\Omega_{\mathbb{A}^n \times \mathbb{G}_m^s / X}$, which we also denote by $u_i^{-1} du$. These elements form a basis for the right-hand side. With respect to these choices, the map is described by the matrix

$$\begin{pmatrix} a_{11}x_1 & \cdots & a_{1n}x_n & t_1 q_1 & & 0 \\ \vdots & \ddots & \vdots & & \ddots & \\ a_{s1}x_1 & \cdots & a_{sn}x_n & 0 & & t_s q_s \end{pmatrix}.$$

Now we may compute the cotangent complex with help of the distinguished triangle

$$\Omega_{\mathbb{A}^n \times \mathbb{G}_m^s} \rightarrow \Omega_{\mathbb{A}^n \times \mathbb{G}_m^s / X} \rightarrow \mathbf{L}q^* L_X[1].$$

Hence, the derived pull-back $\mathbf{L}q^* L_X$ is given by a two-term complex $\Omega_{\mathbb{A}^n \times \mathbb{G}_m^s} \rightarrow \Omega_{\mathbb{A}^n \times \mathbb{G}_m^s / X}[-1]$ with differential as in the matrix above. This is graded by \mathbb{Z}^s , with $|dx_i| = a_i$, $|dt_j| = q_j$ and $|u_k^{-1} du_k| = 0$, which reflects that the complex lives over $[\mathbb{A}^n / \Delta]$. We obtain the original homogeneous coordinates by slicing at $V(t_j = 1 \mid 1 \leq j \leq s)$ as described before, which gives the result stated in the proposition. \square

Note that if the product $q_1 \cdots q_s$ is invertible in \mathcal{O}_S , then this complex is quasi-isomorphic to the \mathcal{O}_X -module

$$\mathcal{H}^0(L_{X/S}) = \mathcal{O}_X(-a_1) \oplus \cdots \oplus \mathcal{O}_X(-a_n).$$

The fibre of this module in a point ξ , together with the natural action of the stabiliser, coincides with the conormal representation at ξ . If $q_1 \cdots q_s$ is not invertible, this need not be true. Consider, for instance, $X = [\mathbb{A}_S^1/\mu_q]$, where $S = \text{Spec } k$ with k a field of characteristic p and $q = 0$ in k . If μ_q is acting with weight a , we have

$$\mathcal{H}^0(L_{X/S}) = \begin{cases} \mathcal{O}_X(-a) \oplus \mathcal{O}_X & \text{if } a = 0 \text{ in } k, \\ \mathcal{O}_X & \text{otherwise.} \end{cases}$$

We see that, in general, the information about the weight is lost. If one wishes to preserve this information, it is better to look at $[L_{X/S}]$ or, equivalently, the alternating sum $[\mathcal{H}^0(L_{X/S})] - [\mathcal{H}^1(L_{X/S})]$ in the K -group.

Appendix C. Cotangent complex interpretation

In this appendix, we take a brief look at an alternative way to look at the conormal representation in terms of the cotangent complex. An advantage with this point of view is that we can use various distinguished triangles for the cotangent complex in our computations.

Given an algebraic stack X , we consider the triangulated category $\text{Perf}(X)$. This is the full subcategory of the derived category of X whose objects are represented by perfect complexes. Its associated Grothendieck group is denoted by $K_0(\text{Perf}(X))$. A morphism $f: X \rightarrow Y$ gives a morphism $f^*K_0(\text{Perf}(Y)) \rightarrow K_0(\text{Perf}(X))$ induced by the derived pull-back. If X is smooth over a field, then $K_0(\text{Perf}(X))$ is canonically isomorphic to the Grothendieck group $K_0(\text{Coh}(X))$ of coherent \mathcal{O}_X -modules.

In the particular situation described in the beginning of § 6, we have a 2-commutative diagram

$$\begin{array}{ccccc} \text{B}\Delta_\xi & \xrightarrow{\iota} & X_{\bar{k}} & \xrightarrow{f} & X \\ & \searrow & \downarrow & & \downarrow \\ & & \text{Spec } \bar{k} & \longrightarrow & S \end{array}$$

where the square is 2-cartesian. Denote the composition $f \circ \iota$ by g . By using the distinguished triangle for composition and the base-change property for cotangent complexes, we get the identities

$$\iota^*[L_{X_{\bar{k}}/\bar{k}}] - [L_{\text{B}\Delta_\xi/\bar{k}}] + [L_{\text{B}\Delta_\xi/X_{\bar{k}}}] = 0, \quad [L_{X_{\bar{k}}/\bar{k}}] = f^*[L_{X/S}]$$

in $K_0(\text{Perf}(\text{B}\Delta_\xi))$. Since the immersion $\text{B}\Delta_\xi \hookrightarrow X_{\bar{k}}$ is regular, the cotangent complex $L_{\text{B}\Delta_\xi/X_{\bar{k}}}$ is quasi-isomorphic to the complex having $\mathcal{N}_{\text{B}\Delta_\xi/X_{\bar{k}}}$ concentrated in degree -1 . Together with the identities above, this implies that

$$[\mathcal{N}_{\text{B}\Delta_\xi/X_{\bar{k}}}] = -[L_{\text{B}\Delta_\xi/X_{\bar{k}}}] = g^*[L_{X/S}] - [L_{\text{B}\Delta_\xi/\bar{k}}]$$

in $K_0(\text{Perf}(\text{B}\Delta_\xi))$. If \bar{k} has characteristic 0, the complex $L_{\text{B}\Delta_\xi/\bar{k}}$ vanishes since $\text{B}\Delta_\xi$ is étale over \bar{k} . In positive characteristic, the complex $L_{\text{B}\Delta_\xi/\bar{k}}$ need not vanish, but its class in the Grothendieck group always vanishes. This can be seen from the explicit formula derived in Appendix B. In particular, we have the identity $[\mathcal{N}_{\text{B}\Delta_\xi/X_{\bar{k}}}] = g^*[L_{X/S}]$. We summarise the result in the following proposition.

PROPOSITION C.1. *Let $(X, \mathbf{E})/S$ be a standard pair with diagonalisable stabilisers. Furthermore, let $\xi: \text{Spec } \bar{k} \rightarrow X$ be a geometric point, Δ_ξ the stabiliser at ξ and $g: \text{B}\Delta_\xi \rightarrow X$ the induced morphism. Then we have the identity*

$$[\mathcal{N}_{\text{B}\Delta_\xi/X_{\bar{k}}}] = g^*[L_{X/S}]$$

in the group $K_0(\text{Perf}(\text{B}\Delta_\xi))$, which we have identified with $K_0(\text{Coh}(\text{B}\Delta_\xi))$.

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